A Case Study of a Versatile Generator of Repeatable Non-Poisson Sequences of Pseudorandom Pulses

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Abstract—This paper is a case study of a general method for designing generators of repeatable non-Poisson sequences of pseudorandom pulses. A class of normal distributions and some classes of skew distributions are derived. Block diagrams and design guides are given. The study is based both on analytical and simulation approaches. Applications of the proposed system may be found in the field of hardware simulation techniques and in more efficient arrangement of general purpose machines where they are used in simulation studies.

Index Terms—Buffer length, generators of non-Poisson sequences, normal distribution, pseudorandom pulses, regular arrivals/Erlang service, simulation, shifted Poisson, skew distributions.

I. INTRODUCTION

In a previous paper [1] the design criteria for a generator of repeatable non-Poisson sequences of pseudorandom pulses have been discussed, and a formalization of a system realizing non-Poisson distributions of this kind has been given.

Here, in this paper, a case study is presented. Transformation procedures are discussed in detail and block diagrams are given for a class of normal distributions and some classes of skew distributions. The statistical behavior of the system is analyzed and design guides are given. The study of the system presented rests both on analytical and simulation approaches. The flexibility in constructing practical distributions for a wide range of applications is thought to be the main asset of the system as far as its use in hardware simulators. The implementation of the procedures which are discussed here could fall within one or other of the three levels of applications discussed in [1].

II. BACKGROUND

A. Some Comments on the Generator of Hartley

The generator discussed by Hartley is used as the first stage of the system which is described here. Due to the fact that this generator is a clocked system, the time interval between consecutive output pulses is a discrete random variable of the lattice type. Events occur at specific instants which are multiples of the clock period \( h \). This should be taken into account in dealing with the mathematical analysis of the system under consideration in this paper. The mathematical problems involved are of two kinds, namely, transformation procedures and probabilistic (queueing) problems.

As far as the transformations are concerned, if the final random variable is to be also of the lattice type, the transformations should be functions in which both the range and the set of values is the set of the integers. Transformations such as addition, translation, and reflection satisfy the above requirement. Multiple and combined use of these transformations is made in what follows to produce a wide range of new distributions.

With reference to the statistical problems involved, it may be observed that if \( p \) is the probability of a pulse occurring at any instant and \( \alpha \) is the mean pulse rate, then

\[
p = \alpha h.
\]

Now if \( p \ll 1 \), i.e., \( h \ll 1/\alpha \), the discrete time process can be approximated closely by a continuous one [4]. In the generator of Hartley, the probability of success \( p \) is given as

\[
p = \left( \frac{1}{2} \right) N \left( f' + 1 \right) F
\]

where \( N, F, f' \) are constants of a coarse and fine frequency control, taking values \( 1 < N < 8 \), \( 1 < F < 31 \), and \( 1 < f' < F - 1 \). For low rates \( N > 2 \) and \( F > 2 \), the above condition of a very small probability of success is fulfilled. Therefore, the sequences which are produced by this type of generator can be treated as completely random sequences in continuous time. From this point of view, the interevent distribution over a large interval, which strictly speaking is a geometric distribution, can be described to a very good approximation by a negative exponential distribution (Poisson approximation). To allow for the discrete character of the process, if \( p(t) \) is the interevent probability density function (pdf), we can write

\[
p(t) = \begin{cases} 
\alpha e^{-\alpha t} \sum_{n=0}^{\infty} \delta(t - nh) & \text{if } t \geq 0 \\
0 & \text{if } t < 0
\end{cases}
\]

where \( \delta(t) \) is the delta-function and \( h \) is the lattice constant.

In practice, the distributions which are produced by the generators under discussion are truncated [Fig. 1(a)], and therefore a realistic way of describing them is
Accordingly, the pdf for the gaps between an arrival and the kth following one, should be a truncated Erlang distribution [Fig. 1(b)] of the type

\[
p_{k}(t) = \begin{cases} 
\frac{\alpha(t-k)^{-1}}{(k-1)!} t^{kN-1} e^{-\alpha t} \sum_{n=0}^{N} \delta(t-nh) & \text{if } 0 \leq t \leq Nh \\
0 & \text{otherwise}.
\end{cases}
\]

(4)

where \( M = kN \).

