Control of Flexible Systems by Mathematical Programming

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Abstract

A class of control problems for a damped distributed parameter system governed by a system of partial differential equations with side constraints (equality and/or inequality) is considered. The proposed approaches approximate each control force of the system by a Fourier-type series. In contrast to standard linear optimal control approaches, the method used here is based on the Mathematical programming approach, in which the necessary condition of optimality is derived as a system of linear algebraic equations. The proposed approach is easy to apply to a large class of control problems.

Introduction

In most control problems, systems to be controlled are distributed parameter systems which are usually formulated in terms of partial differential equations. Several methods from classical and modern control theory have been used to derive a continuous optimal control law which optimizes the system response according to a specified quadratic criterion. In the present paper, a class of control problems for self-adjoint systems described by systems of linear partial differential equations is considered. A method is proposed to damp the undesirable vibrations in the structures actively by means of a distributed force control (open-loop). Actual implementation of an active distributed control to a dynamical system is usually carried at by optimally placing a finite number of force actuators on the structure ([4], [6]). Ref. [9] provides a comparative study of various actuators used in active structural control. The conditions were obtained in [7] for the existence of a control which approximates a distributed control in the best way possible by applying control at a finite number of points. Methods were formulated in [3] to determine the optimal location of actuators for the control of distributed parameter systems.

The problem of damping out the vibrations of a distributed parameter system described by a single partial differential equation [13] by means of distributed control forces is solved by using the results of the theory [11]. A computational method is also applied to the control problem and the same results are obtained as those from the maximum principle approach [11]. In the present paper the computational method [13] is applied for solving a class of control for self-adjoint systems described by a system of evolution-type linear partial differential equations. The problem of damping out the vibrations of such structures by means of open-loop controls is studied. The basic control problem is to minimize a performance index of the structure in a given period of time with the state and control variables subject to inequality and/or equality constraints. For self-adjoint systems, the eigenfunctions are orthogonal where a linear combination of these eigenfunctions is used to convert the problem to that of control of lumped-parameter systems. Then the proposed method approximates each control function as a Fourier series with unknown coefficients. The coefficients are then determined by solving a system of algebraic
equations in such a way that the necessary conditions for minimizing the performance is imposed.

The present control has been studied in [14], where a maximum principle was formulated to relate control forces to adjoint variables. A direct method is applied to the same type of problems considered in [14] where each control variable of a structural model is approximated by a Fourier-type series. In contrast to the maximum principle approach [14], necessary conditions of optimality are derived as a system of linear algebraic equations by a mathematical programming approach. The approach of this study possesses advantages over the maximum principles method ([2], [5], [8], [11], [12]) which has limited applications. It is computationally efficient and can give the exact solutions when closed-form solutions are available. In general, however, it yields only approximate results. The most attractive feature of the present method is its ultimate simplicity and convenience. The control problems are reduced to the solution of algebraic equations. The solution of a coupled state-adjoint system with terminal conditions, which are always required when applying the pontryaginis maximum principle to the control problems, therefore is avoided.

Statement of Control Problem

The class of optimal control problems we consider is the following: Determine the active control function \( f \in U_{ad} \) (open-loop) which minimizes the cost functional

\[
J(f) = \int_{\Omega} \left\{ g_1[\mathbf{x}, \mathbf{y}(\mathbf{x}, T)] + g_2[\mathbf{x}, \mathbf{y}(\mathbf{x}, T)] \right\} \, d\mathbf{x} + \int_0^T \int_{\Omega} G_1[\mathbf{x}, \mathbf{y}(\mathbf{x}, t), \mathbf{f}(\mathbf{x}, t)] \, d\mathbf{x} \, dt = \sigma_1
\]

