Edge Disjoint Graph Spanners of Complete Graphs and Complete Digraphs
extended abstract

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Abstract

A spanning subgraph \( S = (V, E') \) of a connected simple graph (digraph) \( G = (V, E) \) is a \( f(x) \)-spanner if for any pair of nodes \( u \) and \( v \), \( d_S(u, v) \leq f(d_G(u, v)) \) where \( d_G \) and \( d_S \) are the usual distance functions in graphs (digraphs) \( G \) and \( S \), respectively. The delay of the \( f(x) \)-spanner is \( f(x) - x \). We investigate the existence of multiple edge-disjoint spanners in complete graphs and complete digraphs.

1 Introduction

Interconnection networks (the topological structure of parallel and distributed systems) are generally modelled as graphs. Consequently, researchers have been investigating those structural properties of graphs that correspond to useful properties of interconnection networks. One such property that has recently been investigated is the existence of spanners in a graph.

A spanner is a spanning subgraph in which the distance between any pair of vertices approximates the distance in the original graph. They were introduced by Peleg and Ullman [11], who used them for efficient simulation of synchronous distributed systems on asynchronous ones. Richards and Liestman [12] suggested the use of spanners as network topologies: if one has an expensive desired topology, often a sparse (and therefore less expensive) spanner can be substituted, retaining a similar network structure for only a slight increase in communication costs. In a series of papers, Liestman and Shermer [5, 6, 7, 8, 9] continued the study of spanners as network topologies, and introduced a more general definition of spanner. Heydemann, Peters, and Sotteau [4] also investigated spanners as network topologies.

In this paper we begin the investigation of edge-disjoint spanners of a given graph. One possible use of edge-disjoint spanners is to partition a parallel computer for several independent users or processes without significantly decreasing the performance of each "virtual computer"; this would be most useful when the network is asynchronous (making timeslicing on the edges of the network difficult) or when the speed of communication is much lower than the speed of the processors. Another situation in which this type of partition could be useful for allowing several simultaneous independent processes on a network that uses wormhole routing—the communication paths set up by one process would not be interfered with by the operation of the other processes.

We characterize a spanner by its delay, a number which represents how closely it models its underlying graph. We study the number of edge-disjoint spanners of each delay for complete graphs and complete digraphs. Although for application purposes it is unlikely that a parallel machine would have a complete
interconnection network, our results are the first step in tackling this problem for more general graphs and digraphs, and do provide bounds on this more general problem. For instance, we can upper-bound the number of edge-disjoint spanners of any graph by the number of edge-disjoint spanners (of a different delay) in the complete graph (see Lemma 1.4).

A network is represented by a connected simple graph (or digraph) $G = (V,E)$. We use $d_G(u,v)$ to denote the distance from vertex $u$ to vertex $v$ in graph (digraph) $G$. In [6], Liestman and Shermer introduced a general definition of graph spanner which is easily modified to describe digraph spanners: A spanning subgraph (subdigraph) $S$ of a connected simple graph (digraph) $G$ is an $f(x)$-spanner if for any pair of nodes $u$ and $v$, $d_S(u,v) \leq f(d_G(u,v))$. We call $d_S(u,v) - d_G(u,v)$ the delay between vertices $u$ and $v$ in $S$, denoted $ds(u,v)$. For an $f(x)$-spanner $S$, we let $f'(x) = f(x) - x$ and refer to $f'(x)$ as the delay of the spanner. Note that $f'(x)$ is an upper bound (but not necessarily a tight bound) on the maximum delay in $S$ between any pair of vertices at distance $x$ in $G$.

We are interested in constructing multiple edge disjoint $(x+c)$-spanners of a graph (digraph) $G$ on $n$ vertices, for appropriate constants $c$. The maximum value that $c$ can reach will depend on $G$. Let $EDS(G,c)$ denote the maximum number of edge disjoint $(x+c)$-spanners of $G$ for $n \geq 2$ and $c \geq 0$. (Note that for all graphs (digraphs) $G$, $EDS(G,0) = 1$.) The maximum value that $c$ can reach will depend on $G$.

We use $K_n^*$ to denote the complete directed graph on $n$ vertices and $K_n$ to denote the complete graph on $n$ vertices. In these graphs, $d_G(u,v) = 1$ for any pair of vertices $u$ and $v$ and in any connected subgraph $S$, $ds(u,v) \leq n - 1$. Thus, we need only consider values of $c$ in the range $0 \leq c \leq n - 2$.

We begin with a few simple observations:

**Lemma 1.1** $EDS(G,c) \leq EDS(G,c+1)$ where $0 \leq c \leq n - 2$ and the graph (or digraph) $G$ contains $n \geq 3$ vertices.

**Proof** By definition, any $(x+c)$-spanner is also an $(x+c+1)$-spanner. □

**Lemma 1.2** For any connected graph $G$ on $n$ vertices, $EDS(G,c) \leq \left\lceil \frac{c}{2} \right\rceil$ for all $0 \leq c \leq n - 2$, where $n \geq 2$.

**Proof** Any spanner of $G$ must be connected, and thus must contain at least $n - 1$ edges. Since $G$ contains at most $\frac{n(n-1)}{2}$ edges, the result follows. □

**Lemma 1.3** For any connected digraph $G^*$ on $n$ vertices, $EDS(G^*,c) \leq n - 1$ for all $0 \leq c \leq n - 2$, where $n \geq 2$.

