The Serial Solution of the Diffusion Equation Using Nonstandard Hybrid Techniques

SHU-KWAN CHAN

Abstract—In the conventional approach to the serial solution of the one-dimension diffusion equation using continuous space-discrete time (CSDT) techniques, there is a large amount of positive feedback in the resultant analog programming. For all practical purposes, the presence of this positive feedback makes the conventional analog method useless.

In this paper, a new direction of attack is proposed utilizing the equivalence between the ordinary differential equation and the corresponding integral equation. The application of this technique to two demonstration problems, though relatively simple ones, is entirely satisfactory. Furthermore, the method demonstrated here can be applied to situations where the coefficient of the diffusion equation is a spatial function. At the end of the report, a brief discussion is also given on possible modifications to the technique used, so as to shorten the solution time.

Index Terms—Continuous space-discrete time solution, heat equation, instability, integral equation, modified Neumann series, partial differential equation, variable coefficients.

I. INTRODUCTION

A CLASSICAL way of solving the diffusion equation by analog computer is the so-called "parallel approach" which is free of stability problems as the boundary conditions in space are always satisfied in the simulation [1], [2]. The method replaces the space derivative $\frac{\partial u}{\partial x}$ by a difference scheme such as the central difference quotient while keeping the time derivative continuous. Intrinsic to such an approach, the amount of equipment required grows proportionally larger as the quantization of the space variable $x$ is made finer to reduce truncation error. Despite the rapidity with which solutions can be generated, it is obviously not a very efficient way from the equipment point of view, not to mention the cost involved. There are the additional difficulties of keeping a large number of analog components operational.

In the era of hybrid computation, an alternate method known as the serial approach or CSDT (continuous space-discrete time) approach has been proposed. As implied by the terminology, this approach uses a difference quotient approximation for $\frac{\partial u}{\partial t}$ while replacing the space variable $x$ by machine time $\tau$, thereby permitting the time-sharing of a basic analog block while marching forward in time in discrete steps. To implement such a technique, it is necessary that some form of analog memory be available. In conjunction with a digital computer, a sample stored-continuous playback analog memory can easily be achieved, and thus the requirement for an analog memory presents no great problem. The major difficulty, however, is that in solving the basic spacewise sweep from boundary to boundary, one usually is faced with an extremely severe problem of stability.

In order that the CSDT approach be truly serial so that a basic analog block can be time-shared, one must use some form of backward difference for the term $\frac{\partial u}{\partial t}$. Denoting $u(x, t_k)$ by $u_k(x)$ where $t_0 = 0$ and $t_k = k\beta^2$, $\beta^2$ being the time increment in the difference quotient scheme, the diffusion equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial u}{\partial t}$$

becomes a system of difference-differential equations:

$$\frac{d^2 u_k}{d\tau^2} = \frac{\alpha^2}{\beta^2} [u_k(\tau) - u_k-1(\tau)] \quad \text{(with } \tau = x)$$

Here we see that a component of the homogeneous solution of (2) is $e^{\beta^2 \tau}$, a growing exponential. To keep the truncation error in the difference quotient scheme small, one must keep $\beta^2$ small. The consequence of this, however, is a faster growing exponential term in the analog solution. This, not the parasitic instability of the analog components due to the high gains, is the problem with which one has to deal.

Some investigators have reported successful results with the CSDT approach [3]–[5] while others [6] have concluded that the CSDT approach is not suitable and results cannot be obtained. The purpose of this paper is to explain the seemingly contradictory conclusions drawn by different investigators, and to offer an alternative method, the integral equation method, that will bypass the stability problem. Perhaps more important than that, the main thesis of the integral equation method is that, in our view, it can handle a linear diffusion equation whose coefficients are functions of the spatial variable as well. For this reason, we shall defer reference to some of the studies whose methods, in the author's view, can only succeed if the equation is one

1 While the final draft of this paper was being prepared, it was brought to the author's attention that Hara and Karplus reported a different way of handling the instability problem by transforming the problem into one of optimal control. For details, see 1968 Fall Joint Computer Conf., AFIPS Proc., vol. 33, pt. 1. Washington, D. C.: Thompson, 1968, pp. 565-574.
with constant coefficients only. We shall return to a fuller discussion on this aspect at the end of the paper.

II. MATHEMATICAL ANALYSIS OF THE CSCT Approach

We consider here the solution of the diffusion equation, (1), with \( \alpha = \text{constant} \), normalized to unity:

\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}; \quad u = u(x, t),
\]

satisfying the following initial and boundary conditions:

\[
\begin{align*}
  u(x, 0) &= f(x) & (0 \leq x \leq l) \\
  u(0, t) &= w_0(t); \quad u(l, t) &= w_l(t)
\end{align*}
\]

with the requirement that \( u(x, t) \) as a function of space \( x \) and time \( t \) is continuous over \( D \), the closure of \( D: (0 \leq x \leq l, 0 \leq t \leq T) \), except at most a finite number of points. Here we do not demand that \( f(0) = w_0(0) \) or \( f(l) = w_l(0) \), so that more general physical situations can be included.

In the CSCT approach, one takes \( (1/\beta^2) \left[ u(x, t_k) - u(x, t_{k-1}) \right] \) as an approximation to \( (\partial u/\partial t) \) at \( t_k \), with \( t_k = k \beta^2 \) and \( \beta^2 \) a small positive number setting the length of the time quantization. Denoting \( u(x, t_k) \) by \( u_k(x) \), (3) is then reduced to a system of difference-differential equations

\[
\frac{d^2 u_k}{dx^2} = \frac{1}{\beta^2} \left[ u_k(x) - u_{k-1}(x) \right]
\]

satisfying the boundary condition

\[
u_k(0) = w_0(k \beta^2); \quad u_k(l) = w_l(k \beta^2)
\]

with \( u_0(x) = f(x) \) given. Equations (5) and (6) thus approximate the solution of (3) and (4) serially, stepping up \( t \) one step at a time.

Presumably, the application of the finite difference technique in time for the study of the diffusion equation was first due to Rothe. Smirnov [7] also gives a detailed account of this method showing the convergence of \( u_k(x) \) to \( u(x, t_k) \) as \( \beta \to 0 \).

Let the error in the functional value of \( u \) and \( \partial u/\partial t \) be, respectively,

\[
\gamma_k(x) = u_k(x, t_k) - u_k(x)
\]

and

\[
\eta_k(x) = \frac{\partial u}{\partial t}
\]

\[
\bigg| \frac{d}{dt} \bigg|_{t=t_k} \left( u(x, t_k) - u(x, t_{k-1}) \right) - \frac{1}{\beta^2}.
\]

By choice, we have \( \gamma_k(x) = 0 \). Furthermore, without loss of generality, as shall be seen later, we can take the boundary condition as identically zero, i.e., \( w_0(l) = w_l(t) = 0 \), and we also assume \( l = 1 \). Now, letting \( t = t_k \) in (3) and subtracting (2) from it, we have

\[
\frac{d^2 \gamma_k(x)}{dx^2} = \frac{1}{\beta^2} \left[ \gamma_k(x) - \gamma_{k-1}(x) \right] + \eta_k(x).
\]