Subject to the system of partial differential equations:

\[
A[\mathbf{y}_{tt}(\mathbf{x}, t)] + M[\mathbf{y}_t(\mathbf{x}, t)] - L[\mathbf{y}(\mathbf{x}, t)] = \mathbf{f}(\mathbf{x}, t) \quad \text{for} \ (\mathbf{x}, t) \in \Omega \times [0, T] \tag{2.2}
\]

with homogeneous boundary conditions

\[
B[\mathbf{y}] = [0] \quad \text{on} \ \partial \Omega \times [0, T] \tag{2.3}
\]

initial conditions

\[
\mathbf{y}(\mathbf{x}, 0) = \mathbf{y}^0(\mathbf{x}), \quad \mathbf{y}_t(\mathbf{x}, 0) = \mathbf{y}^1(\mathbf{x}) \quad \text{on} \ \Omega \tag{2.4}
\]

with the following integral constraints:

\[
I_1 = \int_{\Omega} h_1[\mathbf{x}, \mathbf{y}_t(\mathbf{x}, T)] \, d\mathbf{x} + \int_0^T \int_{\Omega} G_1[\mathbf{x}, \mathbf{y}(\mathbf{x}, t), \mathbf{f}(\mathbf{x}, t)] \, d\mathbf{x} \, dt = \sigma_2
\]

\[
I_2 = \int_{\Omega} h_2[\mathbf{x}, \mathbf{y}(\mathbf{x}, T)] \, d\mathbf{x} + \int_0^T \int_{\Omega} G_2[\mathbf{x}, \mathbf{y}(\mathbf{x}, t), \mathbf{f}(\mathbf{x}, t)] \, d\mathbf{x} \, dt = \sigma_2
\]

\[
I_3 = \int_0^T \int_{\Omega} G_1[\mathbf{x}, \mathbf{y}(\mathbf{x}, t), \mathbf{f}(\mathbf{x}, t)] \, d\mathbf{x} \, dt \leq \sigma_i \quad 3 \leq i \leq n^* \tag{2.5}
\]

\[
I_4 = \int_0^T \int_{\Omega} G_1[\mathbf{x}, \mathbf{y}(\mathbf{x}, t), \mathbf{f}(\mathbf{x}, t)] \, d\mathbf{x} \, dt = \sigma_i \quad n^* \leq i \leq n
\]

where

(i) \( \mathbf{y}, \mathbf{y}_t \) and \( \mathbf{f} \) are column vectors \( 1 \times p \),

(ii) \( (\quad)_t \) means differentiation with respect to time \( t \),

(iii) \( \Omega \) is an open connected bounded region in \( \mathbb{R}^p \), with boundary \( \partial \Omega, \mathbf{x} \in \mathbb{R}^p, T \) some fixed real number (time),

(iv) A positive definite constant matrix,

(v) \( M, L, B \) are matrix linear spatial differential operators,

(vi) \( g_i \) and \( h_i \) \( (i = 1, 2) \) are defined on \( \bar{\Omega} \times \mathbb{R}^p \) \((\bar{\Omega} = \Omega \cup \partial \Omega)\), \( G_j \) defined on \( \bar{\Omega} \times \mathbb{R}^p \)
\( \mathcal{D} = \mathbb{R} \times [0, T] \), 
0 \leq j \leq n,

(vii) \( \sigma_j \) are some prior given constants, \( n^* \) and \( n \) are some natural numbers with \( n^* \leq n \).

(viii) \( U_{ad} = \{ f_i \mid f_i \in L^2(\mathcal{D}) \text{ and } f_i \leq \gamma \text{ i.e.,} \}
1 \leq i \leq p \} \) where \( L^2(\mathcal{D}) \) denote the Hilbert space of all real-valued square integrable functions on the region with usual inner product and norm defined by

\[
\langle f, g \rangle = \int_0^T \int_\Omega f(x, t) g(x, t) \, dx \, dt
\]

\[
||f||^2 = \langle f, f \rangle
\]

Throughout the paper, we make the following assumptions:

(A1) \((M, B)\) and \((L, B)\) are self-adjoint.