**Proof** In any spanner, each vertex must have out-degree $\geq 1$ and, thus, the spanner must contain at least $n$ edges. Since there are at most $n(n-1)$ edges in $G^*$, there can be at most $n - 1$ edge disjoint spanners of $G^*$. □

**Lemma 1.4** For any graphs (or digraphs) $G_1$ and $G_2$ on the same vertex set, with $G_2$ a supergraph of $G_1$, $EDS(G_1,c) \leq EDS(G_2,c + d - 1)$, where $d$ is the diameter of $G_1$.

**Proof** Consider any delay $c$ spanner $S$ of $G_1$; $S$ is a subgraph of $G_1$ having diameter at most $d + c$. Any two vertices $u$ and $v$ of $G_2$ have delay at most $d + c - d_G_2(u,v)$ in $S$. Since $d_G_2(u,v) \geq 1$, the delay of $S$ as a spanner of $G_2$ is at most $d + c - 1$. A collection of such delay $c$ spanners of $G_1$ is therefore a collection of delay $d + c - 1$ spanners of $G_2$, so the maximum size of the latter is an upper bound on the maximum size of the former. □

For our purposes, the most interesting special case of the previous lemma is when $G_2$ is the complete graph, yielding $EDS(G,c) \leq EDS(K_n,c + d - 1)$ where $d$ is the diameter of the graph $G^*$ and $|V(G)| = n$. Thus, $EDS(K_n,c)$ gives an upper bound for any graph $G$ on $n$ vertices. Similarly, $EDS(K_n^*,c)$ gives an upper bound for any digraph.

In this paper, we give bounds on and exact values for $EDS(K_n,c)$ and $EDS(K_n^*,c)$. In Section 2, we consider the problem in complete (undirected) graphs. In Section 3, we give general bounds on $EDS(K_n^*,c)$. In Section 4, we give some particular values of $EDS(K_n^*,c)$.

2 Spanners of complete graphs

**Lemma 2.1** $EDS(K_n,2) = \left\lceil \frac{n}{2} \right\rceil$ for $n \geq 4$.

**Proof** For even $n = 2p$, label $n$ vertices with the ordered pairs $(a,b)$ where $a \in \{0,1\}$ and $b \in \{1,2,\ldots,p\}$. For each $1 \leq i \leq p$, we construct $S_i$, an $(x+2)$-spanner of $K_n$, as follows: Include in $S_i$ the edge between $(0, i)$ and $(1, i)$. Add edges between $(0, i)$ and all vertices $(0, j)$ for $i < j \leq p$ and edges between $(0, i)$ and all vertices $(1, j)$ for $1 \leq j < i$. Add edges between $(1, i)$ and all vertices $(1, j)$ for $i < j \leq p$ and
edges between \((1, i)\) and all vertices \((0, j)\) for \(1 \leq j < i\).
It is easy to see that \(S_i\) is an \((x + 2)\)-spanner of \(K_n\)
since any two vertices of \(S_i\) are at distance at most 3.
Further, for any \(i \neq j, S_i\) and \(S_j\) are edge disjoint.
Thus, \(EDS(K_n, 2) \geq \frac{n}{2}\).

For odd \(n = 2p + 1\), we can obtain the same result by simply adding one additional vertex \((0, p + 1)\),
following the above construction, and adding the edge between \((0, i)\) and \((0, p + 1)\) in each \(S_i\).

The matching upper bound comes from Lemma 1.2. ∎

**Theorem 2.2** \(EDS(K_n, c) = \lfloor \frac{n}{2} \rfloor \) for \(2 \leq c \leq n - 2\) where \(n \geq 4\).

**PROOF** This follows from Lemma 1.1, Lemma 1.2, and Lemma 2.1. □

We now turn our attention to \(EDS(K_n, 1)\), the maximum number of edge-disjoint \((x + 1)\)-spanners of \(K_n\). Note that a graph on \(n\) vertices is an \((x + 1)\)-spanner of \(K_n\) if and only if it is \(K_n\) itself or has diameter 2. \(EDS(K_n, 1)\) is therefore equal to the maximum number of diameter two graphs that can be packed into \(K_n\). Various authors have studied the function \(g(k)\) such that \(K_n\) can be decomposed into \(k\) graphs of diameter two if and only if \(n \geq g(k)\). Thus, we know that

**Lemma 2.3** \(EDS(K_n, 1) = k\) for \(g(k) \leq n < g(k + 1)\).

Bosák, Rosa, and Znám [3] showed that \(g(2) = 5\) and \(g(3) = 12\) or 13, and that \(15 \leq g(4) \leq 24\). Thus,

**Lemma 2.4** \[
EDS(K_n, 1) = \begin{cases} 
1 & 1 \leq n \leq 4 \\
2 & 5 \leq n \leq 11 \\
2 & n = 12 \\
3 & 13 \leq n \leq 15 
\end{cases}
\]

Bosák [2] showed that \(6k - 52 \leq g(k) \leq 6k\) for every \(k \geq 2\). (In fact, he gave various better lower bounds on \(g(k)\) for \(k\) in the range \(4 \leq k \leq 370\).) Thus, we can conclude that

**Theorem 2.5** \(EDS(K_n, 1) \approx \frac{n}{6}\) for large \(n\).

Thus, the situation in complete graphs is reasonably well understood.