Writing \( 1/\beta^2 = m^2 \), it becomes

\[
\frac{d^2 \gamma_k(x)}{dx^2} - m^2 \gamma_k(x) = -m^2 \gamma_{k-1}(x) + \eta_k(x)
\]

with homogeneous boundary conditions at \( x = 0 \) and \( x = 1 \). We further assume that \( u(x, t) \) has continuous partial derivatives with respect to \( t \) in the domain \( D \) considered, including at \( t = 0 \), except at a finite number of points. We have almost everywhere,

\[
u(x, t_{k+1}) = u(x, t_k) + \frac{\partial u(x, t_k)}{\partial t} \beta^2 + \frac{\beta^4}{2} \frac{\partial^2 u(x, t_k)}{\partial t^2}
\]

where \( \xi_k = t_k + \theta_k \beta^2 \) with \( 0 \leq \theta_k \leq 1 \). Now, let \( 2M \) be the upper bound of \( \partial^2 u(x, \xi_k)/\partial t^2 \) for \( D: [0 \leq x \leq 1; 0 \leq t \leq T] \), a compact region. The existence of \( M \) is assured by our assumption of the existence of an almost everywhere continuous partial derivative of \( u(x, t) \) with respect to \( t \). In other words, we have for all \( k \),

\[
| \eta_k(x) | \leq 2M \beta^2 \quad 0 \leq x \leq 1.
\]

To complete the proof that \( u_k(x) \to u(x, t_k) \) as \( \beta \to 0 \), we must first show the following lemma, namely, the solution of

\[
\frac{d^2 y}{dx^2} - m^2 y = -\pi(x)
\]

with

\[
y(0) = y(1) = 0
\]

also satisfying

\[
| y(x) | \leq \frac{1}{m^2} \max | \pi(x) | \quad 0 \leq x \leq 1.
\]

First, let us assume that \( \pi(x) \geq 0 \) for \( 0 \leq x \leq 1 \). With the help of the Green's function \( G(x, \xi) \) of the differential operator \( [(d^2/dx^2) - m^2] \), satisfying (12a), we have

\[
y(x) = \int_0^1 G(x, \xi) \pi(\xi) d\xi
\]

where

\[
G(x, \xi) = \left\{ \begin{array}{ll}
\frac{1}{m \sinh m} & 0 \leq \xi \leq x \leq 1 \\
\frac{m \sinh m}{m \sinh m} & 0 \leq \xi \leq 1,
\end{array} \right.
\]

if \( G(x, \xi) \geq 0 \) in \( 0 \leq x \leq 1 \) and \( 0 \leq \xi \leq 1 \), and \( \pi(x) \geq 0 \) by assumption. Equation (14) states that \( y(x) \geq 0 \) for \( 0 \leq x \leq 1 \). Thus, \( y(x) \) assumes a positive maximum value, say \( 0 \leq x = x_0 \leq 1 \). At that point, \( y''(x_0) \leq 0 \), (12) now shows

\[
^3 \text{Strictly speaking, } M \text{ is the } L_0^\infty \text{ norm. But in practice, we can take it as the sup norm by simply saying that the continuity is everywhere instead.}
\]
\[ |y''(x_0)| + m^2 y(x_0) = \pi(x_0) \geq m^2 y(x_0), \]
i.e.,
\[ 0 \leq y(x) \leq y(x_0) \leq \frac{1}{m^2} \pi(x_0) \leq \frac{1}{m^2} \max \pi(x_0). \quad (16) \]

In the more general case where \( \pi(x) > 0 \) is no longer assumed, (14), of course, still gives the correct solution, and we conclude that
\[ 0 \leq |y(x)| \leq \int_0^1 G(x, \xi) |\pi(\xi)| d\xi = z(x). \quad (17) \]

But now \( \pi(x) \) is the solution of \( d^2\pi/dx^2 - m^2\pi = -|\pi(x)| \)
satisfying (12a), and by what was shown above, we have
\[ 0 \leq \pi(x) \leq \frac{1}{m^2} \max |\pi(x)|. \]

Together with (17), we have
\[ 0 \leq |y(x)| \leq \frac{1}{m^2} \max |\pi(x)|, \]
showing (13).

This lemma as expressed in (13) is indeed an important one, though possibly rather trivial. Not only is it the basis upon which the validity of the CSDT approach is justified, it also points out, as we shall see, the direction that we should follow in the actual computation so as to bypass the practical difficulty encountered in Ellis' thesis [6].

Now applying the above lemma to (9a), with boundary condition in (12a) for \( k=1, 2, 3, \ldots \), we can conclude, by taking \( |\pi_k(x)| \) as \( |m^2\gamma_{k-1}(x) - \eta_k(x)| \), that
\[ |\gamma_k(x)| \leq \max |\gamma_{k-1}(x)| + M\beta^k \quad (18) \]
in which we have made use of (11), and of \( m^2 = 1/\beta^2 \). In other words, the errors in the finite difference approximation at each time step remains bounded. Denoting \( \delta_k = \max |\gamma_k(x)| \), \( 0 \leq x \leq 1 \), (18) can be written as
\[ 0 \leq \delta_k \leq \delta_{k-1} + M\beta^k. \quad (19) \]

Upon summing (19) on \( k \) from \( k=1 \) to \( k=j \leq N \) where \( T = N\beta^2 \), with \( \delta_0 = 0 \), we obtain
\[ |u(x, t) - u_j(x)| \leq \delta_j \leq M j \beta^k = M\beta^j, \quad (20) \]
showing that the error is bounded by a quantity that increases linearly with time \( t \), keeping \( \beta^2 \) fixed. This is certainly the best estimate one can hope for with the first-order difference quotient approximation. Indeed for any \( t \leq T \leq \infty \) this bound is not to exceed \( M\beta^k T \), showing that \( u_j(x) \to u(x, t) \) as \( \beta^2 \to 0 \).

Finally, we remark that when the boundary condition is not homogeneous, such as that in (4), a simple transformation
\[ v(x, t) = u(x, t) - \frac{w(t) - w_0(t)}{t} x \quad (21) \]
will reduce the equation for \( u \) to the same equation for \( v \), with \( v \) satisfying a homogeneous boundary condition. The above proof can therefore be repeated without change.

III. The Hybrid Implementation

As was discussed at the end of Section II, any diffusion equation with inhomogeneous boundary conditions can be reduced to an equivalent problem with homogeneous boundary conditions. We shall therefore take the homogeneous problem as a basic one. But before a hybrid implementation can be made, the size of \( \beta^2 \) must first be determined. To a certain extent this is somewhat arbitrary, and it is probably because of this reason that few papers on the CSDT approach of (1) give it much thought. Equation (20) states, of course, that the smaller \( \beta^2 \) is, the more accurate is the result. Yet, a smaller \( \beta^2 \) implicitly implies higher gains in the analog block, and gain is inherently limited by the bandwidth of the component. On the other hand, the use of a large \( \beta^2 \), in addition to causing a larger error due to the finite difference scheme, may cause loss of too much of the transient information, thereby making the transient simulation next to useless. It seems, therefore, that some criterion has to be set up for the choice of \( \beta^2 \) if we are to be assured that the transient study by the CSDT approach will be meaningful. Unfortunately, the choice of \( \beta^2 \) cannot realistically be predetermined for a fixed accuracy, but is dependent to a large extent on the characteristics of each individual problem.

