Abstract—Any narrowband signal can be represented by two low-frequency functions, containing only frequencies lower than the bandwidth, which represent the envelopes of the in-phase (cosine) and quadrature (sine) components of the carrier frequency. To obtain the transient response of a narrowband system efficiently by computer simulation, the system can be represented by a low-frequency equivalent system, avoiding time-consuming direct simulation of carrier frequencies. This paper presents techniques by which a wide variety of narrowband systems, expressible mathematically as block diagrams, can be converted into block diagrams consisting only of low-frequency functions. The resulting low-frequency equivalent systems can be simulated on either an analog or digital computer by well-known techniques. Although the low-frequency equivalent representation presented here includes an extension of the bandpass low-pass analogy, it is not subject to many of the restrictions normally applied to this analogy. It is not necessary that the modulation be simple AM, that the filtering be symmetrical, or that only the envelope of the output be of interest.

Index Terms—Asymmetrical filtering, bandpass low-pass analogy, block diagram representation, linear and nonlinear systems, low-frequency equivalent, narrowband systems, simulation techniques, transient response.

Introduction

Simulation on an analog or digital computer has long been recognized as an efficient means of determining the transient response of many physical systems. The system is first expressed mathematically, which is often done in terms of a block diagram in which linear differential equations may be expressed by blocks containing transfer functions. These equations and block diagrams are then mechanized or programmed on an analog or digital computer. For many types of systems, the computer mechanization or programming is quite straightforward, using well-known techniques. However, as the frequency of the signals in the simulated system increases, the computation becomes more difficult and time-consuming. This paper shows that when a system carries signals consisting only of a narrow band of frequencies, it is often possible to represent the system in terms of low frequencies of the order of the bandwidth of the system. This avoids the computation time and difficulty required for direct simulation of the actual frequencies in the narrowband system. The general method is based on the fact that any narrowband signal may be represented by two low-frequency functions of time, representing the envelopes of the in-phase (cosine) and quadrature (sine) components of a carrier frequency. Narrowband filtering and other operations such as modulation and detection can be simulated by use of low-frequency functions alone.

Some of the equations and principles presented here have been applied by previous authors to particular problems, but their applicability to narrowband systems in general does not seem to have been widely recognized. Cherry[1] applied some of these techniques to asymmetrical bandpass filtering of amplitude-modulated waves, giving a rather complete treatment of the case in which only the envelope of the output is of interest. Aigrain, Teare, and Williams[2] showed that, for AM systems, a simple single-channel low-pass analog exists if and only if the filtering is symmetrical about the carrier frequency, or if the modulation is very low. This commonly known bandpass low-pass analogy, which had been described earlier by Landon,[3] is a special case of equations discussed in this paper. The type of equation used here to represent any narrowband signal has had considerable use in the analysis of narrowband noise. [4] This two-channel low-pass-equivalent technique of simulating a narrowband system with symmetrical filtering has been applied by Thaler and Meltzer[5] to a particular problem which used a noise input. The author[6] has performed a similar simulation in which the input included a signal as well as noise, and the narrowband system included a nonlinearity (limiting). All of these applications were for AM systems in which only the envelope of the output was desired. However, this is not a limitation on the applicability of the general methods presented in this paper. The main limitation is that the narrowband approximation be valid. This may be true for practical purposes with many frequency-modulated or phase-modulated signals as well as amplitude-modulated signals, especially when they are passed through narrowband filters. The input may include both deterministic signals and noise. The system may have asymmetrical narrowband filtering and include low-frequency portions as well as narrowband portions. Some important types of nonlinear elements can be simulated. Possible applications include the transient analysis of such systems as AFC loops, AGC loops, phase-locked loops, signal detection systems, pulse compression systems, stagger-tuned amplifiers, and combinations of such systems.

The purpose of this paper is to present an organized description of these simulation techniques, and to point out their general applicability for simulation of a wide variety of systems which are narrowband in whole or in part. Use of these techniques should enhance computer
simulation of some systems for which the transient response has too often been neglected or determined by less efficient methods.

