Robust Control of a Vibrating Link Robot Manipulator

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Abstract

This paper presents an approach for designing robust tracking controller for lumped mass model of a vibrating link (VL) robot manipulator. The control methodology is intuitively simple since it is based on concepts which are familiar to most control engineers. The approach is illustrated by developing a robust tracking controller that achieves global uniform ultimate boundedness (GUUB) stability of the link tracking error in spite of model uncertainty and the additional link dynamics.

Introduction

In recent years control engineers have become increasingly interested in the robot tracking problem. As a result many controllers have been developed which compensate for uncertainty in the nonlinear second order dynamics commonly used to represent rigid link (RL) robots. Most of the more rigorously developed nonlinear controllers for RL robots fall into two categories; adaptive control and robust nonlinear control.

A deficiency associated with many of the previous controllers is that these controllers have been designed utilizing the assumption that the link end-effector and link base are always at the same position. This means that any dynamics associated with the link vibration were neglected. Several researchers have postulated that the detrimental effects of neglected link vibrations are preventing the development of true high performance motion and/or force tracking controllers. Therefore, it is believed that further progress can be made by including the effects of link dynamics in the control synthesis.

The related research is now reviewed. Meckl and Scering [1] have developed a set of shaped force profiles to avoid exciting the fundamental mode of the system. Zalucky and Hardt [2] use dual beam to split up the positioning and load carrying capability. The analysis employs a lumped parameter dynamic model of the system. Hastings and Book [3] have used reduced order observers to estimate modal velocities in the state feedback control of flexible manipulators. The strain gauges are employed to estimate the flexible variables of motion.

The use of positive modal feedback to control structural vibrations has been investigated by Sardar and Paul [4]. The positive modal feedback is shown to drive the poles corresponding to flexible modes into the left half plane. A PD controller with lumped parameter dynamic systems (minimum and non-minimum phase) is used to demonstrate the proposed scheme. Lopez-Linares et al. [5] use inverse dynamic solution to compute the torque that must be applied at the hub to obtain a given trajectory at the tip. To cope with the problems due to disturbances and unknown model parameters, they suggest a closed loop control law using an LQG control similar to the one proposed by Cannon and Schmitz [6] to regulate the position of the arm tip.

The system analyzed in this paper is a linear lumped parameter model with a minimum phase zero in the tip position to hub torque transfer function. A robust tracking controller is developed to obtain the GUUB stability of the link tracking error in spite of model uncertainty and the additional link dynamics.

Vibrating Link Robust Tracking Controller

During the motion of the robot manipulators, link vibrations can often be a potential problem; therefore, the associated control algorithm may have to be designed to actively compensate for these vibrations. To compensate for these link vibrations, a model for these vibrations must first be formulated. In this paper we first select a lumped model so that corresponding robust controller problem is tractable. A linear, lumped parameter model for this problem is shown in Figure 1 and is represented by

\[ m_1 q_1 + b(q_1 - \dot{q}_1) + k(q_1 - q_2) = \tau \]  

and

\[ m_2 \ddot{q}_2 + b(q_2 - \dot{q}_2) + k(q_2 - q_1) = 0 \]

where \( m_1, m_2 \) are positive scaler constants used to represent the lumped mass of the link and that of the end-effector, respectively; \( b \) and \( k \) are positive scaler constants used to represent the lumped damping and lumped spring constant, respectively, for the vibrational effects; \( q_2, q_1 \) are used to represent the position of the end-effector and link, respectively; and \( \tau \) is used to represent the control input torque.

In this paper we solve the tracking problem. That is the error, \( \epsilon \), in the system is given by

\[ e = q_d - q_1 \]

where \( q_d \) is the desired trajectory of the end point of the manipulator. The goal of the control is to make this error, \( \epsilon \), small by controlling the base torque so that the end-effector can track the desired trajectory. The following mathematical development for the control law attempts to meet this objective. The proof of stability is presented at the end of the analysis.