A. A Criterion for the Choice of \( \beta^2 \)

Using the method of separation of variables on (3) with homogeneous boundary conditions, one can easily express the solution \( u(x, t) \) in terms of a Fourier series:
\[ u(x, t) = \sum_{k=1}^{\infty} A_k e^{-n^2 \omega^2 t} \sin n\pi x \quad (22) \]
where the constants \( A_k \) are to be determined by the initial condition \( f(x) \) as follows:
\[ A_k = 2 \int_0^1 f(x) \sin n\pi x \, dx. \quad (23) \]

Equation (22) expresses the fact that the amplitude of each of the normal modes (the \( n\pi x \) terms) of the solution decreases exponentially with time \( t \). In other words, the high-frequency effect of the initial condition \( f(x) \) will decay faster than the low-frequency effects. The entire transient effect will have died out when the lowest frequency term in (22) has decayed away. To observe the transient effect, therefore, the size of the quantization in time must not be too great compared to the largest time constant: \( (1/\pi)^2 \). The actual time step used, of course, should preferably be much shorter than that. In general, it is obvious that if the transient behavior of the \( n \)th normal mode, the \( n\pi x \) term, is to be observed, the size of \( \beta^2 \) should then be not much
larger than \((1/n\pi)^2\), and preferably smaller. It is clear at this point that the size of \(\beta^2\) is dependent upon the highest normal mode that one wants to simulate with some fidelity. Such a decision, in turn, depends on the frequency content of the initial condition which, of course, will vary from problem to problem. For example, if \(f(x) = \sin nx\), then \(\beta^2\) can be chosen to be \(1/\pi^2 \approx 0.1\). On the other hand, if \(f(x) = \sin nx\), \(\beta^2\) will have to be shorter, say \(1/4\pi^2 = 0.025\), if we are to observe the transient effect meaningfully. In a more general situation, the choice of \(\beta^2\) will then have to be chosen in accordance with the frequency distribution of \(f(x)\), and the investigator's estimate of an acceptable error in the higher frequency transient behavior.

B. The Analog Memory

Regardless of which of the various methods is to be used, the CSDT approach inherently requires some form of an analog memory. It is therefore appropriate to describe briefly here the "sample stored-continuous playback" analog memory used in our study.

Briefly, an analog-to-digital converter is used to encode samples of an analog signal at a predetermined rate. These sampled points are stored sequentially in the digital core memory. To play the signal back, the sequence of samples are fed into the analog domain where linear interpolation is performed between points by an analog integrator. The gain of the integrator is dependent upon the rate at which the sampled data are put out. This rate may be the same as or different from the read-in rate mentioned above. This feature is very important in our study.

As set up, the memory has a bandwidth of about 1 kc at 1 percent accuracy and is compatible with the computing range of the GPS T200, the analog computer used in this study. The major limitation of speed for the memory is due to the settling time for the sample-and-hold gates (a 15-\(\mu\)s pulse is required to activate them), the cycle time of the PDP-1 digital computer, the inherent drift and gain errors of the T200 components, and the rise time of the D/A converter, 30 \(\mu\)S.

The idea of the system is very simple indeed. Suppose the signal is to be played back at a rate predetermined by the operator; the time increment \(\Delta\) between any two successive samples is therefore determined. Note that this increment is constant throughout, although it can be changed from signal to signal. Let the analog signal be denoted by \(f(i)\). After sampling, encoding, and storing, we have, in the digital core memory, a table of sampled values for \(f\), say \(f_i = f(i\Delta)\) with \(i = 0, 1, 2, \ldots, N, N+1\) being the number of samples. Continuous playback is achieved by a linear analog interpolation between each pair of successive samples. In the simplest scheme, the digital computer would compute the sample difference \((f_{i+1} - f_i)\) which is then converted to an analog voltage by the D/A converter and stored at the input to an analog integrator having a gain of \(1/\Delta\). The output of the integrator after a time interval \(\Delta\) will change by an amount \((f_{i+1} - f_i)\). Assuming that the integrator output was at the value of \(f\) at the start of the integration interval, it ends up at \(f_{i+1}\) at the termination of the interval. A linear interpolation is thus accomplished. An open-loop scheme such as this, however, results in a cumulative buildup of errors due to integrator drift, offsets, etc. For this reason the project adopted a closed-loop scheme using the arrangement in Fig. 1. The sample-hold gate \(T_1\) stores the initial condition \(f_0\). The digital program then loads the accumulator with successive values of \(f_i\), storing the corresponding analog voltages on \(T_2\) and simultaneously strobing \(T_2\). The net input to the integrator is ideally the difference voltage \((f_{i+1} - f_i)\). Errors tend to be self-correcting. Suppose that at \(t = t_i = i\Delta\) the integrator output has a voltage equal to \(f_i\). At the same time, the accumulator has the value \(-f_{i+1}\), and \(T_1\) and \(T_2\) are being strobed simultaneously. The input to the integrator therefore becomes \(-f_{i+1} - f_i\). At the end of a \(\Delta\) interval, i.e., at \(t = t_{i+1}\), the output of the integrator is ideally

\[
\frac{f_i + (f_{i+1} - f_i)}{\Delta} t_{i+1} = f_{i+1},
\]

achieving a linear interpolation between the points \(f_i\) and \(f_{i+1}\).

A detailed description of the memory, the sample-hold gates, and the ESL hybrid system can be found in Bjørge [8] and Connelly and Federoff [9], respectively.

C. The Conventional CSDT Approach

The heart of the CSDT approach lies in the system of difference-differential equations, (5), with the boundary condition of (6). As is common in the solution of boundary value problems by an analog computer, it amounts to finding the right initial conditions at the near end so that the boundary condition at the far end is satisfied. The process usually involves iteration. In principle, then, the main problem is to find a criterion that uses the desired value of the function itself or its derivatives at the far end to converge iteratively on the corresponding condition at the near end. Schematically, therefore, the CSDT setup for the diffusion equation is as shown in Fig. 2. The configuration in Fig. 2, however, involves a positive feedback loop of two integrators with an overall loop gain of \(1/\beta^2\), a very large number if \(\beta^2\) is kept small. Indeed, as Ellis [6] pointed
out, with a time step of $1/100$, such a system will have a component $e^{10r}$ ($r$ is computer time equivalent to $x$) in the solution. Any imperfection of magnitude $\delta$ in the initial condition will result in an additional term $\delta e^{10r}$ added to the correct solution. At the far end, such a term will build up to a magnitude of $\delta e^{10}$. This means, even for a 100-volt machine, that an initial fluctuation as small as 10 mV (which is certainly not uncommon with an analog computer) will cause a deviation at the far end of about 200 volts! This, indeed, was what Ellis observed, namely, the analog computer saturated before it even had a chance to reach the far end!

On the other hand, in a Boeing experiment [4], $1/\beta^2$ was chosen to be only 0.3. This value would cause an error term of about $\delta e^{0.3\beta}$, i.e., a fluctuation of about 0.78 volts at the far end. Now, if the constant $\alpha$ in (1) is unity, the value 0.3 implies a gross time step of 3.33 in normalized time. Certainly, most high-frequency transient information would be lost using such an increment in time.

In fact, in our study even when using $\beta^2 = 0.04$, a time step that is about half of the largest time constant of (22), it was found that the fluctuation at the far end was about 2 volts. This corresponds to an initial condition fluctuation of about 14 mV, which is consistent with the noise level of the 10-volt GPST 200 computer. Shown in Fig. 3 is a picture of such fluctuations taken directly from the scope. Note that the initial slope was held fixed and the far end fluctuation represents the effect of the initial noise of the computer, not a deliberate change in the initial condition.