### 3 Bounds for directed complete graphs

#### 3.1 General results

In \(K_n^+\), the distance between every pair of vertices is 1. Thus, a spanner with delay \(c\) is a digraph on \(n\) vertices with diameter \(c + 1\). Tomová [14] investigated the decomposition of \(K_n^+\) into digraphs with given diameters. Among other results (which don’t pertain here), she showed that \(K_n^+\) could be decomposed into two digraphs of finite diameter \(d\) if and only if \(n \geq d + 1\). Thus, we can conclude:

**Lemma 3.1** \(EDS(K_n^+, c) \geq 2\) for \(n - 2 \geq c \geq 1\).

**Lemma 3.2** \(EDS(K_n^+, c) \geq EDS(K_n, c)\).

**PROOF** Any spanner \(S\) of \(K_n\) can be transformed into a spanner of \(K_n^+\) by replacing each edge of \(S\) with two opposite edges. □

**Lemma 3.3** \(EDS(K_{n+1}^+, c + 1) \geq EDS(K_n^+, c)\).

**PROOF** Given \(K_{n+1}^+\), choose one vertex \(v\) and consider the \(p = EDS(K_n^+, c)\) edge disjoint spanners \(S_1, S_2, ..., S_p\) of the remaining \(n\) vertices. Choose \(p\) distinct vertices \((p < n)\) \(u_1, u_2, ..., u_p\) other than \(v\). To form spanner \(S'_i\) of \(K_{n+1}^+\), simply add edges \((u_i, v)\) and \((v, u_i)\) to \(S_i\). The resulting spanner has delay at most \(c + 1\). □

It is not necessarily true that \(EDS(K_{n+1}^+, c) \geq EDS(K_n^+, c)\). For example, \(EDS(K_{n+1}^+, n - 2) < EDS(K_n^+, n - 2)\) for \(n = 5\) and \(n \geq 7\) as can be seen from Lemma 3.13 and Lemma 3.14. It could be interesting to determine whether this inequality may hold for some large classes of values of \(n\) and \(c\).

**Lemma 3.4** \(EDS(K_n^+, c) \leq n - 3\) for \(n \geq 5\) and \(c \leq n - 3\) and \(EDS(K_n^+, 1) \leq 2\).

**PROOF** If a strongly connected digraph \(G^+\) has \(n\) vertices and \(m > n + 1\) edges then its diameter is at least \(\frac{\sqrt{m} - 1}{2} + 1\) [1]. A spanner with \(n + 1\) edges must, therefore, have diameter at least \(n - 1\) and its delay must be at least \(n - 2\). Thus, a spanner with delay at most \(n - 3\) must contain at least \(n + 2\) edges. So we have \(EDS(K_n^+, c) \leq \left\lfloor \frac{\sqrt{n} - 1}{2} \right\rfloor\) for \(n \geq 4\) and \(c \leq n - 3\) and the given bounds follow. □

**Lemma 3.5** \(EDS(K_n^+, c) < \frac{\sqrt{n}}{c + 1} \) for \(2 \leq c \leq n - 3\).
**Proof** If \( EDS(K_n, c) \geq \left[ \frac{cn}{c+1} \right] \), then there exists a spanner with at most \( \frac{(c+1)n^2(n-1)}{cn} = n + \frac{n - (c+1)}{e} \) vertices of outdegree \( \geq 1 \) and every vertex is of outdegree at most \( n + \frac{n - (c+1)}{e} - (n-1) = \frac{n-1}{c} \).

Let us consider a vertex \( u \) of outdegree \( 1 \) in such a spanner, and let \( v \) be its outneighbor. Let \( k_1 \) be the number of vertices at distance \( i \) from \( u \) for \( 2 \leq i \leq c + 1 \). Since the delay of the spanner is \( c \), we know that \( 2 + k_2 + \ldots + k_{c+1} = n \). A tree of shortest paths rooted at \( u \) contains exactly \( 1 + k_2 + \ldots + k_{c+1} \) edges. Since all of the \( k_{c+1} \) vertices at distance \( c + 1 \) from \( u \) also have outdegree at least \( 1 \) in the spanner, we know that \( 1 + k_2 + \ldots + k_{c+1} \). Hence, \( k_{c+1} \leq n + \frac{n - (c+1)}{e} + (n-1) \) and, therefore, \( k_2 + \ldots + k_{c+1} \geq n - 2 - \frac{n-1}{c} \).

But for any \( i \), \( 3 \leq i \leq c \), \( k_2 + \ldots + k_i \) is less than or equal to the number of edges going out of \( 1 + k_2 + \ldots + k_{i-1} \) vertices (the vertices at distance \( i - 1 \) from \( u \) in the tree), which is necessarily less than or equal to \( 1 + k_2 + \ldots + k_{i-1} + \frac{n - (c+1)}{e} \) (because of the number of edges in the spanner and the fact that each vertex has outdegree at least \( 1 \) at least one).

We use this same argument with the value of \( i \) decreasing from \( c \) to \( 3 \). When \( i = c \), we get \( n - 2 - \frac{n-1}{c} \leq k_2 + \ldots + k_c \leq 1 + k_2 + \ldots + k_{c-1} + \frac{n - (c+1)}{e} \), which implies that \( k_2 + \ldots + k_{c-1} \geq n - 2 - 2\left(\frac{n-1}{c}\right) \). Then, with \( i = c-1 \), we get \( n - 2 - 2\left(\frac{n-1}{c}\right) \leq 1 + k_2 + \ldots + k_{c-1} + \frac{n - (c+1)}{e} \), which implies that \( k_2 + \ldots + k_{c-1} \geq n - 2 - 3\left(\frac{n-1}{c}\right) \). Continuing to decrease \( i \), we eventually obtain \( k_2 \geq n - 2 - (c-1)\left(\frac{n-1}{c}\right) \), that is, \( k_2 \geq \frac{n-1}{c} - 1 \).

Since the outdegree of every vertex is bounded, we also know that \( k_2 \leq \frac{n-1}{c} \). So, all of the vertices other than \( v \) are of outdegree \( 1 \) except perhaps one vertex which may have outdegree \( 2 \).

