REFERENCES


An Iterative Technique for Determining the Minimal Number of Variables for a Totally Symmetric Function with Repeated Variables

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Abstract—Several analytic procedures exist for transforming a partially symmetric switching function to a totally symmetric switching function by judiciously repeating certain variables. Presumably the best totally symmetric representation for a given function would be the one having the fewest variables. This note presents an iterative technique for finding the totally symmetric realization for a given function that has the absolute minimum number of variables.

Index Terms—Combinatorial logic, minimization of switching functions, switching theory, synthesis of switching functions, totally symmetric switching functions, use of partial symmetry information.

INTRODUCTION

At least three analytic procedures [1]–[3] exist for transforming an arbitrary switching function
\[ f(x_1, x_2, \ldots, x_n) \]
(1)
to a totally symmetric switching function of the form
\[ S_A(x_1, x_2, \ldots, x_r, y_1, \ldots, y_i, y_{i+1}, \ldots, y_r, y_{r+1}, \ldots, y_s) \]
(2)
where
\[ r = n - \delta. \]
Each \( x_i \) and \( y_i \) is assigned to be a unique \( x_k \) (or \( x_0 \)), and each \( y_i \) appears as an argument \( \alpha_i \) times in \( S_A \). The \( y_i \) are thus repeated variables and the total number of variables \( N_V \) in \( S_A \) is given by
\[ N_V = \delta + \sum_{i=1}^{\infty} \alpha_i. \]
(3)
The circuit realizations for \( S_A \) obtained using the methods in [3]–[5] generally increase in complexity as the number of variables in \( S_A \) increases. Thus it is desirable to determine the \( S_A \) representation for \( f \) that minimizes \( N_V \). This note describes an iterative procedure for determining the minimal value of \( N_V \). It will be assumed that the reader is familiar with the results in [2].

PROCEDURE

Assume that the method in [2] has been applied to express a given function in the form of (2). Thus the integers in the set \( A \) and all the \( \alpha_i \) have been calculated. Henceforth refer to these initial values of \( A \) and the \( \alpha_i \) as
\[ A^0 = \text{initial } A \text{ set} \]
(4)
and
\[ \alpha_{i_0} = \text{initial value of } \alpha_i, \]
\[ = 2^{i-1}(\delta + 1), \quad i = 1, 2, \ldots, r. \]
Recall that \( \delta \) is the number of variables in \( S_A \) that appear only once. It will be assumed that these nonrepeated variables have been chosen and therefore \( \delta \) is fixed. The problem of finding a minimal \( N_V \) thus reduces to finding the minimal value for \( \sum_{i=1}^{\infty} \alpha_i \), starting with \( A^0 \) and the \( \alpha_{i_0} \). A solution to this problem was developed by the author and will be presented here without proof. For a proof and a more thorough discussion, the reader should refer to [6, ch. 6].

The solution is an iterative procedure which determines the minimal value of \( \alpha_i \) and then forms the corresponding \( A^1 \) from \( A^0 \). Next, the minimal value of \( \alpha_i \) is determined and then \( A^2 \) is formed from \( A^1 \). Each \( \alpha_i \) is found independently of the other \( \alpha_i, j \neq i \), and then \( A^i \) is formed from \( A^{i-1} \). The following theorem formally defines this procedure.

Theorem (Minimal \( \alpha \)-Set Theorem): Given a function \( S_A(x_1, x_2, \ldots, x_s, y_1, \ldots, y_i, y_{i+1}, \ldots, y_r, y_{r+1}, \ldots, y_r) \) where each \( y_i \) appears as an argument \( \alpha_{i_0} = 2^{i-1}(\delta + 1) \) times, then there exists a function \( S_A^*(x_1, x_2, \ldots, x_s, y_1, \ldots, y_i, y_{i+1}, \ldots, y_r, \ldots, y_r) \) where \( y_i \) appears \( \alpha_i \leq \alpha_{i_0} \) times if and only if both
\[ \alpha_i + j + k(\delta + 1), \quad \alpha_i + j + k(\delta + 1) + \beta_i \in A^{i-1} \]
(5)
or if both
\[ \alpha_i + j + k(\delta + 1), \quad \alpha_i + j + k(\delta + 1) + \beta_i \in A^{i-1} \]
(6)
where
\[ \beta_i = \alpha_{i_0} - \alpha_i, \]
(7)
and the \( j, k, \) counters take on all of the values
\[ j = 0, 1, 2, \ldots, \delta_i - 1 = (\beta_i-1 + \beta_i-2 + \cdots + \beta_1) \]
(8)
\[ k = 0, 1 \cdot 2^i, 2 \cdot 2^i, 3 \cdot 2^i, \ldots, 2^r - 2^i. \]
(9)
The set \( A^i \) is formed from \( A^{i-1} \) as follows (\( k \) is defined above):

Manuscript received February 4, 1972; revised April 28, 1972.