Obviously, unless one is willing to increase the time step $\beta^2$ at the expense of losing some of the high-frequency results in the simulation, one must conclude that the conventional method will not serve the purpose. Such a large far-end fluctuation renders the iteration logic, no matter how sophisticated, essentially useless. Since the purpose of this study is to bypass this kind of difficulty, we shall not dwell on the conventional method any longer but turn our attention to the following variations instead.

D. The Green's Function Method

By the proof given in Section II, it is logically concluded that the failure of the conventional method is basically due to the presence of internal noise in the analog system, and as such is rightfully an engineering problem. Unless a noise-free system can be designed, we must look for a different mathematical approach if the CSDT method is to be implemented. Indeed, (13) shows that the value of $u_4(x)$ should be numerically smaller than the initial condition $f(x)$ and the presence of the $\delta e^{10r}$ term in the conventional method is only the result of our studying a boundary value problem as an iterative initial value problem. In other words, if we can solve (5) not as an initial value problem, but as part of a system given by (5) and (12a), incorporating both equations together at the same time, then we should have no stability problem at all. One way to accomplish this is by the Green's function given in (15).

Physically, the Green's function $G(x, \xi)$ is somewhat similar to the so-called impulse response in elementary circuit theory. As a function of $x$ with $\xi$ being looked upon as a parameter, the Green's function satisfies the differential equation, (12), except at $x = \xi$, with the right-hand side of (12) replaced by an impulse function, and the boundary condition, (12a), as well. The physical significance of $G(x, \xi)$ is the response of the system to an impulse excitation at $x = \xi$. Thus, (14) is nothing more than the result of superposition. The advantage here is that the boundary condition is automatically satisfied, i.e., it is already part of the whole system, and therefore, the effect of internal noise that causes a $\delta e^{10r}$ term is greatly suppressed. On the negative side, the use of such a method requires that for every time step $\beta$, a number of integrations of (14) have to be carried out, depending upon the number of samples one wishes to take for the space variable. In other words, each integration of (14) for $\xi$ from 0 to 1 will

8 The Boeing report did not explicitly state the value of $L$, the rod length, but from the way the data were presented, it was conjectured that $L$ was normalized to unity.

4 Calzi-Bini and Turrini [17] reported a different way of solving the two-point boundary value problem that does not require an iteration at the near end; but that too will not work under such circumstances. See detailed discussion in the concluding chapter.

9 Vichnevetsky and Witsenhausen [18]-[20] reported ways that will bypass the stability problem without a significant increase in solution time. However, their methods can only work under rather severe restriction on the form of the differential operator. See detailed discussion in the concluding chapter.
yield only one point of \( u_k(x) \). This certainly will lead to longer solution times. As an estimate, the solution time probably would be about five times as long as the conventional CSDT method when the latter works. We assume here that 25 sample points are used for the Green's function method and that it takes five iterations for the conventional method to converge. The solution for \( u_k(x) \) is now given by
\[
    u_k(x) = \frac{1}{\beta^2} \int_0^1 G(x, \xi) u_{k-1}(\xi) d\xi. \quad (24)
\]
For a specific example, we have taken the initial condition
\[
    f(x) = \sin \pi x. \quad (24a)
\]
Employing 25 sample points for the space variable, the configuration shown in Fig. 4 easily yielded solutions as we stepped up along the \( t \) axis on the \( t-x \) plane. An oscilloscope photograph and a comparison with the analytic solution are shown in Figs. 5 and 6, respectively. We must note that although Fig. 6 also shows the exact solution to the original partial differential equation, the real comparison of the Green's function method should be made between the hybrid solution and the analytic solution, i.e., the solution to the difference-differential systems, (5). The discrepancy between the exact solution and the analytic solution reflects the effect of the finite difference technique, not the effect of the Green's function method. Analytically, with (24a) as the initial condition, (5) gives
\[
    u_k(x) = \left( \frac{25}{25 + \pi^2} \right)^k \sin \pi x \quad (25)
\]
as the approximation to the exact solution \( u(x, t_k) = e^{-\pi^2 t_k} \sin \pi x \). The close agreement between the computer solution and the analytic solution speaks for itself concerning the superiority of this method.

Next, a problem with nonhomogeneous boundary conditions was studied with \( w_0(t) = 0 \) and \( w_1(t) = 1 \) having the same initial condition as in (24a). Using the transformation given in (21), we again obtain a homogeneous problem with
\[
    \bar{j}(x) = \sin \pi x - x \quad (26)
\]
as the initial condition.

Note here that \( \lim_{x\to\infty} u(1, t) \) is not the same as \( \lim_{x\to\infty} u(x, 0) \). In this case, (14) still yields the correct solution. Fig. 7(a) shows the computer solution of \( v(x, t_i) = v_i(x) \) as \( j \) is being stepped forward, while Fig. 7(b) shows the corresponding \( u_i(x) \). The first traces in both (a) and (b) correspond to the case of \( t_i = 0 \), and a swift change of the boundary condition at \( x = 1 \) is noticed in \( t_i = 0.04 \). In Fig. 8, the computer solution, the analytic CSDT solution, and the exact diffusion equation solution are compared. It is seen that the computer solution is in good agreement with the analytic solution, although it differs rather significantly from the exact solution for small \( t_i \), but agrees quite well for large \( t \) (such as \( t = 0.28 \), shown also in the figure), showing the loss of information about the transient behavior even with a time step of 0.04.

Mathematically, the exact solution can be found in a Fourier series form, using (22) and (23):
Fig. 7. Transient solution for the Green's function method. Vertical: 10 volts fs; horizontal: 2.5 ms fs. Different traces represent different time steps in the solution. (a) $v(x, 0) = \sin \pi x - x$; $v(0, t) = v(1, t) = 0$. (b) $u(x, 0) = \sin \pi x$; $u(0, t) = 0$, $u(1, t) = 1$.

Fig. 8. Comparison of the transient solutions.
\[ u(x, t) = x - \left(1 - \frac{2}{\pi}\right) e^{-\frac{2}{\pi} t} \sin \frac{\pi x}{\lambda} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n \pi x}{n} \quad (27) \]

The coefficients of the various normal modes form a series of \( \sum (-1)^n / n \), whose convergence is only conditional. At times very near zero, the attenuation of the high-frequency components for the exact solution is therefore not very significant. But the effect of the quantization in time is somewhat equivalent to a built-in filter, with a fixed bandwidth inversely proportional to \( \beta^2 \). Hence for \( t \to 0 \), the truncation due to quantization will be significant. As \( t \) becomes appreciably greater than 0, the effects of \( e^{-\frac{2}{\pi} t} \) are significant in damping out the higher modes and therefore the frequency content of the solution will eventually be within the bandwidth of the built-in filter. However, the CSDT solution still gives a good qualitative description of the transient effect for \( t \to 0 \). Calculating from the exact solution for \( t \) very small, (27) indeed shows a behavior similar to that of \( u_k(x) \), i.e., that exhibits a dip below \( u(x) = x \). The large discrepancy between \( u_k(x) \) and the exact solution \( u(x, t) \) is, of course, the result of the rather large \( \beta^2 \) used compared to the frequency content of \( f(x) \) given by (26). As can be noted by \( u_k(x) \), this loss of accuracy in the transient behavior becomes less pronounced as the steady state is approached. Here we see that even for such a simple problem, a time step increment of 0.04 is still, strictly speaking, not too accurate for the study of the transient behavior. A much larger \( \beta^2 \), such as 0.1 for example, is almost certain to make the transient observation meaningless. This is the price one pays for the difference quotient approximation. In this respect, however, the CSDT approach surely has an accuracy that is comparable to the parallel approach as long as the size of quantization is comparable. The saving of hardware in the serial approach is very substantial.