If \( k_2 = \frac{n-1}{c} \), then all vertices other than \( v \) are of outdegree \( 1 \), so that if \( n \geq \frac{n-1}{c} + 3 \) there must be an outneighbor of \( v \) of outdegree \( 1 \) adjacent to another vertex of outdegree \( 1 \).

If \( k_2 = \frac{n-1}{c} - 1 \), there is exactly one vertex (other than \( v \)) which is of outdegree \( 2 \) and all the other vertices must have outdegree \( 1 \). In this case, if \( n \geq \frac{n-1}{c} + 5 \), it follows that there must be a vertex of outdegree \( 1 \) (at distance \( 2 \), \( 3 \) or \( 4 \) from \( u \)) which is adjacent to another vertex of outdegree \( 1 \).

In either of these cases, there is a vertex of outdegree \( 1 \) which is adjacent to another vertex of outdegree \( 1 \). Since, we have shown that the outneighbor of a vertex of outdegree \( 1 \) must have outdegree at least \( \frac{n-1}{c} - 1 \), we have a contradiction for \( \frac{n-1}{c} - 1 > 1 \), that is, when \( n \geq 2c + 2 \).

In the case of \( n \leq 2c + 1 \), the argument is slightly different. As before, there exists a spanner with at most \( n + \frac{n - (c+1)}{e} \) edges. However, when \( n \leq 2c + 1 \), this number is \( n + \frac{n - (c+1)}{e} \leq n + 2\left(\frac{c+1}{e}\right) + (c+1) = n + 1 \). Such a spanner has diameter at least \( n - 1 \) (either it is a cycle with \( n \) edges or we can apply the result of [1] as in the proof of Lemma 3.4) and, thus, it has a delay which is at least \( n - 2 \). So, we have a contradiction if \( c \leq n - 3 \).

\( \square \)

**Lemma 3.6** \( EDS(K_n, c) \geq \left[ \frac{cn}{c+1} \right] \) for \( c \geq 2 \).

**Proof** From Lemma 3.2 and \( EDS(K_n, 2) = \left[ \frac{cn}{c+1} \right] \), we conclude that \( EDS(K_n, c) \geq \left[ \frac{cn}{c+1} \right] \) for \( c \geq 2 \).

Combining the last two results, we obtain the following.

**Theorem 3.7** \( \left[ \frac{c}{2} \right] \leq EDS(K_n, c) \leq \frac{cn}{c+1} \) for \( 2 \leq c \leq n - 3 \).

### 3.2 Spanners with delay 1

We now turn our attention to the special case of \( c = 1 \).

**Lemma 3.8** If \( EDS(K_n, 1) > 2 \), then in each spanner the indegree and outdegree must be at least 2 for each vertex.

**Proof** If a vertex \( u \) in such a spanner has outdegree 1, then \( u \)'s outneighbor \( v \) must have outdegree at least \( n - 2 \) so that there is a path of length at most 2 from \( u \) to each vertex of \( K_n^* \). Since \( v \) has at most one outedge not included in this spanner, there can be at most one other spanner, a contradiction. The argument for indegree is similar.

\( \square \)

**Lemma 3.9** \( EDS(K_n, 1) \leq \left[ \frac{c}{2} \right] \) for \( n = 10, 13, 14 \) and \( n \geq 16 \).

**Proof** Assume \( EDS(K_n, 1) \geq \left[ \frac{c}{2} \right] \). For \( n \geq 7 \), \( \left[ \frac{c}{2} \right] \geq 2 \) so, from Lemma 3.8, the indegree and outdegree of each vertex in each spanner must be at least 2. Since \( K_n^* \) has \( n(n-1) \) edges, some spanner \( S \) must have at most \( n(n-1)/\left[ \frac{c}{2} \right] \) \leq 3n - 3 edges. Thus, in \( S \) there must be a vertex \( u \) of outdegree 2 and every vertex can have outdegree at most

\[(n-1) - 2\left(\left[ \frac{c}{2} \right] - 1 \right) \leq \left[ \frac{c}{2} \right] + 1.\]

(1)
In $S$, the number of vertices at distance 2 from $u$ is at most $2\left(\frac{5}{3}\right) + 1$. Since the delay of $S$ is 1, all $n$ vertices must be within distance 2 of $u$, so

$$n \leq 1 + 2 + 2\left(\frac{5}{3}\right) + 1 = 2\left(\frac{5}{3}\right) + 5. \quad (2)$$

However, this is a contradiction for $n = 14$ and for all $n \geq 16$.

When $n = 10$ or $n = 13$, this argument does not hold. In these cases, in fact for any $n = 3p + 1$, (1) becomes $\left(n - 1\right) - 2\left(\frac{5}{3}\right) \leq \left(\frac{5}{3}\right) - 1$ and, thus, (2) becomes $n \leq 1 + 2 + 2\left(\frac{5}{3}\right) = 2\left(\frac{5}{3}\right) + 3$ which is a contradiction for $n = 10$ and $n = 13$. □

A slightly weaker bound can be shown for a larger range of $n$.