Let \( x \) be a new random variable such that

\[ x = M - t. \]

(7)

From (6) and (7) we find for the pdf of \( x \),

\[
z_{k}(x) = \begin{cases} 
\frac{\alpha(M-x)^{k-1}}{(k-1)!} e^{-\alpha(M-x)} \sum_{n=0}^{M} \delta(x-nh) & \text{if } 0 \leq x \leq Mh \\
0 & \text{otherwise}.
\end{cases}
\]

(8)

Consider now the sum of \( t \) and \( x \):

\[ z = x + t. \]

(9)

We assume that \( x \) and \( t \) are statistically independent. Then the pdf of \( z \) is given by the convolution of \( 1_{P_{k}(t)} \) and \( z_{k}(x) \), i.e.,

\[
p_{k}(z) = 1_{P_{k}(t)} \ast z_{k}(x). \]

(10)

From (6), (8), and (10), by putting \( h = 1 \) we obtain (Appendix I)

\[
p_{k}(z) = \begin{cases} 
\frac{e^{-\alpha(M+z)k^{-2}}}{(k-1)^{2}} \sum_{n=0}^{z} (z-n)^{k-1}(M-n)^{k-1} e^{2an} & \text{if } 0 \leq z \leq M \\
\frac{e^{-\alpha(M+z)k^{-2}}}{(k-1)^{2}} \sum_{n=z-M}^{M} (z-n)^{k-1}(M-n)^{k-1} e^{2an} & \text{if } M \leq z \leq 2M
\end{cases}
\]

(11)

where \( k \) is the degree of the forward and reflected Erlang distributions. It can be shown that distribution (11) is a symmetrical one around \( M \), with mean and variance given by

\[
\mu_{k} \approx M \]

(12)

\[
\sigma_{k}^{2} = \frac{2k}{\alpha^{2}}.
\]

(13)
A class of symmetrical distributions of this type can be obtained for different values of the parameter \( k \). Now, it can be observed that the distribution (11) is the result of successive additions of independent random variables. \( k \) additions are performed in the first place on the exponentially distributed interevent time intervals of an initial Poisson process and then another addition follows between an ordinary Erlang distribution and a reflected one. It is reasonable to expect, according to the central limit theorem, that as \( k \) increases the above distribution approaches more and more the normal limit. This is, of course, a general rule. Nevertheless, when a realization of a normal distribution is tried through such a procedure, the question is how fast a given sum of independent random variables converges to the normal limit.

The distance between the functions \( p_k(z) \) and the normal limit \( n(z) \) can be considered a measure of the speed of convergence. The distance \( \phi(p_k, n) \) can be defined in several ways [5]. As an appropriate expression of the distance in the present study, the expression

\[
\phi(p_k, n) = \sum_{z=0}^{2M} |p_k(z) - n(z)|
\]

has been considered. Expression (14) has been calculated numerically for several values of the parameters \( \alpha, M, \) and \( k \).

Table I illustrates the variation of \( \phi(p_k, n) \) with \( k \) for \( \alpha = 0.062 \).

Fig. 2(a), (b), (c) gives a comparative view of the functions \( p_k(z) \) and \( n(z) \). From Table I and Fig. 2 it can be seen that for very low values of \( k \), say \( k = 4 \) or \( 5 \), a very good approximation of the normal distribution can be obtained. The quality improves as \( k \) increases.

From (12) and (13) the normal curve which corresponds to \( p_k(z) \) is

\[
n(z) = \frac{\alpha}{\sqrt{4kn\pi}} \exp \left[ -\frac{(z - M)^2}{2k} \right] \sum_{s=0}^{2M} \delta(z - nh) \quad 0 \leq z \leq 2Mh.
\]

By changing the parameters \( \alpha, M, \) and \( k \), a range of normal distributions can be obtained.

**Realization:** Fig. 3 gives the block diagram of a system realizing the above transformation. A random generator with pseudorandom output of Poisson type [2], [3] is used as the first stage for each of the two channels.

Channel I realizes the ordinary Erlang distribution. The pulses which come from the generator all have the same amplitude \( V \) and the times between them follow a negative exponential distribution of the type described by Fig. 1(a). A counter provides an output pulse every \( k \)th pulse. As a result, the time intervals between the pulses after the counter follow an Erlang distribution of the type described by (6) [Fig. 1(b)].

The next stage is the time-to-voltage conversion unit. The pulses leaving this unit have variable sizes \( V_1 \) which follow the same distribution as the times between them. These pulses are stored temporarily, in order of their arrival, in buffer store 1.

Channel II is a replica of channel I, except that a further stage is added. This is the voltage-transformation unit, which realizes the voltage transformation \( x = M - t \), or \( V_2 = M - V_1 \). We assume that the computation time in this unit is negligible in comparison with the minimum interarrival time. Therefore, the size \( V_2 \) of the pulses leaving this unit follow a reflected Erlang distribution described by (8) [Fig. 1(c)], while the times between them retain the ordinary Erlang distribution. The pulses are stored temporarily in buffer II, again in the order of their arrival. The transformation given by (9) is realized by the adder. Two words are removed at regular intervals \( T \), one from each buffer. The words are added and their sum is stored temporarily in buffer III.

The last stage is the voltage-to-time conversion unit. The words are removed sequentially from buffer III. The absolute magnitude of the word determines the corresponding time interval which is generated by this last stage.