E. The Integral Equation Method

Despite the success of the Green's function method, we shall not as yet draw a positively favorable conclusion. Indeed, as will be shown later, the Green's function has a very severe limitation which makes it, in the most general situation, almost as impotent as the conventional method. Therefore, with possible greater generality in mind, we continued the study of the CSDT approach in the search for yet another mathematical method of attack. With the integral equation method, the basic approach is still the same as with the Green's function. The only variation is a different practical way of solving the same set of difference-differential equations.

Considering homogeneous boundary conditions again, we have to solve

\[
\frac{d^2 u_k(x)}{dx^2} = \frac{1}{\beta^2} \left[ u_k(x) - u_{k-1}(x) \right] \quad 0 \leq x \leq 1. \quad (28)
\]

Taking \( d^2/dx^2 \) as our differential operator, (instead of \( d^2/dx^2 - 1/\beta^2 \) as in the Green's function method), we seek a Green's function satisfying the homogeneous boundary condition. Trivially, it is found to be

\[
K(x, \xi) = \begin{cases} 
(x(1 - \xi)) & 0 \leq x \leq \xi \leq 1 \\
(\xi(1 - x)) & 0 \leq \xi \leq x \leq 1. \end{cases} \quad (29)
\]

The solution to (28) is now

\[
u_k(x) = -\frac{1}{\beta^2} \int_0^1 K(x, \xi) [u_k(\xi) - u_{k-1}(\xi)] d\xi, \quad (30)
\]

or

\[
u_k(x) = \int_0^1 K(x, \xi) u_k(\xi) d\xi - \frac{1}{\beta^2} \int_0^1 K(x, \xi) u_{k-1}(\xi) d\xi. \quad (31)
\]

The major difference between (31) and (24) is that the unknown function \( u_k(x) \) appears not only on the left-hand side but also under the integral sign as \( u_k(\xi) \). It is truly an integral equation in the usual sense; indeed, it is known as the Fredholm equation of the second kind.

While it is commonly said that the conventional method solves a boundary value problem iteratively for the boundary condition, satisfying the differential equation at all times, the corresponding integral equation, on the other hand, generates functions that always satisfy the boundary condition but iterates to satisfy the differential equation as well. One must bear in mind the following difference. As long as the initial value problem is solvable by the conventional method (in the stability sense discussed previously), it will always be possible to get the final solution. For the integral equation, however, iterations may not always lead to the correct solution, i.e., they may not converge at all.

A practical way of solving a Fredholm integral equation of the second kind is the so-called Neumann series. The convergence of the Neumann series, however, is not assured but depends on the size of the parameter, commonly referred to as \( \lambda \) in the study of integral equations. It is equivalent to our \( 1/\beta^2 \) here. Indeed, it has been established that the Neumann series will converge only if \( |\lambda| \) is smaller than a certain positive number, the absolute magnitude of the eigenvalue \( \lambda_1 \) closest to the origin in the \( \lambda \)-plane [7]. In our illustrative example, \( \lambda_1 = \pi^2 \) and \( |\lambda| = 1/\beta^2 = 25 \); therefore, we know for a fact that the Neumann series will not converge if we were to apply it in our case. This indeed is the major problem we must solve; otherwise, the size of \( 1/\beta^2 \) will have to be reduced and we are then back to a situation similar to that faced in the conventional method. Theo-
retically, there exists a resolvent kernel which solves (31) for any \( \lambda \) as long as it is not an eigenvalue of the kernel. However, it is not practicable to find the Fredholm denominator for the resolvent kernel by actual computation. We shall not discuss the concept of the resolvent kernel any more as it is not used explicitly.

The theory of the Fredholm integral equation is well known [7], and we therefore will not discuss it at any length, but will quote the established results when necessary. Even the implementation of the Neumann series has been well studied [10], [11], and we will simply use these results without discussion. However, we will give a brief description of the reasoning that leads to the actual implementation used in this study to avoid the divergence of the Neumann series due to the fact that the value of \( |\lambda| (= 1/\beta^2) \) used here is larger than \( \lambda_1 = \pi^2 \) in magnitude. A more detailed discussion can be found in [12].

Borrowing the symbolism of linear algebra, writing, for example, \( \mathcal{K}f \) for \( \int_0^1 \mathcal{K}(x, \xi)f(\xi)d\xi \) etc., the Neumann series can be written as

\[
\Gamma = I + \lambda \mathcal{K} + \lambda^2 \mathcal{K}^2 + \lambda^3 \mathcal{K}^3 + \cdots \quad (32)
\]

where \( I \) is the identity operator and \( \mathcal{K}^n \) is \( \mathcal{K} \) applied \( n \) times. With \( \Gamma \) as given above, the solution to (31) is then

\[
u_k = g + \lambda \mathcal{K}g + \lambda^2 \mathcal{K}^2 g + \cdots \quad (33)
\]

where we take

\[g(x) = \frac{1}{\beta^2} \int_0^1 \mathcal{K}(x, \xi)u_{k-1}(\xi)d\xi\quad (34a)\]

\[
\lambda = -\frac{1}{\beta^2} \quad (34b)
\]

and, for example,

\[\mathcal{K}^2 g = \int_0^1 d\eta \mathcal{K}(x, \eta) \int_0^1 \mathcal{K}(\eta, \xi) g(\xi)d\xi. \quad (34c)\]

With \( \beta^2 = 0.04 \), the series of (32) will not converge. However, it is well known [7] that as a function of \( \lambda \) and being regarded as a complex variable, (32) has a pole on the complex \( \lambda \) plane located at \( \lambda = \lambda_1 = \pi^2 \). This is the pole nearest to the origin and is a simple pole in our case. In fact, we also know [7] that, as a function of \( \lambda \), \( \Gamma \) is meromorphic, having the eigenvalues of \( \mathcal{K}(x, \xi) \) as the only poles. This implies then that the series

\[
(\lambda - \lambda_2)^{-1} = (\lambda - \lambda_1)(I + \lambda \mathcal{K} + \lambda^2 \mathcal{K}^2 + \cdots )
\]

\[= -\lambda_1 I + \lambda(I - \lambda_1 \mathcal{K}) + \lambda^2(\mathcal{K} - \lambda_1 \mathcal{K}^2) + \cdots \quad (35)\]

will be convergent for any \( \lambda \) such that \( |\lambda| < |\lambda_2| = \lambda_2 = 4\pi^2 \). Note that rearranging the terms of the right-hand side of (35) is not allowed. In other words, (33)

\[\text{can now be written as the following infinite series that is convergent:}\]

\[u_k = \frac{1}{\lambda - \lambda_1} \left\{ -\lambda_1 g + \lambda \left[ (g - \lambda_1 \mathcal{K}g) + \lambda \mathcal{K}(g - \lambda_1 \mathcal{K}g) + \cdots \right] \right\}. \quad (36)\]