The body of the paper will develop first the basic equations for representing narrowband signals in terms of low-frequency functions, then methods of simulating various narrowband circuits by operating on these low-frequency functions

**Basic Equations for Representing Any Narrowband Signal**

Define a narrowband system as one in which the signals contain only frequencies in a band \( \omega_0 - \Delta \omega \leq \omega \leq \omega_0 + \Delta \omega \), where \( \Delta \omega \ll \omega_0 \). It is shown in Appendix I that any such narrowband signal can be expressed in the following form:

\[
v(t) = x(t) \cos \omega_0 t - y(t) \sin \omega_0 t \tag{1}\]

where

\[
\omega_0 = \text{carrier frequency, a constant, in radians per second, and}\]

\[
x(t), y(t) = \text{low-frequency functions of time, containing only frequencies in the band } 0 \leq \omega \leq \Delta \omega.
\]

Furthermore, any signal expressible in this form is a narrowband signal. Two special cases are of interest. If either \( x(t) \) or \( y(t) \) is continuously zero, \( v(t) \) is a simple amplitude-modulated carrier. If both \( x(t) \) and \( y(t) \) are independent, normally distributed, low-frequency random variables, (1) is a representation of narrowband noise commonly used in signal-detection probability theory. \[1\]

An alternate form of expressing the narrowband signal of (1) is as follows:

\[
v(t) = r(t) \cos \left[ \omega_0 t + \theta(t) \right] \tag{2}\]

where

\[
r(t) = \sqrt{x^2 + y^2} \tag{3}\]

\[
\theta(t) = \tan^{-1} (y/x). \tag{4}\]

This is basically a rectangular-to-polar conversion which is established by means of the following trigonometric identity:

\[
\cos [\omega_0 t + \theta(t)] = \cos \theta(t) \cos \omega_0 t - \sin \theta(t) \sin \omega_0 t.
\]

The related polar-to-rectangular conversion is

\[
x(t) = r(t) \cos \theta(t) \tag{5}\]

\[
y(t) = r(t) \sin \theta(t). \tag{6}\]

Note from (3) and (4) that \( r(t) \) and \( \theta(t) \) are basically low-frequency functions of time, since they depend only on the low-frequency functions \( x(t) \) and \( y(t) \). However, because of the nonlinear relationships in (3) and (4), exact maximum frequencies are less easily established for \( r(t) \) and \( \theta(t) \) than for \( x(t) \) and \( y(t) \).

**Amplitude Modulation**

To generate a simple amplitude-modulated carrier, refer to (2), letting \( \omega_0 \) be the carrier frequency, \( \theta \) be any desired constant phase, and \( r(t) \) be the time-varying modulating function. Equations (5) and (6) can be used to obtain \( x(t) \) and \( y(t) \) for use in (1). For many purposes using amplitude modulation, \( \theta \) is arbitrary and can be set equal to zero, thus eliminating the term in \( y(t) \) in (1).

**Envelope Detection**

For narrowband signals as considered in this paper, \( r(t) \) represents the envelope, and (3) can be used to simulate an envelope detector. \[11,12,14,15\] Both linear detectors, producing \( r(t) \), and square-law detectors, producing \( [r(t)]^2 \), as well as detectors of other laws, are possible. A typical envelope detector includes a low-pass filter to remove carrier and higher frequency components from the output. The effect of this filter on low-frequency components may not be significant, but if it is, it can be included simply. Fig. 1 illustrates the method of simulating an envelope detector, using a linear detector as an example.

**Symmetrical Narrowband Filter Centered at \( \omega_0 \)**

Thaler and Meltzer\[16\] have shown that the effect of a narrowband filter, which is expressible as \( F(j\omega - j\omega_0) \), where \( F(j\omega) \) is the equivalent or analogous low-pass filter, can be simulated simply by passing \( x(t) \) and \( y(t) \) each through the analogous low-pass filter. The proof submitted by Thaler and Meltzer is somewhat heuristic, but can be supported by more rigorous derivations.\[7\] This method is shown in block diagram form in Fig. 2. The best-known form of bandpass low-pass analogy\[10,19\] is a special case of this for a simple amplitude-modulated carrier of frequency \( \omega_0 \), using an envelope detector as in Fig. 1, for which the general two-channel equivalent system reduces to a single channel. The analogous low-pass filter is the same, regardless of whether the one-channel or two-channel equivalent system is used. The difference is that the two-channel equivalent system is subject to fewer restrictions on the nature of the signal.

