ABSTRACT

There are two schemes provided via the gap metric for designing finite dimensional robust controllers for distributed parameter LTI systems. The finite dimensional controllers designed according to these two schemes are guaranteed to stabilize the distributed parameter LTI systems and their neighborhoods, and the real closed-loop responses can be estimated via the designed ones. Several related problems are also discussed such as the continuity of the largest robust stability radius, the finite dimensional approximations in the gap metric, the relationship between the largest robust stability radius of a system and that of its optimally robust controllers, etc.

1. Introduction

In this paper, there are two schemes provided via the largest robust stability radius for designing finite dimensional robust controllers for distributed parameter LTI systems. The finite dimensional controllers designed according to these two schemes are guaranteed to stabilize the distributed parameter LTI systems and their neighborhoods, and the real closed-loop responses can be estimated via the designed one. Several related problems are also discussed such as the continuity of the largest robust stability radius, the finite dimensional approximations in the gap metric, the relationship between the largest robust stability radius of a system and that of its optimally robust controllers, etc.

2. The gap metric

Let $\mathcal{H}_\infty$ be the Hardy space corresponding to the open right half plane and $\mathcal{F}$ be the quotient field of $\mathcal{H}_\infty$. Denote by $\mathcal{H}_{\infty}^{nxm}$ (resp. $\mathcal{F}^{nxm}$) the set of all $nxm$ matrices with entries in $\mathcal{H}_\infty$ (resp. $\mathcal{F}$).

We take $\mathcal{F}^{nxm}$ as the set of systems under consideration, because all finite dimensional systems and most of distributed parameter LTI systems have their transfer matrices in this set. Now, consider the following feedback system.

![Figure 1 Feedback System](image)

Figure 1 Feedback System
where $P \in \mathcal{F}^{n \times m}$ is a given system (in general, $P$ is infinite dimensional and unstable), and $C \in \mathcal{F}^{n \times m}$ is a controller. The closed-loop transfer matrix is

$$H(P, C) = \begin{bmatrix} (I+PC)^{-1} - P(I+CP)^{-1} \\ C(I+PC)^{-1} (I+CP)^{-1} \end{bmatrix}.$$  

The controller $C$ is designed for stabilizing $P$ and for achieving certain desired closed-loop responses as well. $C$ is said to be a stabilizing controller of $P$ if $H(P, C)$ belongs to $H_{\infty}^{(n+m) \times (n+m)}$.

3. Optimally robust controllers

For a given system $P_0 \in \mathcal{F}^{n \times m}$, define

$$K(P_0, \varepsilon) = \{ P \in \mathcal{F}^{n \times m} : \delta(P, P_0) < \varepsilon \} \quad \varepsilon > 0.$$  

The largest robust stability radius $\gamma := \gamma(P_0)$ of $P_0$ is defined as the largest $\varepsilon$ such that $K(P_0, \varepsilon)$ can be stabilized by one single controller, i.e., all systems in $K(P_0, \gamma)$ can be stabilized simultaneously by one controller. Any controller stabilizing $K(P_0, \gamma)$ is called an optimally robust controller of $P_0$.

Let $(D_0, N_0)$ be a normalized r.b.f. of $P_0$, define

$$N(P_0, \varepsilon) := \{ P = (N_0 + \Delta_n)(D_0 + \Delta_d)^{-1} : \| \Delta_d \| < \varepsilon \}.$$  

Vidyasagar and Kimura [V-K] showed that to stabilize $N(P_0, \varepsilon)$ if and only if

$$\| \hat{D}_c \Delta_n \| \leq \varepsilon^{-1},$$  

where $(\hat{D}_c, \hat{N}_c)$ is an l.b.f. of $C$.

(3.3) leads one to look for the largest $\varepsilon_{\text{max}}$ and the controllers stabilizing $N(P_0, \varepsilon_{\text{max}})$. Glover and McFarlane [G-M] solved this problem using state space formulas. In [G-M] the formula for $\varepsilon_{\text{max}}$ is given and all controllers which stabilize $N(P_0, \varepsilon_{\text{max}})$ are parameterized.

Georgiou and Smith [G-S] proved the fact:

$$K(P_0, \varepsilon) = N(P_0, \varepsilon) \text{ for all } \varepsilon \geq 0,$$

where $\lambda(P_0) := \inf_{\text{Re}(s) > 0} \sigma_{\text{min}} \left[ \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} \right]$ and $(D, N)$ is a normalized Bezout fraction of $P_0$.

**Theorem 3.1.** Let $P_0 \in \mathcal{F}^{n \times m}$ and $C_g \in R(P_0)$. Then we have

$$\gamma(C_g) \geq \gamma(P_0).$$

Suppose $C_g \in R(P_0)$, and $P \in K(P_0, \gamma)$ is a perturbation of $P_0$, what is the variation of the closed-loop transfer matrix, i.e., what is the distance from $H(P_0, C_g)$ to $H(P, C_g)$? On the other hand, the optimally robust controller $C_g$ is difficult to computed in practice and one often can obtain an approximate one $C_f$. What is the influence of $C_f$ on the closed-loop response, i.e., what is the distance from $H(P_0, C_g)$ to $H(P_0, C_f)$? The answers are the Theorem 3.2 and 3.3.

**Theorem 3.2** Let $P \in K(P_0, \gamma)$. For any $C_g \in R(P_0)$ we have

$$\| H(P, C_f) - H(P, C_g) \| \leq \gamma^{-1} \delta(P, P_0) \left( 1 + \frac{1}{\gamma - \delta(P, P_0)} \right).$$

Now we consider the influence of the perturbation of the optimally robust controllers on closed-loop responses.