Using (34a) for \( g \), we finally have

\[u_k = \frac{1}{\lambda - \lambda_1} \left[ (u_k - 1 - \lambda \mathcal{K}u_{k-1} - \lambda^2 \mathcal{K}^2 u_{k-1}) + \cdots \right] \]

\[= \frac{-\lambda}{\lambda - \lambda_1} \left\{ -u_{k-1} + [u_{k-1} - \lambda \mathcal{K}u_{k-1}] + \cdots \right\}. \quad (37)\]

Now comparing the bracketed portion of (36) with (33), we see that they are indeed very similar in form, the only difference being that \( g \) of (33) is replaced by \( g + \lambda_1 \mathcal{K}g \). In other words, except for an initial transformation for \( g - \lambda_1 \mathcal{K}g \), the standard implementation for the Neumann series can now be applied without modification with, of course, the advantage that we now have a convergent series instead. Equation (37) will in fact be the basis of our computation in this study. The block diagram of the integral equation method is shown in Fig. 9. When a direct application of the Neumann series is used, \( \beta^2 \) is limited not to be smaller than \( 1/2\pi^2 = 0.1 \), and as was explained previously, this is in general not sufficiently small. With this present modification, \( \beta^2 \) can now be taken as small as \( 1/4\pi^2 \approx 0.025 \), though a value too close to 0.025 is not recommended. Nonetheless, it is adequate for \( \beta^2 = 0.04 \), as used previously. A further widening of the range of \( 1/\beta^2 \) is still possible by employing more premultiplied factors. As a matter of fact, for this particular equation, even the computation using the Fredholm series is not too difficult, but we shall not discuss it any further [12].

To show that the modified Neumann series indeed converges, we repeated the experiment performed by the Green's function method, i.e., a homogeneous boundary value problem with initial condition \( f(x) = \sin \pi x \). The result is also shown in Fig. 6, and is seen to have accu-
Fig. 10. Transient solution for the integral equation. Vertical: 10 volts fs; horizontal: 2.5 ms fs. (a) \( u(x, 0) = \sin x; u(0, t) = u(1, t) = 0 \). Traces represent \( u(x, 0), u(x, \Delta t), u(x, 2\Delta t), \ldots \) from top to bottom. \( u(x, n\Delta t) \) for \( n > 2 \) are barely visible. (b) Kernel \( K(x, \xi) \). Each of the triangular-like curves represents \( K(x\xi) \) as \( \xi \) is being stepped up from left to right. Horizontal axis is \( \xi \), which in this case is computer time.

Fig. 11. The sequence of convergence for the integral equation. Vertical: 10 volts fs; horizontal: 2.5 ms fs. Traces show the converging sequence; starting from the top one they show the 5th iteration. About five iterations are needed for convergence.

accuracy comparable to the Green’s function method. The oscilloscope photograph is shown in Fig. 10 with (a) indicating \( u_0(x) \) and (b) showing the kernel \( K(x, \xi) \).

Finally, with the same initial condition but a nonhomogeneous boundary condition, zero at \( x = 0 \) and unity at \( x = 1 \), the result plotted in Fig. 8 was obtained, with Fig. 11 showing the convergence sequence toward \( u_1(x) \). Note that the convergence shown is for the computation of the bracketed quantity in (37) with \( u_0(x) \) (appearing as the second curve, counting downward) being the input for the computation of \( u_0(x) \). Comparing the different outputs shown in Fig. 8, we see once again that the integral equation method indeed has an accuracy comparable to that of the Green’s function method. The close agreement between the computer result and the analytic solution speaks for itself.

IV. SOME CONCLUDING REMARKS

In this study, we have discussed three different methods to solve the diffusion equation by the CSDT technique. The first, the conventional method, is by far the most commonly discussed. Undoubtedly, when a relatively large time step \( \beta^2 \) is allowed, this method definitely offers a faster solution time than the Green’s function and integral equation methods. In terms of transient study, however, it is almost certainly next to useless. One may conclude, at first glance, that for circumstances such that the constant \( \alpha^2 \) of (1) is very small, a relatively small \( \beta^2 \) may still yield a manageable loop gain for the conventional method. This reasoning is not valid, however. For \( \alpha^2 \) less than unity, the general term in (22) becomes \( A_0 e^{-\alpha^2 t/(4\alpha^2)} \sin \pi x \), and the time constant for the \( n \)th normal mode is then \( \alpha e^{n^2 \pi^2} \), proportionally shorter. That is, what was an adequate \( \beta^2 \) for cases where \( \alpha^2 = 1 \) is now not small enough. Therefore, if one applies the conventional method, he is definitely going to observe very little transient behavior. In fact, what is worse is that the transient he observes may not bear any relevance to the true transient. Despite its superior speed (around 5 ms per time step with the GPS T200 computer), one must reluctantly conclude that the severe limitation on the size of \( \beta^2 \) makes its application impracticable.

For the Green’s function method, our study shows that with the MIT ESL hybrid system, it takes 2.5 ms for each \( \xi \) integration from \( \xi = 0 \) to \( \xi = 1 \), and hence, 62.5 ms per each time step solution. The digital bookkeeping time, being less than 200 ms, is ignored. The speed limitation here is primarily that of the PDP-1. With a faster digital computer, the speed could be reduced by a factor of two, at which point the limitation will be due to the gain of the integrators. This, however, would be just about the top speed of the system used. In this regard, it is not germane to make a comparison between this and the conventional method, since the latter is not usable at all for the size of \( \beta^2 \) being considered.

The solution time for the integral equation method is even longer, for it takes 2.5 ms for each \( \xi \) integration, 62.5 ms for each iteration, and several iterations for one time step solution. Added to this is the bookkeeping time which is no longer negligible.

At this point, therefore, it seems senseless to use the integral equation method at all, for the Green’s function method yields comparable accuracy at a faster rate. We shall defer further discussion on this point until later when we discuss generalizations of the CSDT approach.

Up to this point, all of our discussions and experimentations only dealt with the linear diffusion equation with constant coefficients for which analytic solution either in closed form [7] or in the form of an infinite series is available. (See (22) and (23).) This means that if one is interested in obtaining the solution to such a problem, one could perhaps compute, say, the series solution by a digital computer with better accuracy and at a higher speed. It is therefore a legitimate question to raise, then, what is the purpose of such a study after all. Our answer to this question is that the purpose of this paper is to propose a method which will be useful in a situation where no analytic solution is known. For this reason, therefore, aside from merely pointing out the solution time required for each of the three methods for the examples studied, we do not make a comparison to
the solution time required by computing, for example, the series solution by a digital computer. The latter method will most likely be much faster.

A perceptive reader will probably realize by now that the main theme of this paper lies with the integral equation method despite its very slow speed. Indeed, the so-called conventional method was included in our study only to illustrate its practical uselessness in the handling of the serial solution. By the conventional method we mean the method that sets up the analog block of the second-order ordinary differential equation in a conventional manner, whether or not the far-end boundary condition is met by iterating the near-end boundary condition as is usually done, or by the method of superposition, thereby requiring only two machine runs [17]. The main problem here is that the basic solution has a form that is exponentially growing with a very rapid rate inversely proportional to the square root of the size of the time quantization. Therefore, internal noise of the analog computer alone is sufficient to excite the simulation so that the error is no longer acceptable. In the worst case, of course, the computer may even saturate long before the far end is reached, even if all the initial conditions are set zero. Fig. 3 is a good illustration of what we mean. The spreading out of the solution shown there is due to the noise excitation in repeated runs while the initial condition was supposed to be fixed. Therefore, regardless of how effective the initial condition search routine can be for the matching of the far-end boundary condition, such a method is not going to yield an acceptable result, except at the expense of losing fidelity of the transient observation by using a much larger time step.