**Lemma 3.10** $EDS(K_n^*, 1) \leq \left\lceil \frac{n}{3} \right\rceil$ for $n \geq 7$.

**PROOF** The proof is quite similar to that of Lemma 3.9. Assume $EDS(K_n^*, 1) > \left\lceil \frac{n}{3} \right\rceil$ and that $n \geq 7$. The proof proceeds as before except that (1) becomes $\left(n - 1\right) - 2\left(\frac{5}{3}\right) \leq \left(\frac{5}{3}\right) - 1$ and (2) becomes $n \leq 1 + 2 + 2\left(\frac{5}{3}\right) = 2\left(\frac{5}{3}\right) + 1$ which is a contradiction for all $n \geq 7$. □

**Lemma 3.11** $EDS(K_n^*, 1) \geq \left\lceil \sqrt{n} \right\rceil$ for $n \geq 1$.

**PROOF** Let $k = \left\lceil \sqrt{n} \right\rceil$.

We first show that $EDS(K_n^*, 1) \geq k$. Let $k^2$ vertices be labelled with the integer ordered pairs $(i, j)$ with $0 \leq i, j \leq k - 1$. For each $d$, $0 \leq d \leq k - 1$, define spanner $S_d$ of $K_n^*$ to contain only the edges from each $(i, j)$, $0 \leq i, j \leq k - 1$, to the $k$ vertices $((i + j + d) \mod k, l)$ where $0 \leq l \leq k - 1$. Note that these spanners are edge disjoint. There is a path of length 2 from any vertex $(s, t)$ to any other vertex $(u, v)$ by way of $((s + t + d) \mod k, (u - s - t - 2d) \mod k)$.

For $k^2 < n < (k + 1)^2$, we augment each spanner $S_d$ with $n - k^2$ additional vertices as follows: For each such vertex $w$, connect $w$ and each vertex $(i, k - d)$ where $0 \leq l \leq k - 1$ with two edges, one in each direction. In this spanner, there is a path of length 2 from $w$ to any vertex $(u, v)$ where $0 \leq u, v \leq k - 1$ by way of $(u, k - d)$. From any vertex $(u, v)$ where $0 \leq u, v \leq k - 1$, there is a path of length 2 to $w$ by way of $((u + v + d) \mod k, k - d)$. In addition, there is a path of length 2 from $w$ to any other of these additional vertices $x$ via $(l, k - d)$ for any $0 \leq l \leq k - 1$. □

Combining Lemma 3.9 and Lemma 3.11, we obtain the following.

**Theorem 3.12** $\lceil \sqrt{n} \rceil \leq EDS(K_n^*, 1) < \left\lceil \frac{n}{3} \right\rceil$ for $n = 10, 13, 14$ and $n \geq 16$.

Note that for large $n$, by combining Theorem 2.5 with Lemma 3.2 we can improve this lower bound to approximately $\frac{n}{6}$.

### 3.3 Spanners with large delay

**Lemma 3.13** $EDS(K_n^*, n - 2) = \begin{cases} n - 2 & \text{for } n = 4, 6 \\ n - 1 & \text{for other } n > 2. \end{cases}$

**PROOF** For $n = 4$, the lower bound comes from Lemma 3.1. From Lemma 1.3, $EDS(K_4^*, 2) \leq 3$. If $EDS(K_4^*, 2) = 3$, then each spanner would have exactly 4 edges and would, therefore, be a directed cycle. This would contradict the fact that $K_4^*$ cannot be decomposed into 3 Hamiltonian cycles (see [13]).

For $n = 6$, the lower bound comes from Lemma 3.3 and the fact that $EDS(K_6^*, 4) = 4$. From Lemma 1.3, $EDS(K_6^*, 4) \leq 5$. If $EDS(K_6^*, 4) = 5$, then each spanner would have exactly 6 edges and would, therefore, be a directed cycle, contradicting the fact that $K_6^*$ cannot be decomposed into 5 Hamiltonian cycles (see [18]).

For other $n > 2$, the upper bound comes from Lemma 1.3. Tillson showed that $K_n^*$ can be decomposed into $n - 1$ directed Hamiltonian cycles [13] for even $n$ if and only if $n = 2$ or $n \geq 8$. For odd $n \geq 3$, a decomposition of $K_n^*$ into $n - 1$ directed Hamiltonian cycles is obtained directly from a decomposition of $K_n$ into $\frac{n - 1}{2}$ Hamiltonian cycles which has long been known (see, for example, [10]). In a directed cycle on $n$ vertices, the distance between any ordered pair of vertices is at most $n - 1$ and, thus, is a spanner of $K_n^*$ with delay at most $n - 2$. □

**Lemma 3.14** $EDS(K_n^*, n - 3) = n - 3$ for any $n \geq 5$.

**PROOF** The upper bound comes from Lemma 3.4.

For odd $n$, $n = 2p + 1$, $K_n$ can be decomposed into $p$ Hamiltonian cycles $C_1, C_2, ..., C_p$ (each of length $2p + 1$). Any one of these cycles, $C_i$, can be used to obtain 2 spanners of $K_n^*$ with delay $2p - 1$ (that is, $n - 2$) by taking 2 copies of $C_i$ and assigning all edge directions to be clockwise in one copy and counterclockwise in the other. By adding a pair of edges $(x, y)$ and $(y, x)$ to such a cycle (where $x$ and $y$ are not adjacent vertices in the cycle), we obtain a spanner with delay at most $2p - 2$. The resulting graph can be regarded as two directed cycles of length $\leq 2p$, one including $(x, y)$ and the other including $(y, x)$. The distance between
any (ordered) pair of vertices in the same short cycle is $\leq 2p-1$. From any vertex $u$ to any vertex $v$ which is not on the same short cycle as $u$, there is a path of length $\leq 2p-1$ in the original cycle. Thus, the delay in this spanner is at most $2p-2$ (that is, $n - 3$). We can construct $2p-2$ edge disjoint spanners of this type as follows. Use $C_1, C_2, ..., C_{p-1}$ to form $2p-2$ edge disjoint Hamiltonian cycles as described above. To each of these cycles, add a pair of edges $(x, y)$ and $(y, x)$ corresponding to a different edge $(x, y)$ from $C_p$. Since $C_p$ is edge disjoint from each of the other $C_i$, these edges connect vertices of $C_i$ which are not adjacent.