**Narrowband Filter Centered at a Frequency Other Than \( \omega_0 \)**

Within the narrowband limitation, the choice of \( \omega_0 \) in (1) is arbitrary. Consider the case in which a signal, expressed in the form of (1), is passed through a filter having characteristics centered at a frequency \( \omega_1 \), which is different from \( \omega_0 \), by a small amount. The limitation on \( \omega_1 \) is that the difference between the signal frequencies and \( \omega_1 \) must be small compared to \( \omega_0 \). The signal of (1) can be expressed in terms of \( \omega_1 \) rather than \( \omega_0 \) in the following manner:

\[
v(t) = x_1(t) \cos \omega_1 t - y_1(t) \sin \omega_1 t. \tag{7}\]
The equations for \( x_1(t) \) and \( y_1(t) \) can be derived by using the following trigonometric identities:

\[
\cos \omega t = \cos (\omega_1 t - \omega_c t) \cos \omega_1 t + \sin (\omega_1 t - \omega_c t) \sin \omega_1 t
\]

\[
\sin \omega t = - \sin (\omega_1 t - \omega_c t) \cos \omega_1 t + \cos (\omega_1 t - \omega_c t) \sin \omega_1 t.
\]

Substitute (8) and (9) into (1), equate to (7), and solve for \( x_1(t) \) and \( y_1(t) \):

\[
x_1(t) = x(t) \cos (\omega_1 t - \omega_c t) + y(t) \sin (\omega_1 t - \omega_c t)
\]

\[
y_1(t) = -x(t) \sin (\omega_1 t - \omega_c t) + y(t) \cos (\omega_1 t - \omega_c t).
\]

Equations (10) and (11) correspond to a rotation of reference axes through a time-varying angle, \((\omega_1 t - \omega_c t)\). They are represented in block diagram form in Fig. 3. After these relations have been used to express the signal in terms of the center frequency of the filter, \( \omega_c \), the filter is simulated simply as in Fig. 2. The output of the filter will then be expressed in terms of \( \omega_1 \) rather than \( \omega_c \), but another transformation to \( \omega_1 \) or any other carrier frequency in the band of interest can be accomplished in the same way if needed. Frequently only the envelope of the filter output is desired, and this can be obtained without the second transformation. Note that, from (3), (10), and (11),

\[
r(t) = \sqrt{x^2 + y^2} = \sqrt{x_1^2 + y_1^2}.
\]

**General Asymmetrical Narrowband Filtering**

Physically realizable asymmetrical narrowband filtering can usually be simulated by methods presented previously in this paper. Any narrowband transfer function which is a ratio of polynomials in the Laplace variable \( s \) can be factored into the form of cascaded symmetrical narrowband filters, which may be centered at different frequencies. This includes all discrete-element filters.

It is often unnecessary to break down an asymmetrical transfer function into individually symmetrical factors during preparation of a computer simulation, because these factors are often already available from prior design of the system to be simulated. A numerical example of a case in which the transfer function must be factored to give the individually symmetrical factors is given in Appendix II.

**Phase Modulation and Frequency Modulation**

The phase and frequency of a sinusoid are related; the instantaneous frequency is interpreted to be the time derivative of phase. The simulation of frequency modulation can therefore be accomplished by first integrating the frequency to obtain phase and then simulating phase modulation.

Phase modulation is treated here as the introduction of a phase shift of the carrier frequency without changing the envelope of the signal. Consider the general case in which the carrier frequency of any input \( v(t) \) is shifted by the phase \( \phi(t) \) to give the output \( v_1(t) \):

\[
v(t) = r(t) \cos \left[ \omega_c t + \theta(t) \right]
\]

\[
v_1(t) = r(t) \cos \left[ \omega_1 t + \theta(t) + \phi(t) \right] = x_1(t) \cos \omega_1 t - y_1(t) \sin \omega_1 t. \quad (14)
\]

Apply (5) and (6) to the output defined by (14) and then use common trigonometric identities:

\[
x_1(t) = r(t) \cos [\theta(t) + \phi(t)]
\]

\[
y_1(t) = r(t) \sin [\theta(t) + \phi(t)]. \quad (15)
\]
Substituting (5) and (6) into (15) and (16) results in the desired equations for a $\phi(t)$ phase shift:

$$x_1(t) = x(t) \cos \phi(t) - y(t) \sin \phi(t) \quad (17)$$
$$y_1(t) = x(t) \sin \phi(t) + y(t) \cos \phi(t). \quad (18)$$

Equations (17) and (18) are represented in block diagram form in Fig. 4.