Equations 1 and 2 can be rewritten in the form shown...
below using a new variable $z$, which represents the difference between $q_1$ and $q_2$.

\[ m_1 \ddot{x} + b \dot{x} + k \dot{x} = \tau - m_1 \dot{q}_2 \]  
(4)

\[ m_2 \ddot{q}_2 = b \dot{q}_2 + k \dot{q}_2 \]  
(5)

where,

\[ z = q_1 - q_2 \]  
(6)

Equation 4 can be rewritten using Equation 5, as shown below.

\[ m_1 \ddot{x} = \tau - b(1 + \frac{m_1}{m_2}) \dot{x} + k(1 + \frac{m_1}{m_2}) \dot{x} \]  
(7)

The dynamics of the error can be obtained from Equation 5 and is presented below.

\[ m_2 \ddot{e} = W_2 - b \dot{z} - k \dot{z} - m_2 \dot{e} \]  
(8)

where,

\[ W_2 = m_2 \ddot{q}_2 = m_2 \dot{e} \]  
(9)

$W_2$ is a disturbance input and is upper bounded as shown in the Lemma A.1 in Appendix A, and $a$ is a scaler, positive constant.

Equation 8 describes the dynamics of the error, $e$. The control objective is to determine a control law for torque ($\tau$) which makes the error $e$ small. However, there is no control term in the equation for the dynamics of the error (Equation 8). A fictitious control term $u_x$, given by

\[ u_x = K_2 \dot{e} \]  
(10)

where $K_2$ is a scalar constant, is introduced in Equation 8. The resulting equation for the error dynamics is given below.

\[ m_2 \ddot{e} = -m_2 \dot{e} + W_2 - b \dot{z} - k \dot{z} - m_2 \dot{e} \]  
(11)

Equation 11 can be rewritten in the following form

\[ m_2 \ddot{e} = -m_2 \dot{e} + W_2 - b K_2 \dot{e} - k K_2 \dot{e} + c \eta \]  
(12)

where

\[ \eta = u_x - c \eta; \quad \eta = [\eta]; \quad c = [k \ b] \]  
(13)

The error system given by Equation 12 for the manipulator can also be written in state space form

\[ \dot{\eta} = A_1 \eta + B_1 (W_2 + c \ \Pi) \]  
(14)

where

\[ A_1 = \begin{bmatrix} 0 & 1 \\ 0 & -m_2 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ m_2 \end{bmatrix} \]  
(15)

Having determined the error system in the above state space form, we now find the state space form for the dynamics of $\eta$. From the definition of $\eta$, given in Equation 13, we can write

\[ m_1 \ddot{\eta} = m_1 \dot{u}_z - m_1 \ddot{z} \]  
(16)

The above equation, in combination with Equation 7 gives following dynamics for $\eta$

\[ m_1 \ddot{\eta} = m_1 \dot{u}_z - \tau \dot{z} + K_2 \dot{z} + m_1 (b \dot{z} + K_2 z) \]  
(17)

or

\[ m_1 \ddot{\eta} = -a m_1 \dot{\eta} - \tau + W_1 \]  
(18)

where,

\[ W_1 = a m_1 \dot{\eta} + b \dot{z} + K_2 \dot{z} + m_1 (b \dot{z} + K_2 z) + m_2 \dot{z} \]  
(19)

**Control Law**

The following control law is proposed for the robust control of the flexible manipulator.

\[ \tau = K_p \eta + K_d \dot{\eta} + V_1 \]  
(20)

where

\[ V_1 = \frac{(a \eta + \dot{\eta})^2}{[a \eta + \dot{\eta}]^2} - \rho_1 \geq |W_2| k \ \epsilon \geq 0 \]  
(21)

$K_p$ and $K_d$ are proportional and derivative gains, $\rho_1$ is an upper bound on disturbance $W_2$, and $\epsilon$ is a small, positive scaler constant. Based on this control torque, Equation 18 can be written in state space form as shown below.