For even $n = 2p$, consider the decomposition of $K_{2p-1}$ into $p-1$ Hamiltonian cycles $H_1, H_2, ..., H_{p-1}$. From each $H_i$, $1 \leq i \leq p-1$, construct 2 directed Hamiltonian cycles $C_{2i-1}$ and $C_{2i}$ of $K_{2p-1}$. In particular, let $C_{2i-3}$ be $x_1, x_3, ..., x_{2p-1}$. For each $i$, $1 \leq i \leq 2p-4$, add to $C_i$ a different edge $(x_i, x_{i+1})$ from $C_{2i-3}$ and edges $(x_{i+1}, a)$ and $(a, x_i)$ where $a$ is a vertex not in $K_{2p-1}$. This gives $2p-4$ edge disjoint spanners of $K_{2p}$ with delay at most $2p-3$. Indeed, any two vertices on $C_i$ are at distance at most $2p-2$.

Vertex $a$ forms a directed cycle of length $\leq 2p-1$ with vertices of $C_i$ from $x_1$ to $x_{i+1}$. Note that there is at least one vertex $r$ between $x_i$ and $x_{i+1}$ in $C_i$. Consider any vertex $s$ between $x_{i+1}$ and $x_i$ in $C_i$. There is a path from $a$ to $s$ (also $s$ to $a$) which goes through $x_1$ and $x_{i+1}$ but not $r$. This path has length at most $2p-2$. Thus, the delay of this spanner is at most $2p-3$ (that is, $n - 3$). We construct an additional spanner from $C_{2p-3}$ (which is the “opposite” cycle of $C_{2p-3}$) by adding the remaining edges $(x_2, x_{2p-2})$, $(x_{2p-2}, x_2)$, and $(x_2, x_{1})$ from $C_{2p-3}$. We also add edges $(a, x_{2p-3})$, $(x_{2p-2}, a)$, $(a, x_{2p-2})$, $(x_{2p-2}, a)$, $(a, x_{2p-2})$, and $(x_1, a)$. As before, the delay is $\leq 2p-3$.

3.4 Product constructions

Lemma 3.15

If $EDS(K_n^*, c_1) \geq E_1$ and $EDS(K_n^*, c_2) \geq E_2$, then $EDS(K_{n_1n_2}, 2c_1 + c_2 + 2) \geq n_1 \min(E_1, E_2)$ for $n_1 \leq n_2$ and $n_2$ prime.

Proof. Consider the vertices of $K_{n_1n_2}^*$ placed in an $n_1 \times n_2$ array, $t_{i,j}$ with $0 \leq i \leq n_1 - 1$ and $0 \leq j \leq n_2 - 1$. Let $D(m, j, j')$ denote the diagonal of slope $m$ in the table containing $t_{i,j}$; i.e., $D(m, j, j') = \{t_{i,j} | j = (j + m) \text{mod} n_2, 0 \leq i \leq n_1 - 1\}$, for $0 \leq m \leq n_1 - 1$ and $0 \leq j \leq n_2 - 1$. (Note that a diagonal that runs off one side of the table reenters from the opposite side).

Observe that $D(m, j, j') = \emptyset$ when $j \neq j'$. We can furthermore show that any two diagonals of different slopes have at most one common element: $|D(m, j, j') \cap D(m', j_i)| \leq 1$ when $m \neq m'$. To see this, suppose without loss of generality that $m < m'$ and that two table vertices $t_{i,j}$ and $t_{i',j'}$ (without loss of generality, $i < i'$) are in both diagonals. We have then that

$$j = j_1 + mi \mod n_2$$

$$j' = j_1 + m'i \mod n_2$$

$$j = j_1' + m'i \mod n_2$$

$$j' = j_1' + m'i' \mod n_2$$

Subtracting (2) and (3) from (4), and then adding (1), we get $0 = (m' - m)(i' - i) \mod n_2$. Thus, either $m' - m = 0$ or $i' - i = 0$, contradicting our assumptions. Otherwise, by prime factorization and the fact that $n_2$ is a prime, $n_2$ must divide either $m' - m$ or $i' - i$. However, $0 \leq m' - m \leq m' \leq n_1 - 1$ and $0 \leq i' - i \leq i' \leq n_1 - 1$, so $n_2$ could not divide either factor.

Thus, we have shown that $|D(m, j, j') \cap D(m', j_i)| \leq 1$ whenever $(m, j, j') \neq (m', j_i')$. We call this latter property the combinatorial property of diagonals.

Let $D(K_{2p}, i)$ denote the $i$th column containing $t_{i,j}$, where $0 \leq j \leq n_2 - 1$ for $0 \leq i \leq n_1 - 1$. It is trivial that $|D(0, i) \cap D(m, j)| \leq 1$ for $m \neq \infty$. Let $C(m, j) \in D(K_{2p}, i)$ denote the complete directed graph of $D(m, j) \in D(0, i)$. Each such $C(m, j) \in D(0, i)$ is a subgraph of $K_{n_1n_2}$ and, furthermore, they are all edge disjoint by the combinatorial property of diagonals. We divide these cliques into $n_1$ collections $C_0, C_1, ..., C_{n_1-1}$ where $C_i$ contains the cliques $C(0, i)$ and $C(i, j)$ for $0 \leq j \leq n_1 - 1$. Note that each $C_i$ spans $K_{n_1n_2}$.