**Theorem 3.3** Let $P_0 \in \mathcal{F}^{n \times m}$ and $C_g \in$
R(P₀). For any C ∈ K(C₀, γ), we have

(3.8) \[ \| H(P₀, C) - H(P₀, C₀) \| ≤ \gamma⁻¹\delta(C,C₀) \left( 1 + \frac{1}{\gamma - \delta(C,C₀)} \right). \]

4. Finite dimensional stabilizing controller design

SCHEME 4.1 Let P₀ ∈ \( \mathbb{F}^{n×m} \).

Step 1 Find the largest robust stability radius \( \gamma = \gamma(P₀) \) and an optimally robust controller \( C₀ \) of \( P₀ \).

Step 2 Find a finite dimensional approximation \( C_f \) of \( C₀ \) such that

\[ \delta := \delta(C_f,C₀) < \gamma. \]

THEOREM 4.2 The finite dimensional controller \( C_f \) obtained from Scheme 4.1 will stabilize the system \( P₀ \) and its neighborhood \( K(P₀,\varepsilon₀) \), where \( \varepsilon₀ = \gamma-\delta(C_f,C₀) \). Moreover,

(4.1) \[ \| H(P₀, C₀) - H(P₀, C_f) \| ≤ \gamma⁻¹\delta(C_f,C₀) \left( 1 + \frac{1}{\gamma - \delta(C_f,C₀)} \right). \]

SCHEME 4.3 Let P₀ ∈ \( \mathbb{F}^{n×m} \).

Step 1 Find a finite dimensional approximation \( P_f \) of \( P₀ \) such that

\[ \delta := \delta(P₀,P_f) < \gamma(P_f). \]

Step 2 Compute an optimally robust controller \( C_f \) for \( P_f \).

THEOREM 4.4 The finite dimensional controller \( C_f \) obtained from Scheme 4.3 will stabilize \( P₀ \) and its neighborhood \( K(P₀,\varepsilon₀) \), where \( \varepsilon₀ = \gamma(P_f)-\delta(P₀,P_f) \). Moreover,

(4.2) \[ \| H(P₀, C_f) - H(P₀, C_f) \| ≤ \gamma⁻¹\delta(P_f,P₀) \left( 1 + \frac{1}{\gamma - \delta(P_f,P₀)} \right). \]

PROOF By definition, \( C_f \) stabilizes \( K(P_f,\gamma(P_f)) \). Since
\[ \delta(P_f,P₁) ≤ \delta(P_f,P₀)+\delta(P₀,P₁) < \delta+\varepsilon₀ = \gamma(P_f) \]
holds for any \( P₁ \in K(P₀,\varepsilon₀) \), \( C_f \) stabilizes \( K(P₀,\varepsilon₀) \). (4.2) follows from (3.7). This completes the proof.

The first step of the Scheme 4.1 is to design an infinite dimensional optimally robust controller \( C₀ \) for \( P₀ \), and if this step can be done the second step is just to compute a finite dimensional approximation for \( C₀ \). In this scheme the approximation is postponed to the last stage and one may expect a better design. The key factors in this scheme are 1) how to perform a design of an infinite dimensional optimally robust controller \( C₀ \), and 2) how to obtain a finite dimensional approximation in the gap metric? The second issue will be addressed later.

The Scheme 4.3 has been used in [Zh. 3] for designing a finite dimensional robust controller for Euler–Bernoulli Beam, and we shall discuss in detail how to implement this scheme. The key step in this scheme is to find a finite dimensional approximation \( P_f \) in the gap metric and the rest is in finite dimensional setting. There are two crucial points in this scheme: 1) Is there a finite dimensional approximation \( P_f \) to \( P₀ \) in the gap metric and what is the efficient way to compute \( P_f \)? 2) Does \( \gamma(P_f) \) approach to zero when \( P_f \) approaches to \( P₀ \) in the gap metric? These two problems will be dealt...
information about model, i.e., no analytical expression of transfer matrix or state space operators needed. As stated in [G-K-L]: a finite dimensional approximation can be obtained just via numerical frequency response data. Further, it is clear that there is no requirements of stabilizability and detectability on the model.

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with successively in the follows.

For systems with finitely many unstable poles, [Zh. 3] has shown that the approximation in the gap metric is equivalent to the approximation in $L_\infty$- norm.

THEOREM 4.5 Suppose that a sequence of systems $\{P_k\}$ and $P_0$ have only finite many unstable poles and without imaginary axis poles. Then the following statements are equivalent:

1) $\{P_k\}$ converges to $P_0$ in the gap metric;

2) $\{P_k\}$ converges to $P_0$ in $L_\infty$- norm and $P_k$ has the same number of unstable poles as $P_0$ when $k$ is sufficiently large.

According to the last theorem, finding a finite dimensional approximation in the gap metric is equivalent to the computing a finite dimensional approximation in $L_\infty$- norm, which has been studied by many scholars [C-G, G-C-P, G-K-L, Pa. etc.]

Note that (4.4) can also be used as an estimate for $\gamma(P)$. By Theorem 4.8 $\gamma(P_k)$ will approach to $\gamma(P_0)$ as $P_k$ approaches to $P_0$ in the gap topology. Hence, as long as $\{P_k\}$ converges to $P_0$ in the gap metric, there will be a $K$ such that $\delta(P_0,P_k) < \gamma(P_k)$ when $k > K$. Thus, Scheme 4.3. can be implemented as long as a finite dimensional approximation in $L_\infty$- norm can be obtained.

Finally, we point out that another advantage of frequency domain approach for designing finite dimensional controllers is that it can handle highly complex systems. Actually, one do not need complete
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