The inadequacy of the conventional method therefore leads us to search for a different method of attack. The Green's function method was therefore developed in this paper, not for its own sake, but as a motivation for the integral equation method. Witsenhausen [18] proposed the use of the Green's function as a way of bypassing the instability mentioned above. The method he illustrated was further developed by Vichnevetsky [19], [20]. The solution time of their Green's function method is much faster than that used in this study. Certainly, if the Green's function method is the preferred mode in this study, it would be well-advised to follow their line of attack rather than the cumbersome way of integration used here. In fact, Witsenhausen and Vichnevetsky made a very clever use of the Green's function and developed it into what they termed as the two different directions of causality [18], which amounts to doing two integrations at high speed, one along the forward direction from the near end to the far end, and the other along the backward direction. Therefore, solutions will be obtained in a matter of a couple of milliseconds, say, instead of the 62.5 ms as in our study (per time step). The success of the Green's function method, however, whether in our form or in the original form as developed by Witsenhausen, depends on our ability to obtain, in fact, the two homogeneous solutions to the ordinary differential equation. However, once that is assumed, then the concept of "variation of parameters" discussed in almost every elementary text on differential equations will suffice, and the more sophisticated concept of the Green's function will not be needed at all. Therefore, call it Green's function or variation of parameters, they yield the same solution at the same rate in the same manner. However, it is of course not true that one can always write down the explicit form of all of the homogeneous solutions to any given linear differential equation. For example, think about those that have variable coefficients. One can, of course, argue that the separation of two directions of integration as developed by them will avoid the stability problem, while at the same time one does not have to be able to write down the homogeneous solutions explicitly. However, a review of their papers shows that the success of such a method depends on two assumptions. First, the differential operator in question is factorizable into two factors, each one of them is stable in each of the two directions. By factorizable we mean here not the existence of such factors but our ability to write these factors down explicitly. Second, these two factors commute. Needless to say, these assumptions impose a very strong restriction on the kind of differential operators that we can handle, and cannot be done in general unless the differential operator is one with constant coefficients in which case, then, the analytic solution can be obtained.

For the reasons discussed above, it is not the purpose of this paper to emphasize the role of the first two methods illustrated here. Also for these reasons, we preferred to do the Green's function method in a way that will motivate the integral equation method. This is why we did not discuss until now the concept of the two directions of integration and the like. With these remarks, we shall then devote the rest of this paper to discussing the concept of the integral equation method, having in mind that the method is developed primarily for a differential operator with variable coefficients. While the demonstration problem is one with constant coefficients, the method used, however, was developed in a manner that does not depend on the fact that the coefficients of the equation are constants. Therefore, it is clear that such a method will be useful as well for an equation with variable coefficients.

As was pointed out in the previous section, the kernel of the integral equation \( K(x, \xi) \) is nothing more than a Green's function. The only difference between \( K(x, \xi) \) and \( G(x, \xi) \) is that while the latter is the Green's function of the operator \( d^2/dx^2 - 1/\beta^2 \), the former is the

---

\( ^6 \) Though Vichnevetsky [20] remarked that there are other means of computing the homogeneous solutions even if the factors do not commute; however, since he limited his partial differential equation to have coefficients that are not spatial-dependent, it is not clear that the situation will arise where the factors do not commute. The resultant ordinary differential equations belong to the type with constant coefficients.
Green’s function of only the leading term $d^2u/dx^2$. Therefore, we may not be able to write down $G(x, t)$ in general, we can always write $K(x, t)$, as in (29), which is of course all that is needed for the integral equation. In the example illustrated, $K(x, t)$ itself is the kernel of the integral equation; but in a more general case as we will shortly see, $K(x, t)$ is only a factor of the appropriate kernel. To solve the integral equation, one almost always applies the Neumann series, which as we know is not always convergent. Therefore, we demonstrated a method that will enlarge the radius of convergence of the Neumann series and obtain (36) or (37), which we shall hereafter refer to as the modified Neumann series. Within this framework, we can then apply the method to the linear diffusion equation with coefficients that are dependent on the spatial variable.

In the following, we shall first of all discuss several rather straightforward extensions, assuming for the time being that the coefficients are constants. After that, we shall then discuss how the integral equation method can be applied to a more general situation.

An immediate generalization is for the case when the boundary conditions are time-dependent. Though we have studied only non-time-dependent boundary conditions, (21) actually is for time-dependent boundary conditions. In this case, we need only take

$$v_k(x) = u_k(x) - \left[ w_{0k} \frac{w_{1k} - w_{0k}}{t} x \right]$$

(38)

where

$$w_{0k} = w_0(t_k) \text{ and } w_{1k} = w_1(t_k).$$

Next, the CSSTT approach can immediately be generalized to a diffusion equation with a source or a sink represented by

$$\frac{\partial^2 u}{\partial x^2} = \alpha^2 \frac{\partial u}{\partial t} + h(x, t) \quad \alpha^2 = \text{constant.}$$

(39)

Here, the corresponding difference-differential equation system is

$$\frac{d^2 u_k}{dx^2} = \frac{\alpha^2}{\beta^2} [u_k(x) - u_{k-1}(x)] + h_k(x)$$

(40)

with

$$h_k(x) = h(x, t_k).$$

(41)

The additional term $h_k(x)$ in (40) presents no problem at all as a review of either the Green’s function method or the integral equation method reveals.

For more general boundary conditions such as that in (50), everything that has been done up to this point remains valid provided that an appropriate $G(x, t)$ or $K(x, t)$ is used. As long as the equation under study is of the form of (39), it is only a trivial matter to obtain either the appropriate $G(x, t)$ or $K(x, t)$.

Now, if the material under study is not homogeneous, we will end up with an equation that is similar to (40) except that the coefficient will now be a spatial function. To begin with, the heat equation for an inhomogeneous material will be of the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left[ \kappa(x) \frac{\partial u}{\partial x} \right] + h(x, t) \quad \kappa(x) > 0$$

(42)

where $\kappa(x)$ is a function of the spatial variable. After the standard quantization of the time variable, we will then obtain an ordinary differential equation of the form

$$\frac{d}{dx} \left[ \kappa(x) \frac{du_k}{dx} \right] = \frac{1}{\beta^2} [u_k(x) - u_{k-1}(x)] + h_k(x)$$

(43)

with the appropriate boundary conditions. This system must of course be solved serially as before. By the transformation

$$\eta = \int \frac{dx}{\kappa(x)}$$

(43a)

(43) is then reduced to

$$\frac{d^2 \tilde{u}_k(\eta)}{d\eta^2} = \alpha^2(\eta) \left[ \tilde{u}_k(\eta) - \tilde{u}_{k-1}(\eta) \right] + \tilde{h}_k(\eta)$$

where

$$\alpha^2(\eta) = \kappa(\eta)$$

and

$$\tilde{u}_k(\eta) = u_k(\eta),$$

which is of the same form as that of (40) now with $\alpha^2$ as a function of the spatial variable. This is of course the prime object of our study even though the demonstration problem is of a form with constant coefficients. For such a situation, the explicit form of the Green’s function will in general not be available, nor will the differential operator be factorizable into two factors that commute. It is for circumstances like this that we are forced to use the integral equation method. For, unlike $G(x, t)$ which is the Green’s function of the differential operator $d^2/dx^2 - \alpha^2(x)/\beta^2$, $K(x, t)$ is independent of $\alpha^2$, being the Green’s function of $d^2/dx^2$. The story is not complete yet, however. As in our integral method, we made use of the knowledge that the eigenvalues of $K(x, t)$ is $\pi^2/t^2$ to obtain the modified Neumann series to incorporate a finer time step $\beta^2$ with $\lambda_1 < 1/\beta^2 = |\lambda| < \lambda_2$. With $\alpha^2$ dependent on $x$, the integral equation becomes

$$y(x) = f(x) + \lambda \int_x^1 K(x, t) \alpha^2(t) y(t) dt.$$  

(44)