We will partition each $C_i$ into $\min(E_1, E_2)$ spanners as follows. Since $EDS(K_{2p}^*, c_2) \geq E_2$, we can color the edges of $C(0, i)$ with the $E_2$ colors $1, 2, ..., E_2$ such that the edges of each color class form a $(c_2 + z)$-spanner of $C(0, i)$. Similarly, we can color the edges of each $C(i, j)$ with the $E_1$ colors $1, 2, ..., E_1$ such that the edges of each color class form a $(c_1 + x)$-spanner of $C(i, j)$. Consider the edges of $C_i$ that are of color $p$ where $1 \leq p \leq \min(E_1, E_2)$.

We claim that these edges form a $(2c_1 + c_2 + 2 + z)$-spanner $S$ of $K_{n_1n_2}$. First, observe that $C(0, i)$ intersects each $C(i, j)$ in exactly one vertex $w_{ij}$. Let $u, v$ be two vertices in $K_{n_1n_2}$. The vertex $u$ is in some $C(i, j)$ and $v$ is in some $C(i', j')$. In $S$ there is a path from $u$ to $v$ consisting of a path from $u$ to $w_{ij}$, followed by a path from $w_{ij}$ to $w_{ij'}$, followed by a path from $w_{ij'}$ to $v$. The length of this path is at
most \((c_1 + 1) + (c_2 + 1) + (c_1 + 1) = 2c_1 + c_2 + 3\), yielding a delay of at most \(2c_1 + c_2 + 2\). Thus, \(S\) is a 
\((2c_1 + c_2 + 2 + x)\)-spanner of \(K_{n_1n_2}\) as claimed.

To complete the proof, we note that there are \(n_1\min(E_1, E_2)\) such spanners, \(\min(E_1, E_2)\) for each 
\(C_i\). Furthermore, these spanners are edge disjoint as 
the \(C_i\) are. □

**Lemma 3.16**

If \(EDS(K_{n_1}^{*}, c_1) \geq E_1\) and \(EDS(K_{n_2+1}^{*}, c_2) \geq E_2\), then \(EDS(K_{n_1n_2+1}^{*}, 2c_1 + c_2 + 2) \geq n_1\min(E_1, E_2)\)

for \(n_1 \leq n_2\) and \(n_2 + 1\) prime.

**PROOF** The proof is omitted but is similar to the proof of Lemma 3.15. □

By applying Lemma 3.13, we get \(EDS(K_{n}^{*}, n-2) = n - 1\) and \(EDS(K_{n+1}^{*}, n - 1) = n\) for \(n \geq 7\). Using 
Lemma 3.15 with \(n_1 = n\) and \(n_2 = n + 1\) a prime, we get \(EDS(K_{n+2}^{*+n}, 3n - 3) \geq n(n - 1)\). Previously, the
best lower bound was \(EDS(K_{n+2}^{*+n}, 3n - 3) \geq \left\lfloor \frac{n^2 + n}{2} \right\rfloor\)
from Lemma 3.6. The improvement is significant for large \(N\), that is, if \(N\) is large, the new bound says that delay \(\sqrt{N}\) allows roughly \(N - \sqrt{N}\) spanners, compared to the \(\left\lfloor \frac{N^2}{2} \right\rfloor\) bound which was known before. The first 
value of \(N\) for which such an improvement is realized is \(N = 42\). The lower bound from Lemma 3.6 is 21. Using 
\(n_1 = 6, c_1 = 4, n_2 = 7,\) and \(c_2 = 3,\) Lemma 3.15 gives \(EDS(K_{42}^{*}, 13) \geq 24\).

**4 Specific bounds for complete digraphs**

Using the results from the previous section, one can construct a table showing bounds on \(EDS(K_{n}^{*}, c)\) for 
various values of \(n\) and \(c\). Although for some values of 
\(n\) and \(c\) we know \(EDS(K_{n}^{*}, c)\) exactly, for many \(n\) and 
\(c\) the bounds are not tight. In this section, we tighten 
the bounds on \(EDS(K_{n}^{*}, c)\) for small \(n\). The resulting 
bounds are shown in Table 1 for \(n \leq 17\).

In this extended abstract, we omit the proofs of 
many of the lemmas in this section.

**Lemma 4.1** \(EDS(K_{7}^{*}, 1) = 2\).

**PROOF** From Lemma 3.1 we know that \(EDS(K_{7}^{*}, 1) \geq 2\). Suppose, by way of contradiction, 
that \(EDS(K_{7}^{*}, 1) \geq 3\). From Lemma 3.8, we know 
that in such a spanner, each vertex must have indegree 
and outdegree at least 2. Since the indegree and 
outdegree of each vertex in \(K_{7}^{*}\) is 6, we can have at 
most 3 such spanners and, if they exist, every vertex 
in each spanner would have indegree and outdegree 
exactly 2.