\[ \dot{\eta} = A_2 \eta + B_2 (W_1 - V_1) \]  
(22)

where,
It should be noted here that the control law is a function of \( u_2 \) since

\[
\eta = u_2 + z_{K_p} e - z \quad \text{and} \quad \dot{\eta} = \dot{u}_2 + z_{K_p} \dot{e} - \dot{z}.
\]

**Lyapunov Proof for Stability of the System**

We now analyze the stability of the system given by Equations 14 and 22 for the control law given by Equation 20. The goal is to show that \( e \) and \( \eta \) are Globally Uniformly Ultimately Bounded (GUUB). Let the Lyapunov function be given by

\[
V = \frac{1}{2} e^T P_1 e + \frac{1}{2} \eta^T P_1 \eta
\]

where,

\[
P_2 = \begin{bmatrix}
K_{\mu_2} + aK_{\mu_2} + a^2 m_2 & a m_2 \\
- a m_2 & m_2
\end{bmatrix} \quad P_1 = \begin{bmatrix}
K_{\mu_2} + aK_{\mu_2} + a^2 m_1 & a m_1 \\
- a m_1 & m_1
\end{bmatrix}.
\]

The Lyapunov function can also be written as

\[
V = \frac{1}{2} X^T P X
\]

where,

\[
X = \begin{bmatrix}
e \\
\eta
\end{bmatrix} \quad P = \begin{bmatrix}
P_2 & 0 \\
0 & P_1
\end{bmatrix}.
\]

It can be shown that the matrix \( P \) is PD if the following conditions are satisfied

\[
0 < a < 1; \quad K_{\mu_2} + aK_{\mu_2} > m_1 (1 - a^2); \quad K_{\mu_2} + aK_{\mu_2} > m_2 (1 - a^2).
\]

The time derivative of the Lyapunov function can be written as

\[
\dot{V} = \frac{1}{2} e^T P_2 e + \frac{1}{2} \dot{e}^T P_2 e + \frac{1}{2} e^T P_2 \dot{e} + \frac{1}{2} \dot{\eta}^T P_1 \dot{\eta}
\]

Substituting from Equations 14 and 20 into Equation 30 gives the following form for time derivative of the Lyapunov function

\[
\dot{V} = -e^T Q e - \eta^T Q \eta + e^T P_2 B_1 (W_2^T C \eta) + \eta^T P_1 B_2 (W_1 - V_1)
\]

where,

\[
Q_1 = \frac{1}{2} (A_1^T P_2 A_1 + P_2 A_2) = \begin{bmatrix}
a K_{\mu_2} & 0 \\
0 & K_{\mu_2}
\end{bmatrix}
\]

\[
Q_2 = \frac{1}{2} (A_2^T P_1 A_2 + P_1 A_1) = \begin{bmatrix}
a K_{\mu_2} & 0 \\
0 & b K_{\mu_2}
\end{bmatrix}
\]

Note that

\[
P_2 B_2 + P_1 B_1 = \begin{bmatrix} a \\ 1 \end{bmatrix}
\]

therefore, the bound on the time derivative of the Lyapunov function is given by

\[
\dot{V} \leq -X^T Q X + \|e\|^2 \left[ 2 \|W_2^T C \|_1 + \|1\| \|1\| \right] (\alpha \eta + \dot{\eta}) + \eta^T (W_1 - V_1)
\]

where,

\[
Q = \begin{bmatrix}
Q_2 & 0 \\
0 & Q_1
\end{bmatrix}
\]

Each of the three terms on right hand side of Equation 35 can be further shown to be bounded as described below.

\[
-X^T Q X \leq -\lambda_{\mu_2}(Q) \| X \|^2
\]

\[
(\alpha \eta + \dot{\eta}) + \eta^T (W_1 - V_1) \leq \mu_0 \|\eta\| + \mu_1 \|\dot{\eta}\| + \mu_3 \|\eta\| + \mu_4 \|\dot{\eta}\| + \mu_5 \|\eta\| + \mu_6 \|\dot{\eta}\|
\]

where,

\[
\mu_0 = 1 \quad \mu_1 = 1 \quad \mu_2 = \|C\| \quad \mu_3 = \|1\| \quad \mu_4 = \|1\| \quad \mu_5 = \|1\| \quad \mu_6 = \|1\|
\]