The above transformation procedure has been simulated in ATLAS AUTO CODE. The pattern of pseudorandom arrivals of Poisson type, was provided by a method which remodelled exactly the hardware generator described in [2] and [3]. The length of the simulation was 10 000 epochs. The number of Poisson arrivals was 1117. The results for a set of parameters are shown in Fig. 4(a), (b), and (c). A test has been carried out for goodness of fit of a normal distribution. The results are shown in Table II. It is evident that from \( k = 5 \) at the 5 percent level of significance the distribution obtained is consistent with the hypothesis that it belongs to a population which is normally distributed. A comparison between Tables I and II shows a close correspondence between the theoretical and experimental results. It is also an indirect test of the pseudorandom generator as for its suitability for realizing the distribution under consideration.

**Design Aspects:** In [1] the philosophy of designing the present system has been laid down. The factors affecting the output distribution were considered and it was shown that the timing of the system as well as the length of the buffers determine the quality of the distribution obtained.

**Timing:** As far as the timing is concerned, it has been shown that whenever a queue is developed, the ratio \( \rho \) of the output—input mean rates (traffic intensity) should be as high as possible while remaining less than unity. It should be recalled also that the system operates in real time and therefore the above statement concerning \( \rho \) refers to the ratio of real mean rates.

Suppose that the output clock period of the above system is required to be \( \tau_0 \). If the mean interval between the pulses is required to be \( M \) epochs, then the real-time mean rate of the output distribution is going to be

\[
m = M \cdot \tau_0.
\]
Fig. 2. (a) Comparison between the function $p_3(z)$ and the corresponding normal limit $n(z)$. (b) Comparison between the function $p_3(z)$ and the normal limit $n(z)$. (c) Comparison between the function $p_5(z)$ and the normal limit $n(z)$. 
Fig. 3. Block diagram of a generator of non-Poisson sequences of pseudorandom pulses. If \( k_1 = k_2 \) and \( k_1 \geq 5, k_2 \geq 5 \) the output is a normal distribution. If \( k_1 \neq k_2 \) the output is a skew distribution. If only channel IIa operates and \( k_2 = 1 \), the output is a shifted negative exponential distribution.

Fig. 4. Typical set of results for (a) \( k = 3 \), (b) \( k = 5 \), and (c) \( k = 10 \). Length of test: 10 000 epochs. Poisson arrivals: 1117.

<table>
<thead>
<tr>
<th>( k )</th>
<th>3</th>
<th>5</th>
<th>10</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 )</td>
<td>12.15</td>
<td>9.42</td>
<td>3.52</td>
<td></td>
</tr>
<tr>
<td>( x^2 ) 5 percent</td>
<td>11.07</td>
<td>5</td>
<td>21.03</td>
<td>12</td>
</tr>
</tbody>
</table>

\( x^2 \) test for goodness of fit of a normal distribution.
\( v \) represents the number of degrees of freedom.
If \( \mu \) and \( \tau \) are the corresponding parameters of the pattern arriving at buffer III, then the following relation must be satisfied,

\[
\rho_1 = \frac{M \tau_0}{\mu \tau} \Rightarrow 1. \tag{17}
\]

Now the pattern \( \mu, \tau \) is the output pattern for the buffers I and II. It has already been explained that the input pattern for both the buffers as far as timing is concerned is the same. Let \( \mu_1, \tau_1 \) be the corresponding parameters of the patterns entering the buffers I and II. Then

\[
\rho_2 = \frac{\mu \tau}{\mu_1 \tau_1} \Rightarrow 1. \tag{18}
\]

From the previous analysis it can be seen that the number \( M \) (the required mean number of epochs between successive pulses) is the truncation point of the initial Erlang distribution, and \( \mu_1 \) is the mean of this distribution. It is obvious that \( \mu_1 \) should be much smaller than \( M \). The relation between \( M \) and \( \mu_1 \) may be used to ensure that the error due to truncation is small. It has been found that for

\[
M \geq 4\mu_1 \tag{18a}
\]

the error becomes negligible. It can also be seen from the description of the system that a convenient value for \( \mu \) is \( \mu = 1 \).

**Statistics of the system:** There are three buffers in the system under discussion. Buffers I and II behave in the same way, given that in both of them the arrivals occur according to an Erlang distribution \( (E_k) \) and the departures are regular \((D)\). The queueing system to be analyzed here is of the type \( E_k/D/1 \). In buffer III the arrivals are regular \((D)\) and the departures occur according to a normal distribution \((N)\).

For both these queueing systems, the necessary capacity (buffer length) has to be determined for a given traffic intensity \( \rho \) and an acceptable loss \( R_L \). Also, for the given value of \( \rho \), the probability of an empty queue has to be determined for different capacities [1].