**Phase Detection**

Either of two approaches to the simulation of phase detection may be used: a theoretical equation or a more direct representation of the actual circuit. Examples of both are given here.

A commonly used theoretical equation for phase detection, also used for product demodulation, considers the output to be the sine of the phase angle between the two carrier-frequency input signals. Let $v_1(t)$ and $v_2(t)$ be the input signals and $v_0(t)$ be the output:

$$v_1(t) = r_1(t) \cos [\omega_0 t + \theta_1(t)]$$
$$= x_1(t) \cos \omega_0 t - y_1(t) \sin \omega_0 t \quad (19)$$
$$v_2(t) = r_2(t) \cos [\omega_0 t + \theta_2(t)]$$
$$= x_2(t) \cos \omega_0 t - y_2(t) \sin \omega_0 t \quad (20)$$
$$v_0(t) = \sin [\theta_1(t) - \theta_2(t)]. \quad (21)$$

By means of a common trigonometric identity and (5) and (6), (21) becomes

$$v_0(t) = \sin \theta_1(t) \cos \theta_2(t) - \cos \theta_1(t) \sin \theta_2(t)$$
$$= \frac{y_2(t)x_1(t) - x_2(t)y_1(t)}{r_2(t)r_1(t)}. \quad (22)$$

Equation (22) may be used to simulate phase detection without the ambiguities which would arise if an attempt were made to find $\theta_1(t)$ and $\theta_2(t)$ individually by an arctangent method. As with envelope detectors, if low-pass filtering to remove high-frequency ripple from the output of a phase detector is significant, it may be included easily in the simulation.

A more direct simulation of the actual phase detector may be quite simple, depending on the type of circuit used. An example of a common type of phase detector circuit is shown in Fig. 5. The narrowband voltage $v_1(t)$ is added to and subtracted from the narrowband voltage $v_2(t)$ in the top and bottom halves of the circuit, respectively. The rectifiers and low-pass filters act as detectors to obtain the envelopes of the sum and difference voltages. The output is the difference of the envelopes. A method of simulating this phase detector is shown in Fig. 6. If it is desired to make this circuit produce an output similar to (21) and (22), a 90° phase shift must be introduced in one signal (a lag in $v_2$ or a lead in $v_1$), in addition to making $v_1$ very large compared to $v_2$.

**Frequency Discrimination**

As with phase detection, two different approaches to the simulation of frequency discrimination are possible: a theoretical equation or a more direct simulation of the actual discriminator circuit. The theoretical approach is to consider instantaneous frequency to be the time derivative of phase, so the difference between the instantaneous frequency and $\omega_0$ is the derivative of $\theta(t)$. An equation for this derivative can be obtained by differentiating (4):

$$\theta(t) = \frac{x'y - y'x}{x^2 + y^2}. \quad (23)$$

A dot above a variable denotes differentiation with respect to time. It may be difficult to obtain this quantity in an actual simulation without introducing a time lag in obtaining the derivatives. This should not be sur-
prising, because in any actual circuit a voltage proportional to instantaneous frequency is also unobtainable without a time lag. For this reason, it may well be preferable to simulate the actual discriminator circuit instead of using (23).

Simulation of two common discriminator types will be discussed here. A description and analysis of the operation of these circuits which is compatible with this presentation has been given by Grant.\[1\]

The Round–Travis discriminator shown in Fig. 7 operates by comparison of the envelopes of voltages from two tuned circuits with different center frequencies. Assuming loose coupling in the input transformer, the upper and lower halves of the circuit each form an independent single-tuned circuit followed by an envelope detector. The low-pass analog of a narrowband (high Q) single-tuned circuit is a simple time lag. Thus, this discriminator circuit can be simulated by combining previously described techniques, as shown in Fig. 8. Fig. 8 assumes identical bandwidths for the two tuned circuits, a common but not necessary relation.

The Foster–Seeley discriminator shown in Fig. 9 operates by subtraction of the envelopes of the sum and difference of a capacitor-coupled voltage and a transformer-coupled tuned-circuit voltage. As before, loose coupling is assumed in the transformer, so that the secondary forms part of a single-tuned circuit. A 90° phase lag occurs in the tuned circuit at its center (resonant) frequency, but no phase shift occurs in the capacitor-coupled voltage. The tuned circuit is simulated by means of its equivalent low-pass filter by the method of Fig. 2. The resulting simulation is a combination of previously described techniques, as shown in Fig. 10.