(A2) There is a complete set of orthogonal eigenfunctions \( \{ \phi^k(\chi) \} \) of \( M \) and \( L \) such that;

\[
M [ \phi^k(\chi)] = \lambda^k \phi^k(\chi)
\]

\[
L [\phi^k(\chi)] = \Gamma^k \phi^k(\chi), \quad k = 1, 2, \ldots
\]

where \( \lambda^k \) and \( \Gamma^k \) are discrete symmetric matrices;

\[
\phi^k(\chi) = [\phi_{-1}^k(\chi), \ldots, \phi_p^k(\chi)]^T, \quad \text{where } ^T \text{ denote transpose of a vector.}
\]

(A3) \( \psi \in L^2(\mathcal{D}), \psi^i \in L^2(\Omega), \quad i = 0, 1. \)

(A4) To each \( f_i \in U_{ad} \), there corresponds a solution \( \psi_i(\chi, t) \) satisfying (2.1) \(-(2.5).\)

Method of Solution

We now consider the system (2.2) \-(2.5) and replace the cost functional (2.1) by a more general case where we include a number of side constraints by using the method of Lagrange multipliers. We proceed in the following way: In order to minimize the cost functional \( J(f) \) given in (2.1) subject to (2.2) \)-(2.5) we minimize the augmented functional

\[
L(f, \lambda) = J(f) + \sum_{i=1}^n \lambda_i (I_i - \sigma_i)
\]

where \( J(f) \) is given by (2.1),

\[
\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]^T, \quad \text{with the state functions subject to (2.2) \)-(2.4).}
\]

Since \( f \in U_{ad} \), then the following converges in \( L^2(\mathcal{D}) \) componentwise,

\[
f(\chi, t) = \sum_{k=1}^\infty Q^k(t) \phi^k(\chi)
\]

where

\[
Q^k(t) = \text{diag} [q^k_{11}(t), \ldots, q^k_{pp}(t)]
\]

\[
q^k_j(t) = \int_\Omega f_j(\chi, t) \phi^k_j(\chi) \, d\chi,
\]

\( 1 \leq j \leq p \)

\( Q^k(t) \) is the only unknown here and we can consider it as a control function for the posed problem. Clearly, we can approximate the unknown control function \( Q^k(t) \) anyway we like, for example by trigonometric approximation

\[
q^k_j(t) = \sum_{r=0}^m \left( a^k_{jk} \sin rt + b^k_{jk} \cos rt \right),
\]

where \( a^k_{jk} \) and \( b^k_{jk} \) are constants to be determined optimally.

One can increase the accuracy of the approximations by simply increasing the number of available terms.

In particular, we take

\[
q^k_j(t) = \sum_{r=1}^m H^k_{jk} q^r_{jk}(t)
\]

where \( q^r_{jk} \) are some known functions, \( H^k_{jk} \)
are unknown constants to be determined.

Inserting (3.2) \-(3.5) into (3.1), we obtain the augmented functional

\[
L(f, \lambda) = L(q^r_{jk}, H^k_{jk}, \lambda)
\]
Now the problem is reduced to that of a mathematical programming problem given by

$$\min \frac{L}{H(j, k; \lambda)}$$

Indeed, a vast number of solution techniques is available to solve (3.7). The method of setting

$$\frac{\partial L}{\partial H^{(r)}} = 0, \quad \frac{\partial L}{\partial \lambda_i} = 0$$

for \(r = 1, ..., N, \quad j, k = 1, ..., p, \quad i = 1, 2, ..., n, \) and solving the resulting algebraic equations for \(H^{(r)}\) and \(\lambda_i\) is only one of the available methods and we can use it here because of the special form of \(q^{(k)}(t)\), \(j, k = 1, ..., p\). Thus, we can determine the constants \(H^{(r)}\) and \(\lambda_i\) from (3.8) and obtain the solution of the problem (2.1) – (2.5).

Conclusions

This paper presents an efficient alternative to standard approaches for the problem of suppressing the vibrations of structures by means of distributed control forces. The dynamic behavior of these structures is described by a system of partial differential equations. The objective of the control is to minimize a prescribed performance index of the structure in a given period of time with side conditions on the state and control variables. The approach relies on a Fourier-based approximation of the control vector and converts the optimal control problem into a simple mathematical programming problem. This approach avoids formulation of the costate equations as the case with the maximum principles in [2, 5, 8, 10, 11]. The major advantage of the approach is that it is computationally easier to handle than the standard approaches for solving optimal control problems. A second advantage of the approach is that it is easily adaptable to many problems in structural mechanics. But obviously there is a certain price that has been paid for using the direct approach since it doesn’t provide sufficient conditions of optimality while the standard control approaches sometimes provide sufficiency.

References