**Buffers I and II—The System \( E_k/D/1 \):** A detailed treatment of this system has already been given in [6]. The dependence of the fractional loss \( R_L = R_L(L) \) for different values of \( \rho \) is given there for a range of values of the parameter \( k \). Here, for the same range of \( k \), the dependence of the probability of an empty queue \( Q(0) \) on the available capacity of the system (buffer length) is given, for a range of traffic intensities \( \rho \). Fig. 6(a), (b), (c), (d). The evaluation of these curves is based on the equations which are given in [6].

**Buffer III—The System \( D/E_k/1 \):** In buffer III the arrivals are regular and the departure times are normally distributed \((N)\). Under these circumstances the queue length being a non-Markov process is not easily treated analytically as such, especially since explicit results are required for design purposes. Instead the system \( D/E_k/1 \) is analyzed (see Section II-B). The analysis of this system is feasible, meaningful results are obtained, and the \( D/N/1 \) system can be approximated by the \( D/E_k/1 \) system for a relatively high value of \( k \).

The analysis of the \( D/E_k/1 \) model is done here on the same lines on which the analysis of the dual model \( E_k/D/1 \) is done in [6]. The method of stages for the Erlang distribution is applied and an imbedded Markov chain is defined which is made up of the instants just before the arrivals. It is assumed that the words are removed on a first-come first-served basis. The results of the analysis (see Appendix III) are illustrated in Figs. 6 and 7. Fig. 6(a), (b), (c), (d) shows the dependence of the mean fractional loss \( R_L \) on the buffer length \( L \) for different traffic intensities.

Fig. 7(a), (b), (c), (d) shows the dependence of the probability of zero queue length on the buffer length for different values of \( \rho \). It can be observed from the above curves that the buffer length necessary for minimum loss is already low even for small values of \( k \). The observed trend provides the designer with valuable information as far as a worse case design is concerned. A length of 15 to 20 words would be sufficient for the application under discussion.

**Determination of System Parameters:** The determination of the parameters of the system proceeds as follows.

For specified values of \( M \) and \( \tau_0 \) (see above) a value of \( \tau \) is determined from (17) where \( \mu \) is taken equal to unity, and \( \rho_1 \) is given a specific value close to unity. For this value of \( \rho_1 \) and an acceptable value of \( Q(0) \) a first estimate of \( L \) of the buffer length is determined from Fig. 7(d). Then for a given loss \( R_L \) and the chosen value of \( \rho_1 \), a buffer length, \( L_3 \geq L \) is determined from Fig. 6(d). Then from (16) of (1) the initial queue length \( L_{3,0} \) is determined and therefore buffer III should have a length \( L_3 + L_{3,0} \). Subsequently a corresponding value of \( \mu_1 \) is determined from (18a) and a value \( \rho_2 \) is specified for which the corresponding values of \( L_3, L_{2,0}, L_{1,0} \) are obtained from Fig. 5, Fig. 6, and the corresponding figures of [6]. Then, from (18) a value of \( \tau_1 \) is found. Now, if \( k \) is the degree of the Erlang distribution suitable for the required accuracy (Tables I and II) then it is obvious that the mean rate of the initial exponential distribution is going to be

\[
\lambda = \frac{\mu_1}{k}
\]

and the corresponding clock rate will be equal to \( \tau_1 \).

**Example:** If \( M = 300, \tau_0 = 1 \mu s, k = 5 \), and the length of the run is 2 seconds. Let \( \mu_1 = 60 \).

**Buffer III:** For \( \rho_1 = 0.990, Q(0) = 1 \times 10^{-3} \), and \( R_L = 10^{-5} \) we determine the following values: \( \tau = 303 \mu \), \( L_3 = 18, L_{3,0} = 6 \). Therefore the length of buffer III is \( L_3 + L_{3,0} = 24 \).

**Buffers I and II:** For \( \rho_2 = 0.990, Q(0) = 10^{-2} \), and \( R_L = 10^{-5} \) we obtain the following values \( \tau_1 = 51 \mu s, L_1 = L_2 = 90, L_{1,0} + L_{2,0} = 66 \). Therefore the corresponding buffer lengths should be \( L_1 + L_{1,0} = L_2 + L_{2,0} = 155 \).

Finally, the mean value of the initial random process which is given by \( \alpha = \mu_1/k \) is found to be \( \alpha = 12 \).