The bounds on \( W_2 \), \( W_1 \) are shown in Lemmas A.1 and A.2 and the bound given in Equation 38 is derived in Lemma A.3 in Appendix A. Thus we can write that

\[
\dot{V} = -\lambda_{\mu_2}(Q) \| X \|^2 - \lambda_{\mu_2}(Q) \| \eta \|^2 + \| e \|^2 - \mu_0 \|\eta\| + \mu_1 \|\dot{\eta}\| + \mu_2 \|\eta\| + \mu_3 \|\dot{\eta}\|
\]

Equation 41 above can also be written as
where  

$$\dot{X} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} \lambda_{\text{min}}(Q) & -\mu_3 \\ -\mu_3 & \lambda_{\text{min}}(Q) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \nu_0 \lambda_{\text{min}}(Q)$$

(42)

or

$$\dot{X} = (\lambda_{\text{min}}(Q) - \mu_3) \dot{X}^2 + \mu_3 \dot{X}^2 + \nu_0 \lambda_{\text{min}}(Q)$$

(43)

where,

$$Q' = \begin{bmatrix} \lambda_{\text{min}}(Q) & -\mu_3 \\ -\mu_3 & \lambda_{\text{min}}(Q) \end{bmatrix}$$

(44)

From Equations 27 and 43, the state $X$ and hence error $e$ are GUUB in the sense of Lemma B.1 given in Appendix B if $K_p > K_q$, $K_p$ and $K_q$ are so adjusted that

$$\lambda_1 = \lambda_{\text{min}}(Q') - \mu_1 > 0.$$  

(45)

**Simulation Results**

The controller was simulated with the following parameters

$$m_1 = 1; \quad m_2 = 1; \quad k = 1; \quad b = \frac{1}{2} q_2 \sin(t); \quad q_1(0) = 1; \quad q_2(0) = 0; \quad a = 1; \quad \mu_1 = 20; \quad \nu_0 = 1;$$

(46)

The desired trajectory ($q_d$) of the tip is chosen to be sinusoidal as it is a difficult trajectory to track for a vibrating link due to the fluctuating nature of the trajectory. The simulation results are shown in Figures 2 and 3. Figure 2 shows that the error ($e = q_d - q_2$) and its derivative ($\dot{e} = \dot{q}_d - \dot{q}_2$) decay exponentially and stabilize within a small bound as time approaches infinity. The required hub torque ($\tau$) is shown in Figure 3. The results of the simulation show a Globally Uniformly Ultimately Bounded (GUUB) tracking error with the proposed controller.

**Conclusions**

In this paper, we have designed a robust tracking controller for vibrating link robot manipulators. The implementation of the proposed controller requires measurement of the position and velocity of the base and tip of the link. The controller achieves GUUB stability of the link tracking error in spite of model uncertainty and the additional link dynamics due to vibrational effects.

**References**


**Appendix A**

**Lemma A.1**

The disturbance input $W_2$ is given as

$$W_2 = m_2 \ddot{q}_d + m_3 a \dot{e}.$$  

(A.1)

From Equation A.1, it is obvious that

$$|W_2| \leq q_0 + \frac{q_1}{2} \dot{q}_1; \quad \dot{q}_1 > 0; \quad t > 0.1$$  

(A.2)

Thus the disturbance input $W_2$ is bounded by the error, as shown in Equation A.2.

