A Real Implementation of the Robust Pole Assignment Algorithms

Alexander Yu. Evnin
Department of Applied Mathematics
Chelyabinsk State Technical University
Chelyabinsk 454080, Russia

Abstract
In this paper new conditions for the existence of solutions to the output feedback pole assignment problem are established. This generalizes well-known results of Kimura, Porter, Bradshaw [1] and allows us to build robust pole assignment algorithms [2] in the real arithmetics.

1 Introduction
The problem of pole assignment by state feedback for multivariable control systems is underdetermined. J. Kautsky et al. [2] have demonstrated that the extra degrees of freedom in the problem may be used to determine a robust, or well-conditioned, solution such as to minimize the sensitivities of the closed-loop poles to perturbations in the system and gain matrices.

The aim of this paper is to prove new criterion of the existence of solutions to the output feedback pole assignment problem. It allows us to implement well-known algorithms of J. Kautsky et al. [2] in the real arithmetics.

2 The main result
The theoretical foundation of mentioned algorithms is the following theorem.

Theorem 1. Suppose that B and C are full rank n x m and p x n matrices, respectively (n > m, n > p). There exists a real matrix K such that the spectrum of $A + BKC$ coincides with the spectrum of a given matrix $L$ if and only if there exists a nonsingular matrix $S$ such that
1) $\text{Im}(AS - SL) \subseteq \text{Im}(B);$ 
2) $\text{Im}(A'T - TL') \subseteq \text{Im}(C'),$ where $T'S = I$. Then

$$K = B^+(SLT' - A)C^+. \quad (1)$$

(Notation: $M'$ is transposed matrix; $M^+$ is Moore-Penrose inverse; $\text{Im}(M)$ is the range of $M$, that is, the linear span of the columns of $A$).

To prove this theorem we first require the following Lemma. If $\text{Im}(X) \subseteq \text{Im}(B), \text{Im}(Y) \subseteq \text{Im}(B), \text{Im}(Q_b) = \text{Im}(B), Q_bQ_b^* = I_m$, then $X = Y$ if and only if $Q_b^*X = Q_b^*Y$.

Proof. A necessity is obvious. We’ll prove a sufficiency. Lemma condition implies that matrices $X$ and $Y$ are presented as $Q_bT_x$ and $Q_bT_y$, respectively, for some matrices $Q_b$ and $T_x, T_y$. If $Q_bX = Q_bY$, then $T_x = T_y$, whence $X = Y$, Q.E.D.

With help of transposition the following reciprocal assertion is derived.

Corollary. If $\text{Im}(X') \subseteq \text{Im}(C'), \text{Im}(Y') \subseteq \text{Im}(C'), \text{Im}(Q'_b) = \text{Im}(C'), Q_bQ_b^* = I_p$, then $X = Y$ if and only if $XQ'_b = YQ'_b$.

Proof of Theorem 1.
A necessity. Let matrix $A + BKC$ is similar to $L$.
It means the existence matrix $Y$ such that

$$A + BKC = YLY^{-1}$$

or

$$BK = YLY^{-1} - A. \quad (2)$$

Since $(YL - AY)Y^{-1} = BKC$, we have

$$\text{Im}(YL - AY) \subseteq \text{Im}(B).$$

Since $Y(LY^{-1} - Y^{-1}A) = BKC$, we obtain

$$\text{Im}(A'(Y^{-1})' - (Y^{-1})'L') \subseteq \text{Im}(C').$$

Thus conditions 1), 2) from Theorem 1 statement are satisfied with $S = Y$.
Since $B$ and $C$ are full rank matrices, we have

$$B^+B = I_m, CC^+ = I_p.$$ 

Multiplying both sides of equality (2) from the left-hand side by $B^+$ and from the right-hand side by $C^+$, we obtain (1).

A sufficiency. At first, we’ll formulate the following simple assertion:
Let $n \times m$ matrix $Q(n > m)$ has biorthogonal normed columns: $Q'Q = I_m$. Then $Q^+ = Q'$.

It should be obtained from formula for pseudoinverse matrix of full rank $n \times m$ matrix $Q(n > m)$[3]:

$$Q^+ = (Q'Q)^{-1}Q'.$$

Let us consider QR-decompositions of matrices $B$ and $C'$:

$$B = Q_bR_b, C' = Q'_cR'_c$$

(here $Q'_cQ_b = I_m, Q_cQ'_b = I_p, R_b$ and $R_c$ are nonsingular quadratic matrices upper- and lowtriangular).