Consider the tree of height 2 of edges going out from 
any vertex in such a spanner. This tree must contain 
7 vertices and must, therefore, be a balanced binary 
tree of height 2. Label the root of such a tree \(a\), with 
\(a\)'s outneighbors being labelled \(b\) and \(c\). Label \(b\)'s outneighbors \(d\) and \(e\) and label \(c\)'s outneighbors \(f\) and \(g\). 
Since \(b\) must also be the root of such a tree, \(a, c, f,\) and 
\(g\) must be outneighbors of \(d\) and \(e\). Likewise, since \(c\)
must be the root of such a tree, \(a, b, d,\) and \(e\) must be outneighbors of \(f\) and \(g.\) Without loss of generality, assume that \(a\) is an outneighbor of \(d\) and \(g.\) Vertex \(c\) can not be an outneighbor of \(d\) since, otherwise, \(d\)'s tree would contain \(c\) twice. So, \(c\) must be an outneighbor of \(e.\) Similarly, \(b\) must be an outneighbor of \(f.\) Now, either \(f\) or \(g\) must be an outneighbor of \(d.\) If \(f\) is an outneighbor of \(d,\) then \(d\)'s outtree contains two paths of length 2 to \(b.\) If \(g\) is an outneighbor of \(d,\) then \(d\)'s outtree contains paths of length 1 and 2 to \(a.\) Since we cannot construct such a spanner, \(EDS(K_7^+, 1) \leq 2.\)

\[\text{Lemma 4.2} \ EDS(K_7^+, 2) = 3.\]

\[\text{Lemma 4.3} \ EDS(K_7^+, 3) = 4.\]

\(\text{Proof}\) \(EDS(K_7^+, 3) \leq 4\) comes from Lemma 3.4 and four spanners of delay 3 are shown in Figure 1. \(\Box\)

\[\text{Lemma 4.4} \ EDS(K_8^+, 1) = 2.\]

\[\text{Lemma 4.5} \ EDS(K_8^+, 2) = 4.\]

\[\text{Lemma 4.6} \ EDS(K_8^+, 4) = 5.\]

\[\text{Figure 2: Five spanners of complete directed graph on 8 vertices with delay 4.}\]

\(\text{Proof}\) \(EDS(K_8^+, 4) \leq 5\) comes from Lemma 3.4 and five spanners of delay 4 are shown in Figure 2. \(\Box\)

\[\text{Lemma 4.7} \ EDS(K_9^+, 3) = 6.\]

\[\text{Lemma 4.8} \ EDS(K_{11}^+, 1) = 3.\]

\[\text{Lemma 4.9} \ EDS(K_{13}^+, 8) \geq 8.\]

\(\text{Proof}\) We show how to construct 8 spanners with delay 8. Label the vertices of \(K_{13}^+\) with 0, 1, 2, ..., 10, 11, \(\infty.\) We will perform additions on the vertex labels using modulo 12 addition (leaving 8 unaffected by the addition).

Decompose \(K_{13}^+\) into 12 directed Hamilton cycles as follows: Let \(C_0\) be the cycle visiting the vertices in the order \(\infty, 0, 1, 11, 2, 10, 3, 9, 4, 8, 5, 7, \infty.\) For \(1 \leq i \leq 5,\) \(C_i\) is the cycle visiting the vertices in the order obtained by adding \(i\) (modulo 12) to the vertex list for \(C_0.\) The 6 cycles \(C_i \) are obtained by visiting the vertices of \(C_i\) in the reverse order.

Eight spanners will be constructed from the cycles \(C_i\) and \(C_i'\) where \(0 \leq i \leq 3\) by adding to each of them four different edges from the remaining cycles \((C_i\) and \(C_i', i = 4, 5)\) as follows:

Spanner \(S_0\) is constructed by adding edges \((\infty, 4), (4, \infty), (5, 3),\) and \((3, 5)\) to \(C_0.\) Spanner \(S_1\) is \(S_0 + 1.\)
That is, it can be constructed as follows: If \((i, j)\) is an edge in \(S_0\), then \((i + 1, j + 1)\) is an edge in \(S_1\). In other words, this means that \(S_1\) is constructed by adding edges \((\infty, 5), (5, \infty), (6, 4),\) and \((4, 6)\) to \(C_1\).

Spanner \(S_2\) is constructed by adding edges \((\infty, 10), (10, \infty), (9, 11),\) and \((11, 9)\) to \(C_2\). Spanner \(S_3\) is \(S_2 + 1\).

Spanner \(S_4\) is constructed by adding edges \((6, 2), (2, 6), (1, 8),\) and \((8, 1)\) to \(C_4\). Spanner \(S_5\) is \(S_4 + 1\).

Spanner \(S_6\) is constructed by adding edges \((0, 8), (8, 0), (1, 7),\) and \((7, 1)\) to \(C_6\). Spanner \(S_7\) is \(S_6 + 1\).

\[\Box\]

**Lemma 4.10** \(EDS(K_{15}, 1) \leq 4\).

**Lemma 4.11** \(EDS(K_{16}, 1) = 4\).

5 Conclusions

We have investigated the existence of edge disjoint spanners in complete graphs and complete digraphs. In the case of complete graphs, we found, among other things, that \(EDS(K_n, c) = \left\lceil \frac{n}{c} \right\rceil \) for \(2 \leq c \leq n - 2\) where \(n \geq 4\). and that \(EDS(K_n, 1) \approx \frac{n}{c+1}\) for large \(n\). The situation in complete digraphs is not as well understood. In addition to some more specific results, we have bounded the number of spanners as follows:

\[\left\lceil \frac{n}{2} \right\rceil \leq EDS(K_n, c) \leq \frac{cn}{c+1} \] for \(2 \leq c \leq n - 3\) and \[\left\lceil \sqrt{n} \right\rceil \leq EDS(K_n, 1) \leq \left\lceil \frac{n}{3} \right\rceil \] for \(n \geq 7\) and \(n \neq 9\) or 12.

We find these bounds surprisingly large; the number of edge-disjoint spanners in the complete graph and complete digraph is in most cases at least some fixed fraction of \(n\). These results indicate that, by using spanners, a large number of simultaneous processes can be executed on a machine with a complete interconnection network without edge contention between processes. Although our results give only upper bounds on other, more interesting, network topologies, they suggest that further study of these topologies is warranted.

References