**Lemma A.2**

The disturbance input $W_1$ is given as

$$W_1 = a m_1 \dot{q}_1 + b \dot{z} + Kz + \frac{m_1}{m_2} (b \dot{u}_2 + Ka) - (1 + \frac{m_1}{m_2}) (b \dot{u}_2 + Ka) + (1 + \frac{m_1}{m_2}) (b \dot{u}_2 + Ka) + m_1 \dot{q}_3.$$  

(A.3)

The above Equation A.3, can be rewritten as

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Using Equation 12, Equation A.4 can be rearranged as

\[ W = a_m \dot{\eta} - (1 + \frac{m_1}{m_2}) \mathbf{C} \mathbf{C}^* (1 + \frac{m_1}{m_2}) K_p C \mathbf{e} + m_1 K_p \ddot{e} \]  

(A.4)

where,

\[ \beta = (K_p^2 - 1) \frac{m_1}{m_2} - 1 \]  

(A.5)

Using Lemma A.1, the following upper bound holds for disturbance input \( W_1 \)

\[ |W_1| \leq \zeta_3 + \zeta_3 \ln \zeta_4 + K_p \zeta_4 \text{ for } \zeta_4 > 0; \quad i = 2, 3, 4 \]  

(A.7)

**Lemma A.3**

\[ (\alpha \eta + \eta')^T (W_1 - V_1) \leq \varepsilon \]  

(A.8)

\[ |\alpha \eta + \eta'| |W_1 - (\alpha \eta + \eta')^T (\alpha \eta + \eta') | \leq \varepsilon \]  

(A.9)

\[ |\alpha \eta + \eta'| \rho_1 - |(\alpha \eta + \eta') | \rho_1^* \leq \varepsilon \]  

(A.10)

\[ \varepsilon \leq |(\alpha \eta + \eta') | \rho_1^* \leq 1 \]  

(A.11)

The above equation is true as

\[ |(\alpha \eta + \eta') | \rho_1^* \leq 1 \]  

(A.12)

**Appendix B**

Let \( V(.) \) be a Lyapunov function for any given continuous time system with the following properties

\[ \gamma_1 (x(t)) \leq V(x(t)) \leq \gamma_2 (x(t)) \]  

(B.1)

\[ \dot{V}(x(t)) \leq -\gamma_3 (x(t)) + \gamma_4 (t) \]  

(B.2)

where \( \eta \) is a positive constant, \( \gamma_1(.) \) and \( \gamma_2(.) \) are continuous strictly increasing functions, and \( \gamma_3(.) \) is a continuous and non-decreasing function. If \( \dot{V}(x(t)) < 0 \) for \( |x(t)| > \eta \), then the given system has a uniform ultimate boundedness property, that is, if \( x(t) \) is a solution to the system with initial state \( x(t_0) \), then given a

\[ d' > (\gamma_1^{-1} \circ \gamma_2)(\eta) \]  

(B.3)

one can write

\[ |x(t)| < d' \]  

for every \( t \in [t_0 + T, \infty) \)

(B.4)

where

\[ T = \frac{\gamma_3 (\mathbf{y}_z (\mathbf{y}_z)) - \gamma_1 (\mathbf{y}_z (\mathbf{y}_z))}{\gamma_3 (\mathbf{y}_z (\mathbf{y}_z)) - \gamma_1 (\mathbf{y}_z (\mathbf{y}_z))} \]  

(B.5)

and

\[ \eta' = (\gamma_2^{-1} \circ \gamma_3)(d'). \]  

(B.6)

Given Equation 45, the state \( X \) and hence error \( e \) are GUUB in the sense of above Lemma B.1 with

\[ \gamma_1 (x(t)) = \frac{1}{2} \lambda_{\omega^2} |x|^2, \quad \gamma_2 (x(t)) = \frac{1}{2} \lambda_{\omega^2} |x|^2 \]  

(B.7)

\[ \eta = \left( \lambda_{\omega^2} - \frac{\mu_0}{4 \lambda_{\omega^2}} \right)^{\frac{1}{4}} + \frac{\mu_0}{2 \lambda_{\omega^2}}, \quad \gamma_3 (x(t)) = \lambda_{\omega^2} - \frac{\mu_0}{2 \lambda_{\omega^2}} \]