Therefore, the initial random generators which are going to supply the system should have an output clock period of \( \tau_1 = 51 \mu \) and a mean interevent rate of 12 epochs. A final
Fig. 5. Dependence of the probability of an empty buffer $Q(0)$ on the buffer length ($L$) for different values of the traffic intensity $\rho_k$ for an Erlang input of degrees $k = 2, 3, 4, 5$. 
Fig. 6. Dependence of the mean fractional-loss $R_L$ on the buffer length $L$ for different values of the traffic intensity $\rho_k$ for an Erlang output of degrees $k = 2, 3, 4, 5$. 
Fig. 7. Dependence of the probability of an empty buffer $Q(0)$ on the buffer length $L$ for different values of the traffic intensity $\rho_k$, for an Erlang output of degrees $k = 2, 3, 4, 5$. 
point which must be made is that, since statistical independence between the two components has been assumed, the outputs of the two initial random generators should be uncorrelated. This can be achieved in two ways, either by setting different initial patterns in the two generators or by using a delayed output from the same generator to feed the second channel. The amount of delay required can be deduced from the autocorrelation curves of the pseudorandom generators [2], [3].

B. Class I of Skew Distributions

If in (6) and (8) the parameter \( k \) assumes different values \( k_1, k_2 \), the range of the variables \( t \) and \( x \) becomes \( (0, M_1) \) and \( (0, M_2) \), respectively. In this case, the same procedure which was applied in Section III-A leads to the following expressions for the probability density function of the sum of \( x \) and \( t \) (see Appendix II):

I. \( M_2 < M_1 \)

\[
p_{k_1,k_2}(z) = \frac{e^{-a(z+M_2)}a^{k_1+k_2}}{(k_1-1)!(k_2-1)!} \sum_{n=0}^{z} (z-n)^{k_1-1}(M_2-n)^{k_2-1}e^{2an} \quad \text{for } z \leq M_2
\]

II. \( M_1 < M_2 \)

\[
p_{k_1,k_2}(z) = \frac{e^{-a(z+M_2)}a^{k_1+k_2}}{(k_1-1)!(k_2-1)!} \sum_{n=0}^{z} (z-n)^{k_1-1}(M_2-n)^{k_2-1}e^{2an} \quad \text{for } M_2 < z < M_1
\]

\[
p_{k_1,k_2}(z) = \frac{e^{-a(z+M_2)}a^{k_1+k_2}}{(k_1-1)!(k_2-1)!} \sum_{n=M_1}^{z} (z-n)^{k_1-1}(M_2-n)^{k_2-1}e^{2an} \quad \text{for } M_1 < z \leq M_2
\]

It can be shown that the distributions \( P_{k_1,k_2}(z) \) are not symmetrical. They are skew distributions with mean

\[
m \simeq M_2 + \frac{k_1 - k_2}{a}
\]

and variance

\[
\sigma^2 \simeq \frac{k_1 + k_2}{a^2}.
\]

A measure of the skewness is defined by Pearson [7] as

\[
\text{skewness} = \frac{\text{mean-mode}}{\text{standard deviation}}
\]

where mode is the value of the variable corresponding to the maximum of the theoretical curve.

It is evident that the distributions \( P_{k_1,k_2}(z) \) and \( P_{k_2,k_1}(z) \) are images of each other. By varying the parameters \( k_1, k_2 \), \( M_1 \), and \( a \), a whole class of skew distributions can be produced, which approximates a variety of practical distributions.

It can be said that the flexibility of the procedure described in generating such a variety of distribution functions is the main asset of the system described by the block diagram of Fig. 3.

A sample of skew distributions of this type is shown in Fig. 8(a), (b), (c), where the pairs \( (p_{2,3}(z), p_{3,3}(z)), (p_{2,5}(z), p_{5,3}(z)), (p_{2,8}(z), p_{8,3}(z)) \) have been plotted.

The transformation procedure under discussion has been simulated using ATLAS AUTOCODE, with the pseudorandom generators providing the initial patterns. The histograms of the resulting skew distribution are shown in Fig. 9(a), (b).

The realization of these distributions can be achieved with the same hardware which is shown in Fig. 3 by varying the order of the two counters \( k_1, k_2 \).

The statistics of the system in its new setting, as far as the buffers I and II are concerned, are governed by exactly the same laws as in the configuration employed for the normal distribution. As for buffer III, the approximation procedure which has been established for dealing with the normal distribution is also valid here.

C. Class II of Skew Distributions

Another class of skew distributions can be produced by considering the range of the reflected Erlang distributions. These distributions may be realized by activating only channel II of the block diagram of Fig. 3. The adder in this case would act merely as a sampling unit supplying buffer III and the stages which follow.

In such a setting the statistics of the system are again determined by the theory which has been developed above. Fig. 10(a), (b), (c) give a sample of distributions of this kind for some values of \( k \).
Fig. 8. Typical theoretical skew distributions for different combinations of $k_1$, $k_2$.

Fig. 9. Typical set of results of skew distributions. Length of test: 10 000 epochs. Number of Poisson arrivals: 1117.
The only where required adding transformation.