**Frequency-Independent Nonlinearity**

Techniques for simulating nonlinear circuits depend on the nature of the nonlinearity and the circuit following it. Envelope detection, previously discussed, is a nonlinearity which is followed by a low-pass circuit. Simulation of a slowly time-varying gain which is independent of the signal is straightforward and needs no discussion. The following discussion is restricted to the important case in which the output of the nonlinearity is a function only of the value of the input, independent of its past history, and the output is to be passed through a narrowband circuit. The narrowband circuit will remove harmonics of the input frequency as well as any low-frequency component, so only the fundamental input-frequency component of the output is of interest. Consider an input in the form of (2), approximating \( r \) and \( \theta \) by constants during each cycle of the carrier frequency. For each cycle, the output of this type of nonlinearity will be expressible as a Fourier cosine series, of which the fundamental component is a term in \( \cos (\omega t + \theta) \), the same frequency and phase as the input. The amplitude of this component, the only component of interest, is a function of \( r \), the envelope of the input signal. The effect of this type of nonlinearity is therefore simply that of a gain which is a function of the envelope.
The gain function required to simulate the nonlinearity is the ratio of the amplitude of the fundamental component of the output to the amplitude of a sinusoidal input. This is called the “describing function,” a well-known function in feedback control engineering. A description of the method of calculation of a describing function, together with the equation for the describing function of a combination of limiting and dead zone, can be found, for example, in Truxal. Fig. 11 shows how the describing function can be used to simulate a nonlinearity which is to be followed by a narrowband circuit.

**Other Nonlinearities Representable by Describing Functions**

When the output of a nonlinear element depends on past values of the input as well as the present value, the describing function generally includes a phase shift as well as a gain. If an expression for the describing function can be obtained, such a nonlinear element may be simulated by including the phase shift by the method of Fig. 4. This method shows possibilities for simulating such phase-shift-producing nonlinearities as hysteresis. However, a describing function for a frequency-dependent nonlinearity often is difficult to compute.

**Conclusions**

The transient response of a wide variety of systems which are narrowband in whole or in part can be computed efficiently by computer simulation, using the techniques described in this paper. The principal limitations are that the system be physically realizable and expressible mathematically in block diagram form, and consist of portions for which narrowband approximations are valid, as well as possible low-frequency portions. The methods presented here are useful in converting block diagrams of such systems to block diagrams consisting only of low-frequency functions for which well-known simulation methods are available.

**Appendix I**

**Derivation of Equation Representing Any Narrowband Signal**

Equation (1) may be derived in more than one way. The following method was chosen here for reasons of simplicity and physical significance, avoiding unnecessary use of complex numbers and negative frequencies.

Any function of time which has a finite number of maxima and minima and a finite number of ordinary discontinuities in the interval $-T \leq t \leq T$, can be represented in this interval by the Fourier series

$$v(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi f}{T} \right) + b_n \sin \left( \frac{n\pi f}{T} \right) \right]$$

(24)

with

$$a_n = \frac{1}{T} \int_{-T}^{T} v(t) \cos \left( \frac{n\pi f}{T} \right) dt$$

$$b_n = \frac{1}{T} \int_{-T}^{T} v(t) \sin \left( \frac{n\pi f}{T} \right) dt, \quad (n = 0, 1, 2, \cdots).$$

Let $v(t)$ be a narrowband function, containing no frequency components outside the band between $\omega_c - \Delta \omega$ and $\omega_c + \Delta \omega$, where

$$\Delta \omega \ll \omega_c.$$

For the equations to follow, $\omega_c$ and $\Delta \omega$ should be chosen so that $(\omega_cT/\pi)$ and $(\Delta \omega T/\pi)$ are integers. This requirement can always be met if $T$ is large enough to define a narrowband function. Furthermore, this limitation disappears if $T$ is considered arbitrarily large. Changing the summation limits of (24) to include only the non-zero terms, and noting that $a_0 = 0$ for a narrowband function,

$$v(t) = \sum_{n=(T/\pi)(\omega_c+\Delta \omega)}^{n=(T/\pi)(\omega_c-\Delta \omega)} \left[ a_n \cos \left( \frac{n\pi f}{T} \right) + b_n \sin \left( \frac{n\pi f}{T} \right) \right],$$