That is, the effective kernel becomes $K(x, t) \alpha^2(t)$, and the eigenvalues are no longer equal to $\pi^2/t^2$. This means that somehow we must determine at least the first eigenvalue if a small time step $\beta^2$ is to be used. Such a task involves more work, naturally, but poses no serious threat. Going back now to the differential equation which is of the form
where we take\(^7\) \(a^2(x) > 0\) for \(0 \leq x \leq 1\). Thus, \(a^2(x)\) attains a maximum \(M > 0\) and a minimum \(m > 0\) in the compact domain \(0 \leq x \leq 1\). Here, of course, we assume \(a^2\) is continuous or at least piecewise continuous. Now, the comparison theorem [14], [15] in the development of the Sturm–Liouville problem states that the eigenvalues of (42) lies between those of

\[
\frac{d^2y}{dx^2} + \lambda y = 0
\]

(45)

where \(\lambda\) is the eigenvalue.

The modified integral equation method that is proposed here does not call for any extra equipment while offering a reduction in solution time. Instead of iterating through the standard Neumann series as suggested previously, we compute, either by performing the integration or by algebraic reduction, the following integral:

\[
F_1 = \frac{1}{2} - \frac{1}{\beta^2} \int_0^1 K \left( \frac{1}{2}, \xi \right) f(\xi) d\xi.
\]

The value \(F_1\), which is a constant, is equivalent to the first iterated solution at the midpoint of the hot rod. Using this value, we now go back to the conventional CSDT method, solving two boundary value problems:

\[
\begin{align*}
\lambda_0' - \frac{1}{\beta^2} \gamma &= f; \\
\gamma(0) &= A; \\
\gamma(1) &= B.
\end{align*}
\]

(49a)

(49b)

The exponentially growing factor is, of course, still present, but the maximum value it attains will be much smaller: only \(e^{1/\beta^2}\), i.e., \(e^{1.5} = 12\) instead of \(e^5 = 150\). The solution thus obtained in a matter of few milliseconds will in general be piecewise differentiable, possibly having no first-order derivative at \(x = 1/2\). This solution will now be used as \(y_1(x)\), the first iterated solution. Equations (48) and (49) are repeated again, replacing \(\lambda_0\) by \(\lambda_1\), to obtain \(y_2(x)\). As long as the Neumann series converges, it seems that this iterative method will also converge, even though we have not had time to carry out a rigorous analysis to support this claim.

For the more general case where \(a = a(x)\), and, e.g., \(\lambda_1 < 1/\beta^2 < \lambda_2\), this modified technique evidently remains valid, as remarked previously, provided that we now use the modified Neumann series that is the equivalent of (35), corresponding to the appropriate kernel of (44).

This modification has not yet been put on a solid mathematical basis as far as convergence is concerned, though intuitively it would seem to be convergent. It has not been experimentally tested either. Since the modification is truly a simple one, it can easily be experimented with in the future as circumstances allow.

To conclude this paper, we should like to point out that the method as discussed is only pertinent to the linear diffusion equation, though not necessarily one with constant coefficients. We did not attempt to look into the situation where the equation is nonlinear. True enough, formally, an integral equation of similar form can be set up, even though it will be a nonlinear one. Also true enough, a formal iteration scheme not unlike that of the Neumann series can be applied. However, the convergence of such an iteration scheme is by no
means assured. In fact, only under rather restricted conditions can the convergence be proved. Our understanding of nonlinear equations, whether partial or ordinary, is still merely a very meager collection of certain limited facts, and for this reason, we do not venture into saying anything on a nonlinear situation in this study.

**APPENDIX**

**A Brief Summary of the Green’s Function**

For an ordinary differential equation

$$L(D)y = 0$$  \hspace{1cm} (50)

where $L$ is a linear operator and $D = d/dx$, satisfying homogeneous boundary conditions at $x = a$ and $x = b$,

$$\alpha_1 y(a) + \beta_1 y’(a) = 0 \quad \alpha_2 y(b) + \beta_2 y’(b) = 0.$$  \hspace{1cm} (51)

(The problem is assumed to be self-adjoint.) The Green’s function is a function of two variables, say $x$ and $\xi$, and satisfies the following conditions.

1) $G(x, \xi)$ is symmetric in $x$ and $\xi$.
2) $G(x, \xi)$ as a function of $x$, holding $\xi$ as a parameter, satisfies (50).
3) $G(x, \xi)$ as a function of $x$ satisfies (50) for every $x \neq \xi$.
4) $G(x, \xi)$ as a function of $x$ is continuous even at $x = \xi$.
5) $G(x, \xi)$ as a function of $x$ has continuous derivatives up to order $n-1$ where $n$ is the highest order derivative appearing in $L(D)$.
6) As a function of $x$, $d^{n-1}G/dx^{n-1}$ is continuous except at $x = \xi$ at which it suffers a discontinuity of the simplest kind, i.e.,

$$\left. \frac{d^{n-1}G}{dx^{n-1}} \right|_{x=\xi^-} + \left. \frac{d^{n-1}G}{dx^{n-1}} \right|_{x=\xi^+} = - \frac{1}{p(\xi)}$$

where we assume the leading term of $L(D)$ is $p(x)(d^n y/dx^n)$.

Then the solution to

$$L(D)y = -f(x, y)$$  \hspace{1cm} (52)

is given by

$$y(x) = \int_a^b G(x, \xi)f(\xi, y(\xi))d\xi.$$  \hspace{1cm} (53)

When $f(x, y)$ is a function of $x$ explicitly, (53) of course, will immediately give the correct solution, as in the case of our Green’s function method. When $f(x, y)$ depends on $x$ implicitly through $y$ as well, such as in the integral equation method with $f(x, y) = (-1/\beta^2)[y - f(x)]$, (53) becomes a Fredholm equation of the second kind, as shown previously.

In the CSDT approach studied above, $G(x, \xi)$ and $K(x, \xi)$ are, of course, not unrelated. The reader who is familiar with the theory of integral equations will probably realize that $G(x, \xi)$ of (15) is indeed the resolvent kernel $G(x, \xi; \xi; \lambda)$ of $K(x, \xi)$ for $\lambda = -(1/\beta^2)$; i.e., the analytic continuation of (32) to $\lambda = -(1/\beta^2)$.

Finally, if the problem of (50) and (51) is not self-adjoint, we can construct a function $H(x, \xi)$, satisfying the adjoint problem and with similar properties. The solution $y(x)$ will then be given by (53) again, with $H(x, \xi)$ replacing $G(x, \xi)$.

**Acknowledgment**

The author wishes to thank the reviewers for some valuable suggestions, most of which have been incorporated into this paper.

**BIBLIOGRAPHY**