**Fig. III-A**

Consider the distribution (4). Let

\[ t' = t + b, \]

where \( b = \text{constant}, \) be a transformation of the variable \( t. \)

The pdf of the variable \( t' \) [Fig. 1(d)] will be

\[ p(t') = \begin{cases} 
ae^{-at} \sum_{n=b}^{N+b} (t' - nh) & bh \leq t' \leq (N+b)h \\
0 & \text{otherwise.} 
\end{cases} \]

**Fig. 11.** Block diagram of a system generating shifted negative exponential distributions.

In this case the same simulation procedure as in Sections III-A and III-B was applied.

**D. A Shifted Poisson Distribution**

Consider the distribution (4). Let

\[ t' = t + b, \]

where \( b = \text{constant}, \) be a transformation of the variable \( t. \)

The pdf of the variable \( t' \) [Fig. 1(d)] will be

\[ p(t') = \begin{cases} 
ae^{-at} \sum_{n=b}^{N+b} (t' - nh) & bh \leq t' \leq (N+b)h \\
0 & \text{otherwise.} 
\end{cases} \]

The present study demonstrates the main feature of the method which has been outlined in [1], namely, the versatility of the system which can be designed on these lines. A wide range of distributions is derived for different settings of a relatively low cost hardware equipment. The flexibility of the device in constructing distributions close enough to practical requirements should be noticed. This is of primary importance in simulation studies, where real life distributions rarely coincide with explicit mathematical forms.

Thus it is possible over a wide range of applications to find a set of values for \( \alpha, k_1, k_2, M_1, \) and \( M_2 \) such that desired probability distributions are obtained.

Further development of the system is also possible. New transformations may be added and new combinations found so that new classes of distributions may be developed.

**APPENDIX I**

Consider a truncated Erlang distribution of the type

\[ f(t) = \frac{\alpha(t)^{k-1}e^{-at}}{(k-1)!} \sum_{m=0}^{M} \delta(t - mh) \quad 0 \leq t \leq Mh. \]

Let \( x \) be a new random variable, such as
x = M - t.

From (26) and (27) it follows that

\[ zP_k(x) = \frac{\alpha(x(M - x))^{k-1}e^{-at(M - x)}}{(k - 1)!} \cdot \sum_{n=0}^{M} \delta(x - nh) \] 0 ≤ x ≤ Mh.  \hfill (28)

\[ p_k(z) = \frac{e^{-at(M + 1)}a^{2k}}{(k - 1)!} \sum_{n=0}^{z} (z - n)^{k-1}(M - n)^{k-1}e^{2zn} \] 0 ≤ z ≤ M

\[ z - n ≥ 0 \] 0 ≤ z ≤ 2M

\[ z = \text{integer}. \] \hfill (32)

Now (32) can be written as

\[ p_k(z) = \frac{e^{-at(M + 1)}a^{2k}}{(k - 1)!} \sum_{n=0}^{M} (z - n)^{k-1}(M - n)^{k-1}e^{2zn} \] M ≤ z ≤ 2M. \hfill (33)

Consider the sum of t and x, i.e.,

\[ z = t + x. \] \hfill (29)

Assuming statistical independence of t and x if \( P_k(z) \) is the pdf of z, then from (26), (28), and (29) we obtain the relation

\[ p_k(z) = \int_{0}^{z} tP_k(t)zP_k(z - t)dt \] 0 ≤ z ≤ 2Mh. \hfill (30)

From (26), (28), and (30) we obtain the expression

\[ p_k(z) = \int_{0}^{z} xP_k(x) \sum_{M=0}^{z} \frac{(z - n)^{k-1}(M - n)^{k-1}e^{2zn}}{(k - 1)!} \] \hfill (31)

or by putting \( h = 1 \)

\[ p_k(z) = \sum_{n=0}^{M} \frac{\alpha(x)^{k-1}e^{-at}}{(k - 1)!} \sum_{m=0}^{M} \delta(t - mh) \] \hfill (33)

or

\[ p_k(z) = \sum_{n=0}^{M} \alpha(M - n)^{k-1}e^{-at} \sum_{m=0}^{M} \delta(z - n + m) \] \hfill (34)

or

\[ p_k(z) = \sum_{n=0}^{M} \alpha(z - n)^{k-1} \sum_{m=0}^{M} \delta(z - (n + m)) \] \hfill (35)

or

\[ p_k(z) = \sum_{n=0}^{M} \alpha(z - n)^{k-1} \sum_{m=0}^{M} \delta(z - (n + m)) \] \hfill (36)

or

\[ p_k(z) = \sum_{n=0}^{M} \alpha(z - n)^{k-1} \sum_{m=0}^{M} \delta(z - (n + m)) \] \hfill (37)

Consider the distributions

\[ zP_k(t) = \frac{\alpha(x)^{k-1}e^{-at}}{(k - 1)!} \sum_{m=0}^{M} \delta(t - mh) \] 0 ≤ t ≤ Mh \hfill (38)

and

\[ zP_k(x) = \frac{\alpha(M - x)^{k-1}e^{-at}}{(k - 1)!} \sum_{m=0}^{M} \delta(x - mh) \] 0 ≤ x ≤ Mh. \hfill (39)