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where
\[ A^i = A^i_1 \cup A^i_2 \quad (10) \]

and
\[ A^i_2 = \{ a - \beta / a \in A^{i-1} \text{ but } a \notin A^i_1 \} \quad (11) \]

Considerable insight into the process of deriving the conditions stated in (5) and (6) can be gained by considering the special case \( r = 1 \). Then
\[ \delta + 1 \text{ times} \]
\[ f = S_A(x_1, \ldots, x_n, y_1, \ldots, y_l) \quad (13) \]
can be expanded about \( \beta \) of the \( y_i \) variables as
\[ f = y_1 S_A(x_1, \ldots, x_n, y_1, \ldots, y_l, 0, \ldots, 0) \]
\[ + y_0 S_A(x_1, \ldots, x_n, y_1, \ldots, y_l, 1, \ldots, 1) \quad (14) \]
\[ = y_1 S_A(x_1, \ldots, x_n, y_1, \ldots, y_l) \]
\[ + y_0 S_A(x_1, \ldots, x_n, y_1, \ldots, y_l) \quad (15) \]

where
\[ A_0 = A \cap \{ 0, 1, \ldots, \delta + \alpha_1 \} \]

and
\[ A_1 = (A - \beta) \cap \{ 0, 1, \ldots, \delta + \alpha_1 \}. \]

If the conditions of (5) and (6) are satisfied, then
\[ y_0 S_A(x_1, \ldots, x_n, y_1, \ldots, y_l) \]
\[ = S_{A_1}(x_1, \ldots, x_n, y_1, \ldots, y_l) \]

where
\[ A_1 = A \cap \{ \delta + 1, \ldots, \delta + \alpha_1 \}. \]

This is readily verified by considering the form of the minterms for integers in \( A_0 \) and \( A_1 \) that are \( \leq \delta \) and that are \( \geq \delta \). For example, if \( a \in A_0 \) and \( a > \delta \), then the minterms that \( a \) represents in \( S_A \) are of the form \( x_1^* x_2^* \cdots x_n^* y_1 \). But then \( y_0 x_1^* x_2^* \cdots x_n^* y_1 = 0 \). Thus and

\[ y_0 S_{A_0} = S_{A_1}. \]

The conditions in (5) and (6) are necessary in order that
\[ S_{A_0} + S_{A_1} = S_{A^1} = S_A \quad (17) \]

where \( A^1 = A_0 \cup A_1 \). Cases 1 and 2 elaborate on the necessity.

Case 1: Suppose \( \alpha_1 + j \in A \) but \( \alpha_1 + j + \beta_1 \notin A \) where \( 0 \leq j \leq \delta - \alpha_1 \). Then there are minterms in \( S_A \) of the form \( x_1^* x_2^* \cdots x_n^* y_1 \), where \( \alpha_1 + j \) of the \( x_i^* = x_i \) but no minterms \( x_1^* x_2^* \cdots x_n^* y_1 \), where \( j \) of the \( x_i^* = x_i \). However, \( \alpha_1 + j \in A_0 \). Thus \( \alpha_1 + j \in A' \) and there would be minterms in \( S_A' \) of the form \( x_1^* x_2^* \cdots x_n^* y_1 \), where \( j \) of the \( x_i^* = x_i \). But then \( S_{A'} \neq S_A \).

Case 2: Suppose \( \alpha_1 + j \in A \) but \( \alpha_1 + j + \beta_1 \in A \) where \( 0 \leq j \leq \delta - \alpha_1 \). Then \( \alpha_1 + j \in A_0 \), \( A_1 \) and \( \alpha_1 + j \in A' \). Originally \( S_A \) contains minterms of the form \( x_1^* x_2^* \cdots x_n^* y_1 \), where \( j \) of the \( x_i^* = x_i \), whereas \( S_{A'} \) will not contain these minterms. Thus \( S_{A'} \neq S_A \).

In Case 1, one might propose that in forming \( A_0 \) the integer \( \alpha_1 + j \) be deleted. This is not possible as \( \alpha_1 + j \leq \delta \), and since \( \alpha_1 + j \in A \), necessarily \( \alpha_1 + j + \beta_1 \in A \). In Case 2, the integer \( \alpha_1 + j \) might be included in \( A_0 \) and thus in \( A' \). But then there are minterms in \( S_A' \) of the form \( x_1^* x_2^* \cdots x_n^* y_1 \), where \( \alpha_1 + j \) of the \( x_i^* = x_i \), which are not in the original function \( S_A \). Thus the conditions in (5) and (6) are both necessary and sufficient.

Clearly, application of the theorem must start with \( i = 1 \) and if it is desired to find \( \alpha_{\text{min}} \), then apply the tests of (5) and (6) to \( \alpha_1 = 2 \). As soon as one test fails, move on to \( \alpha_1 = 3 \). If all the tests are successful, then \( \alpha_{\text{min}} = 2 \) and \( A^1 \) can be formed from \( A^0 \) as in (10). Once the reader has worked a few concrete examples the conditions that there exist \( \alpha_1 \leq \alpha_{\text{min}} \) and the formation of \( A^1 \) from \( A^{i-1} \) become much less formidable than the notation. The following example demonstrates the application of the theorem.