(25)

$$n = \frac{T}{\pi} (\omega_c - \Delta \omega), \quad \frac{T}{\pi} (\omega_c - \Delta \omega) + 1,$$

$$\frac{T}{\pi} (\omega_c - \Delta \omega) + 2, \cdots.$$  

Letting $k = n - (\omega_cT/\pi)$, (25) becomes

$$v(t) = \sum_{k=-(\Delta \omega T/\pi)}^{k=-(\Delta \omega T/\pi)} \left[ a_k \cos \left( \omega_c f + \frac{k\pi f}{T} \right) + b_k \sin \left( \omega_c f + \frac{k\pi f}{T} \right) \right]$$

(26)

with

$$a_k = \frac{1}{T} \int_{-T}^{T} v(t) \cos \left( \omega_c f + \frac{k\pi f}{T} \right) dt$$

$$b_k = \frac{1}{T} \int_{-T}^{T} v(t) \sin \left( \omega_c f + \frac{k\pi f}{T} \right) dt,$$

$$k = -\frac{T\Delta \omega}{\pi}, \quad -\frac{T\Delta \omega}{\pi} + 1, \quad -\frac{T\Delta \omega}{\pi} + 2, \cdots.$$
By trigonometric identity,
\[
\cos \left( \omega t + \frac{k \pi t}{T} \right) = \cos \left( \omega t \right) \cos \left( \frac{k \pi t}{T} \right) - \sin \left( \omega t \right) \sin \left( \frac{k \pi t}{T} \right)
\]
\[
\sin \left( \omega t + \frac{k \pi t}{T} \right) = \sin \left( \omega t \right) \cos \left( \frac{k \pi t}{T} \right) + \cos \left( \omega t \right) \sin \left( \frac{k \pi t}{T} \right).
\]
Substituting (27) and (28) into (26), and defining \( x(t) \) and \( y(t) \),
\[
\tau(t) = x(t) \cos (\omega t) - y(t) \sin (\omega t) (1)
\]
where
\[
x(t) = \sum_{k=-T}^{k=+T} \left[ a_k \cos \left( \frac{k \pi t}{T} \right) + b_k \sin \left( \frac{k \pi t}{T} \right) \right]
\]
\[
y(t) = \sum_{k=-T}^{k=+T} \left[ a_k \sin \left( \frac{k \pi t}{T} \right) - b_k \cos \left( \frac{k \pi t}{T} \right) \right].
\]
Note that, since \( \cos (-x) = \cos x \) and \( \sin (-x) = -\sin x \), \( x(t) \) and \( y(t) \) are expressible as Fourier series consisting only of low frequencies in the range \( 0 \leq \omega \leq \Delta \omega \). That is,\[
x(t) = \sum_{k=0}^{\infty} \left[ A_k \cos \left( \frac{k \pi t}{T} \right) + B_k \sin \left( \frac{k \pi t}{T} \right) \right]
\]
\[
y(t) = \sum_{k=0}^{\infty} \left[ C_k \sin \left( \frac{k \pi t}{T} \right) - D_k \cos \left( \frac{k \pi t}{T} \right) \right].
\]
Solution of (33) through (35) yields the following transfer function:
\[
\frac{V_2(s)}{V_1(s)} = \frac{M/C_2}{L_1L_2(1 - k^2)s^2 + (L_1R_2 + L_2R_1)s + \left( \frac{L_1}{C_2} + \frac{L_2}{C_1} + \frac{R_1R_2}{C_1} + \frac{R_1}{C_2} + \frac{R_2}{C_1} \right) \left( \frac{1}{s} \right) + \left( \frac{1}{C_1C_2} \right) \left( \frac{1}{s^2} \right)}.
\]
where
\[
A_k = a_k + a_{-k} \quad C_k = a_k - a_{-k}
\]
\[
B_k = b_k - b_{-k} \quad D_k = b_k + b_{-k}.
\]
While this derivation has been for a finite time interval of length \( 2T \), it remains valid if \( T \) is made arbitrarily large. It has therefore been shown that, in general, any narrowband function, (25), can be expressed in the form of (1), where the information is carried by \( x(t) \) and \( y(t) \), which are low-frequency functions as given by (31) and (32). Reversing the derivation, any low-frequency functions \( x(t) \) and \( y(t) \) containing only frequencies in the range \( 0 \leq \omega \leq \Delta \omega \) can be expressed in the form of (31) and (32), which can be converted to the form of (29) and (30). These equations, substituted into (1), can be converted into (25), which is a narrowband function. This proves that, in general, (1) with any low-frequency functions \( x(t) \) and \( y(t) \) represents a narrowband function.