Consider the function \( P_k, k_1(z) \)

\[ P_k, k_1(z) = \sum_{n=0}^{M} \frac{\alpha(z - n)^{k-1} \alpha(M - n)^{k-1}}{(k - 1)!} \] \hfill (40)

or

\[ P_k, k_2(z) = \sum_{n=0}^{M} \frac{\alpha(z - n)^{k-1} \alpha(M - n)^{k-1}}{(k - 1)!} \] \hfill (41)

or

\[ P_k, k_3(z) = \sum_{n=0}^{M} \frac{\alpha(z - n)^{k-1} \alpha(M - n)^{k-1}}{(k - 1)!} \] \hfill (42)

or

\[ P_k, k_4(z) = \sum_{n=0}^{M} \frac{\alpha(z - n)^{k-1} \alpha(M - n)^{k-1}}{(k - 1)!} \] \hfill (43)

The function (38), depending on the relative magnitude of \( M_1 \) and \( M_2 \), assumes the following forms:
I. $M_2 < M_1$

\[ e^{-a(z+M_2)}z^{k_1+k_2} / (k_1 - 1)! (k_2 - 1)! \]

\[ \cdot \sum_{n=0}^{M_2} (z - n)^{k_1-1}(M_2 - n)^{k_2-1} e^{2zn} \quad z \leq M_2 \]

\[ P_{k,1,k}(z) = e^{-a(z+M_2)}z^{k_1+k_2} / (k_1 - 1)! (k_2 - 1)! \]

\[ \cdot \sum_{n=0}^{M_2} (z - n)^{k_1-1}(M_2 - n)^{k_2-1} e^{2zn} \quad M_2 < z < M_1 \]

\[ e^{-a(z+M_2)}z^{k_1+k_2} / (k_1 - 1)! (k_2 - 1)! \]

\[ \cdot \sum_{n=0}^{M_2} (z - n)^{k_1-1}(M_2 - n)^{k_2-1} e^{2zn} \quad M_1 \leq z \]

II. $M_1 < M_2$

\[ e^{-a(z+M_2)}z^{k_1+k_2} / (k_1 - 1)! (k_2 - 1)! \]

\[ \cdot \sum_{n=0}^{M_2} (z - n)^{k_1-1}(M_2 - n)^{k_2-1} e^{2zn} \quad z \leq M_1 \]

\[ P_{k,1,k}(z) = e^{-a(z+M_2)}z^{k_1+k_2} / (k_1 - 1)! (k_2 - 1)! \]

\[ \cdot \sum_{n=0}^{M_2} (z - n)^{k_1-1}(M_2 - n)^{k_2-1} e^{2zn} \quad M_1 < z < M_2 \]

\[ e^{-a(z+M_2)}z^{k_1+k_2} / (k_1 - 1)! (k_2 - 1)! \]

\[ \cdot \sum_{n=0}^{M_2} (z - n)^{k_1-1}(M_2 - n)^{k_2-1} e^{2zn} \quad M_2 \leq z. \]

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**Appendix III**

**Buffer Length for Regular Arrivals and Erlang Service Times—D/E_k/1**

**Formulation**

Let

\[ p(t) = \begin{cases} 
\frac{\alpha^{k+1}}{k!} t^k e^{-\alpha t} & t \geq 0 \\
0 & t < 0 
\end{cases} \]  

be the Erlang distribution of degree $k$ which describes the service process. This is equivalent to saying that a departure occurs over $k$ stages 0, 1, 2, \cdots $k-1$ in each of which the word waits for a time interval the length of which is exponentially distributed (with mean $\alpha$) before proceeding to the next stage. At any given time only one of the stages is occupied, a new word entering stage zero just as its predecessor leaves stage $k-1$. Accordingly, as a word arrives at the queue, the number of the stages of the system increases by $k$.

The state of the system at any time, is defined as the sum of the stages which the words in the system can assume. Thus, if the system is in state $m=nk+l$ where $l<k$, then $n$ words are actually standing in line or being served and another is in the "ith stage of departure."

Let $T$ be the constant interarrival time. If $r$ is the number of stages departing from the system within a time interval $T$, the queuing process can be described by the sequence of random variables $W_0$, $W_1$, $W_2$ \cdots defined recursively by the following equations corresponding to three different ranges of $W$, where $W$ is the number of stages in the system just before the arrivals. It is assumed that the length of the buffer is $L$. In this case, the maximum possible number of stages in the queue is $kL$.