**Example**

As an example of the application of this procedure, consider the function
\[ f(z_1, z_2, z_3, z_4, z_5, z_6) = \Sigma(1-6, 9-14, 17-22, 24-26, 28, 33-38, \] \[ 40-42, 44, 48-50, 52, 56-58, 60) \]

which appears in both [1] and [2] and for which Born and Scidmore [2] determined
\[ f = S_A(z_1, z_2, z_3, z_4, z_5, z_6, \ldots, z_6) \]
with \( A = \{ 2-13, 16-21, 24-25 \} \) and for this function
\[ \delta = r = 3, \]
\[ \alpha_{10} = 4, \]
\[ \alpha_{20} = 8, \]
\[ \alpha_{26} = 16, \]
\[ A^0 = \{2-13, 16-21, 24-25\}. \]

Let \( \alpha_1 = 2 \) and apply the test conditions given in (5) and (6).
\[
k = 0: \quad 2, 4 \in A^0 \quad k = 2: \quad 10, 12 \in A^0 \\
k = 3, 5 \in A^0 \quad 11, 13 \in A^0 \\
k = 4: \quad 18, 20 \in A^0 \quad k = 6: \quad 26, 28 \in A^0 \\
19, 21 \in A^0 \quad 27, 29 \in A^0.
\]

All the tests pass and so \( \alpha_{1 \text{ min}} = 2 \) and
\[
A^1 = \{2-3, 8-11, 16-19, 24-25\} \cup \{2-5, 10-11, 18-19\} \\
= \{2-5, 8-11, 16-19, 24-25\}.
\]

Let \( \alpha_2 = 2 \) and apply the test conditions given in (5) and (6).
\[
k = 0: \quad 2, 8 \in A^1 \quad k = 4: \quad 18, 24 \in A^1 \\
k = 3, 9 \in A^1 \quad 19, 25 \in A^1 \\
k = 4, 10 \in A^1 \quad 20, 26 \in A^1 \\
k = 5, 11 \in A^1 \quad 21, 27 \in A^1.
\]

All the tests pass and so \( \alpha_{2 \text{ min}} = 2 \) and
\[
A^2 = \{2-5, 16-19\} \cup \{2-5, 18-19\} \\
= \{2-5, 16-19\}.
\]

Let \( \alpha_3 = 2 \) and apply the test conditions given in (5) and (6).
\[
k = 0: \quad 2, 16 \in A^2 \\
k = 3, 17 \in A^2 \\
k = 4, 18 \in A^2 \\
k = 5, 19 \in A^2 \\
k = 6, 20 \in A^2 \\
k = 7, 21 \in A^2.
\]

All the tests pass and so \( \alpha_{3 \text{ min}} = 2 \) and
\[
A^3 = \{2-5\} \cup \{2-5\} \\
= \{2-5\}.
\]

Finally then
\[
f = S_{A^3}(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8).
\]

The reader may wish to try finding the minimal \( \alpha_i \) for the function \( S_{A^0}(x_1, x_2, x_3, y_1, \ldots, y_4, y_5, \ldots, y_9) \), where \( A^0 = \{3, 4, 6, 9, 10, 12, 16\} \), \( \alpha_{0 \text{ min}} = 4 \), \( \alpha_{10 \text{ min}} = 8 \), and \( \alpha_{16 \text{ min}} = 16 \). This function was reduced by the author to \( S_{A^0}(x_1, x_2, y_1, \ldots, y_4, y_5, \ldots, y_9) \), where \( A^0 = \{3, 4, 6\} \), \( \alpha_{0 \text{ min}} = 4 \), \( \alpha_{16 \text{ min}} = 26 \), and \( \alpha_{18 \text{ min}} = 6 \).

**Discussion**

In view of the work by Yau and Tang [1], if the original function \( f \) is invariant under the permutation \( y_i \sim y_j \), then once the minimal \( \alpha_i \) is found using the preceding procedure it is obvious that \( \alpha_{i \text{ min}} = \alpha_{i \text{ min}} \). Thus in the example, it was found that \( \alpha_{1 \text{ min}} = 2 \) and it was known that
\[
f \equiv y_1 \sim y_2 \sim y_3 \quad (y_1 = z_4, y_2 = z_5, y_3 = z_6).
\]

Therefore, \( \alpha_{3 \text{ min}} = \alpha_{14 \text{ min}} = \alpha_{1 \text{ min}} = 2 \) and it was not necessary to apply the tests given in (5) and (6) for \( \alpha_2 = 2 \) and \( \alpha_3 = 2 \).

Since the procedure is iterative, it is well suited for programming. A program has been written that will handle incompletely specified functions as well as completely specified functions. The program starts by computing \( A^0 \) and the \( \alpha_0 \) for a given function using the method in [2] and then proceeds to determine the minimal \( \alpha_i \) and the corresponding \( A^i \). For a complete description of the program, see [7].

**References**