**APPENDIX II**

**Numerical Example of an Asymmetrical Narrowband Filter**

A schematic diagram of the double-tuned transformer-coupled circuit chosen for this example is given in Fig. 12. A series rather than parallel-tuned primary circuit was used for simplicity; similar results can be obtained with the parallel-tuned arrangement commonly used as an interstage network in intermediate-frequency amplifiers. The voltages and currents in Fig. 12 are expressed in terms of the Laplace operator \( s \), to permit expressing impedances in similar terms. Conventional frequency response expressions (phasors) can be obtained by letting \( s = j\omega \). Equations for the circuit may be written as follows:
\[
\frac{V_2(s)}{V_1(s)} = \frac{M/C_2}{L_1L_2(1 - k^2)s^2 + (L_1R_2 + L_2R_1)s + \left( \frac{L_1}{C_2} + \frac{L_2}{C_1} + \frac{R_1R_2}{C_1} + \frac{R_1}{C_2} + \frac{R_2}{C_1} \right) \left( \frac{1}{s} \right) + \left( \frac{1}{C_1C_2} \right) \left( \frac{1}{s^2} \right)}.
\]
The following values were chosen to give an asymmetrical narrowband transfer function, using units of ohms, farads, and henries:
\[
L_1 = L_2 = 10^{-6} \quad k = 0.04
\]
\[
R_1 = 0.04 \quad (1/C_1) = 1.01 \times 10^6
\]
\[
R_2 = 0.02 \quad (1/C_2) = 0.99 \times 10^6.
\]
Substituting these values in (36) gives the following transfer function:
\[
\frac{V_2(s)}{V_1(s)} = \frac{0.0396}{0.9984 \times 10^{-12}s^2 + 0.06 \times 10^{-8}s + 2.0008 + 0.0598 \times 10^6(1/s) + 0.9999 \times 10^4(1/s^2)}.
\]
Factoring the denominator by a numerical method, (37) is equivalent to
\[
\frac{V_2(s)}{V_1(s)} = \frac{0.039663 \times 10^{12}}{[s + 0.02638 \times 10^6 + (0.98055 \times 10^9)^2/s][s + 0.03371 \times 10^6 + (1.02061 \times 10^6)^2/s]}.
\]
This may be rearranged into the following factors:

\[ V_2(s)/V_1(s) = 44.6 F_1(s) F_2(s) \]  (39)

where

\[ F_1(s) = \frac{0.02638 \times 10^6}{s + 0.02638 \times 10^6 + (0.98055 \times 10^6)^2/s} \]  (40)

\[ F_2(s) = \frac{0.03371 \times 10^6}{s + 0.03371 \times 10^6 + (1.02061 \times 10^6)^2/s} \]  (41)

The magnitudes of the frequency response of these factors and their product is shown in Fig. 13, as obtained by letting \( s = j\omega \). It can be seen that \( |F_1(j\omega)| \) and \( |F_2(j\omega)| \) are each symmetrical about its own center frequency (within the accuracy of the narrowband approximations), but that the product is asymmetrical. To obtain the equivalent low-pass filter, the approximation is made that \((s^2 + \omega_0^2)/s\) is replaced by \(2s\). With this approximation,

\[ F_1(s + j\omega) \approx \frac{26380}{2s + 26380} \]  (42)

\[ F_2(s + j\omega) \approx \frac{33710}{2s + 33710} \]  (43)

\( \omega_1 \) and \( \omega_2 \) are the center frequencies

\[ \omega_1 = 0.98055 \times 10^6 \text{ rad/s} \]  (44)

\[ \omega_2 = 1.02061 \times 10^6 \text{ rad/s}. \]  (45)

The equivalent low-pass filters are simple time lags

\[ T_1 = \left(\frac{2}{26380}\right) s = 75.81 \mu s \]  (46)

\[ T_2 = \left(\frac{2}{33710}\right) s = 59.33 \mu s. \]  (47)

REFERENCES