---

I. $0 \leq W \leq k-1$

\[ W_{n+1} = \begin{cases} 
W_n + k - r & \text{if } 0 \leq W_n \leq kL - k \text{ and } W_{n+1} \leq k - 1 \\
W_n - r & \text{if } kL - k + 1 \leq W_n \leq kL \text{ and } W_{n+1} \leq k - 1
\end{cases} \] (40a)

(40b)

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II. $k \leq W \leq kL - k$

\[ W_{n+1} = \begin{cases} 
W_n + k - r & \text{if } W - k \leq W_n \leq kL - k \text{ and } k \leq W_{n+1} \leq kL - k \\
W_n - r & \text{if } kL - k + 1 \leq W_n \leq kL \text{ and } k \leq W_{n+1} \leq kL - k
\end{cases} \] (41a)

(41b)

---

III. $kL - k + 1 \leq W \leq kL$

\[ W_{n+1} = \begin{cases} 
W_n + k - r & \text{if } W - k \leq W_n \leq kL - k, kL - k + 1 \leq W_{n+1} \leq kL \\
W_n - r & \text{if } W \leq W_n \leq kL, kL - k + 1 \leq W_{n+1} \leq kL
\end{cases} \] (42a)

(42b)
i.e., the states $kL - k + 1 \leq W \leq kL$ can be achieved just before a new arrival, either when the previous state is $W - k \leq W_0 \leq kL - k$ when a new arrival is possible (+$k$) and the number of departures $r$ is such that $kL - k + 1 \leq W_0 + k - r \leq kL$ (42a), or if the previous state $W \leq W_0 \leq kL$ when no arrival is possible and $r$ is such that $kL - k + 1 \leq W_0 - r \leq kL$ (42b).

Let $q(W)$ be the probability of $W$ stages in the system just before the beginning of the time interval $T$. Similarly let $p(r)$ be the probability of $r$ stages departing within a time $T$. From (40), (41), (42) we derive the corresponding probability expressions for the equilibrium state:

$$q(W) = \sum_{r=0}^{kL-k-1} q(r)p(r - (W - k)) + \sum_{r=kL-k}^{kL-k+1} q(r)p(r - W) \quad 0 \leq W \leq k - 1 \quad (43)$$

$$q(W) = \sum_{r=W-k}^{kL-k} q(r)p(r - (W - k)) + \sum_{r=W}^{kL} q(r)p(r - W) \quad k \leq W \leq kL - k \quad (44)$$

$$q(W) = \sum_{r=W-k}^{kL-k} q(r)p(r - (W - k)) + \sum_{r=W}^{kL} q(r)p(r - W) \quad kL - k + 1 \leq W \leq kL. \quad (45)$$

The variable $r$ is distributed according to the Poisson law and therefore

$$p(r; T) = \frac{(\alpha T)^r}{r!} e^{-\alpha T} \quad \text{if } r \geq 0$$

$$= 0 \quad \text{if } r < 0. \quad (46)$$

By solving (43), (44), (45), (46), the probabilities $q(0)$, $q(1)$, $q(2), \cdots$, $q(kL)$ can be found.

**Fractional Loss Calculation**

Let $\rho$ be the traffic intensity. It is

$$\rho = \frac{k}{T} = \frac{k}{\alpha T} = \text{mean number of arrivals/mean service time (mst).} \quad (47)$$

If $R_L$ is the mean fractional loss then

$$\rho(1 - R_L) = \text{average number of words input to the queue/mean service time.} \quad (48)$$

But during the mst $\alpha/k$ words leave the queue. Therefore

$$\rho(1 - R_L) = \text{probability of removing } \alpha/k \text{ words from the queue/mst} \quad (49)$$

or

$$1 - \rho(1 - R_L) = \text{probability of not removing } \frac{\alpha}{k} \text{ words/mst} \quad (50)$$

$$= \text{probability of not removing } k \times \frac{\alpha}{k} = \alpha \text{ stages/mst}$$

or

$$1 - k/\alpha T(1 - R_L) = \text{probability of number of stages in the queue } < V \quad (51)$$

where

$$V = \left\{ \begin{array}{ll} \alpha - 1 & \text{if } \alpha \text{ is integer} \\ \lceil \alpha \rceil & \text{if } \alpha \neq \text{integer} \end{array} \right. \quad (52)$$

or

$$1 - k/\alpha T(1 - R_L) = \sum_{0}^{V} q(W) \quad (53)$$

or

$$R_L = 1 - \left[ 1 - \sum_{0}^{V} q(W) \right] \frac{\alpha T}{k}. \quad (54)$$

For a particular $k$ and a value of $\rho$ between 0 and 1 the corresponding value of $\alpha T$ is calculated from (47). Then the corresponding probabilities $p(r)$ are found from (46) and the system of equations (43), (44), (45) is solved for different values of $L$. For each $L$, the value of the probability of an empty queue $Q(0)$ is found, and also the corresponding value of $R_L$ is calculated from (54).

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