Then

$$B^+ = R_b^+Q_b^+ = R_b^{-1}Q_b, C^+ = Q'_cR'_c^{-1}. \quad (3)$$
Substitute (3) into (1):

$$K = R_c^{-1}Q_c'(SLT' - A)Q_c'(R_c)^{-1},$$

that is equivalent to

$$R_cK R_c' = Q_c'(SLT' - A)Q_c'$$

or

$$Q_c'R_cK R_c'Q_c' - Q_c'(SLT' - A)Q_c'. \quad (4)$$

We observe that

$$\text{Im}(Q_c'R_cK R_c'Q_c') \subset \text{Im}(Q_c') = \text{Im}(B)$$

and owing to condition 1)

$$\text{Im}(SLT' - A) = \text{Im}((SL - AS)T') \subset \text{Im}(B),$$

whence by Lemma in equality (4) we can omit left factor $Q_c'$, obtained the equivalent relation. Similarly, we have

$$(Q_c'R_cK R_c'Q_c') \subset \text{Im}(Q_c') = \text{Im}(C')$$

and owing to condition 2)

$$(SLT' - A)' = (S(LT' - T'A))' \subset \text{Im}(C').$$

By the corollary from Lemma, we can drop right factor $Q_c'$ in (4), obtained the equivalent relation

$$Q_c'R_cK R_c'Q_c' = SLT' - A$$

or

$$A + BKC = SLT' = SLS'^{-1}. \quad \text{Q.E.D.}$$

As corollary of Theorem 1 we'll prove the assertion from [1] (where there is no proof and (in our terms) matrix $L$ is supposed to be a diagonal one).

Beforehand we define subspaces

$$F(\lambda_i) = \{s \in C^n|(A - \lambda_i)s \in \text{Im}(B)\}$$

and

$$G(\lambda_i) = \{t \in C^n|(A - \lambda_i)t \in \text{Im}(C')\},$$

where $C^n$ is the space of $n$-column vectors with complex entries.

**Theorem 2.** There exists a real matrix $K$ such that self-conjugate spectrum (the set of eigenvalues) of matrix $A + BKC$

$$\sigma(A + BKC) = \{\lambda_1, \ldots, \lambda_n\}$$

if and only if there exist vectors $s_i \in F(\lambda_i), t_j \in G(\lambda_j)$ for $i, j = 1, \ldots, n$ such that

$$t_j's_i = \delta_{ij} \text{ for } i, j = 1, \ldots, n, \quad (5)$$

$$\lambda_k = s_i't_k = t_i \text{ if } \lambda_k = \lambda_i. \quad (6)$$

**Proof.** We construct matrices $S = [s_1 \ldots s_n], \quad T = [t_1 \ldots t_n]$. The orthogonality requirement (5) means that $ST' = I$. Let $L = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Conditions 1) and 2) of Theorem 1 are equivalent to

$$A_{i} - \lambda_i s_i \in \text{Im}(B) \text{ or } s_i \in F(\lambda_i)$$

and

$$A't_i - \lambda_i t_i \in \text{Im}(C') \text{ or } t_i \in G(\lambda_i)$$

respectively. Vectors $s_i$ and $t_i$ become right and left eigenvectors of real matrix $A + BKC$, that implies (5),(6). Q.E.D.

### 3 A real implementation of the algorithms

The possibility to work with real matrix $L$ having given spectrum allows us to build algorithms in the real arithmetics. For example we consider the robust state feedback problem for the time-invariant, linear, multivariable system with dynamic state equation

$$\dot{x} = Ax + Bu, \quad (7)$$

where $x, u$ are $n$- and $m$-dimensional vectors, respectively, and $A, B$ are real, constant matrices of compatible orders. Matrix $B$ is assumed to be of full rank. If a state feedback control

$$u = Kx$$

takes place (it's a particular case of output feedback control $u = Ky, y = Cx$), then the behavior of system (7) is governed by eigenvalues of closed system matrix $A + BK$. Let spectrum of some real matrix $P$ coincides with desired spectrum of closed-loop system. We can choose $P$ as block-diagonal matrix with blocks of form

$$\begin{bmatrix} a_i & b_i \\ -b_i & a_i \end{bmatrix}$$

Thus the spectrum of $P$ consists of real numbers $\lambda_k$ and complex (non-real) numbers $a_i + ib_i$. Developing the ideas of [2], we can get a measure of robustness as the condition number of such matrix $Y$ that

$$(A + BK)Y = YP.$$

Author created the C-program of multivariable modal synthesis. The obtained numerical results for examples from [2] are quite satisfactory.

Notice that the obtained theoretical results may be applied to the problem of designing a stable sliding mode giving robust performance of a variable structure control system [4].

### References


