ABSTRACT

In this paper the well-known Markus-Yamabe example [1] is revisited in a more general setting and in light of the Floquet Characteristic Exponent Theory which leads to some interesting and enlightening results. These results provide deeper insight into and better understanding of the concept of frozen-time eigenvalues and Floquet Characteristic Exponents. They are also useful in the robustness analysis for linear time-invariant systems, since the nominal eigenvalues in that case are the frozen-time eigenvalues of the perturbed systems used in such analysis.

A Case Study Of Frozen-Time Eigenvalues In The Stability Analysis For Periodic Linear Systems

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\[ \mathbf{A}(t) = \begin{bmatrix} -1 + \frac{3}{2}\cos \omega t & 1 - \frac{3}{2}\cos \omega t \sin \omega t \\ -1 - \frac{3}{2}\cos \omega t \sin \omega t & -1 + \frac{3}{2}\sin \omega t \end{bmatrix} \]  

The frozen-time eigenvalues of this system are given as solutions to the equation:

\[ \lambda^2 + \lambda + \frac{1}{2} = 0 \]  

which yields frozen-time eigenvalues:

\[ \lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} \]

The state-transition matrix \( \Phi(t,0) \) of (1) can easily be verified to be:

\[ \Phi(t,0) = \begin{bmatrix} e^{\frac{t}{2}} \cos \frac{t}{2} & e^{-\frac{t}{2}} \sin \frac{t}{2} \\ -e^{\frac{t}{2}} \sin \frac{t}{2} & e^{-\frac{t}{2}} \cos \frac{t}{2} \end{bmatrix} \]

which leads to the interesting observation that although the frozen-time eigenvalues of (1) are fixed in the LHP for all time, the system is actually unstable [1]!

The main purpose of the present work is to take a deeper look into the Markus-Yamabe equation in light of the Floquet Characteristic Exponent Theory. In a nutshell, this theory states that the stability of a periodic linear system is actually governed by its Floquet Characteristic Exponents, which are the constant eigenvalues of an associated time-invariant linear system. This can be summarized as a theorem as follows:

**Theorem 1**: For a periodic linear system of the form \( \mathbf{A}(t+T) = \mathbf{A}(t) \) to be asymptotically stable, it is both necessary and sufficient that all the Floquet Characteristic Exponents have negative real parts.

In particular, the Floquet Characteristic Exponents \( \rho_{1,2} \) of the Markus-Yamabe example are given by:

\[ \rho_1 = -1 \ , \ \rho_2 = \frac{1}{2} \]

which indeed imply instability.

In this paper, we will introduce a parametrized version of the classical Markus-Yamabe example which will allow us to demonstrate:
i) The domain of validity, in the parameter plane, for the frozen-time LHP stability criterion through closed-form solution for the Floquet Exponents for a class of periodic linear systems.

ii) A numerical analysis technique that can be employed for a general periodic linear systems, for which closed-form solutions for the Floquet Exponents are not available, in order to assess the asymptotic stability in light of the Floquet Characteristic Exponent Theory, without recourse to costly, complicated and time-consuming simulations.

This study provides a deeper insight into and a better understanding of the concepts of frozen-time eigenvalues and Floquet Characteristic exponents. It also provides conclusive evidence that, at least for periodic linear systems, stability analysis based on these eigenvalues highly unreliable, unless the parameters of the periodic system sufficiently small and slow variations. The numerical analysis technique used in this study is a well-known one, but its feasibility, with the aid of today's computers, appears to be under-appreciated. The numerical case studies presented in this paper serve to unveil the utility and feasibility of such numerical methods for the stability analysis of general periodic linear systems.

2. MAIN RESULTS

In this paper we examine the following parametrized version of the Markus-Yamabe equation, where the matrix $A(t)$ of (1) takes the form:

$$
A(t) = \begin{bmatrix}
-b + a \cos \omega t & -a \cos \omega t \sin \omega t \\
-b - a \cos \omega t \sin \omega t & b + a \sin \omega t
\end{bmatrix}
$$

Clearly, when $b = 1$, $a = 3/2$ and $\omega = 1$, (4) reduces to the original Markus-Yamabe equation.

From (4), the frozen-time eigenvalues of the generalized Markus-Yamabe equation are given by:

$$
\lambda_1,2 = \left(\frac{a-2b}{2} + \frac{a^2 - 4}{2}\right) e^{\pm \omega t}
$$

which shows that the frozen-time eigenvalues are independent of both $t$ as well $\omega$. According to the "frozen-time LHP stability criterion", the domain of asymptotic stability for the system, in the $a-\omega$ parameter plane is given by:

$$
a < 2b
$$

However, the Floquet exponents of (1) are, as will be shown subsequently, dependent on $\omega$. This means that even for $a < 2b$, a stable, time-invariant linear system ($\omega = 0$) can start wandering into an unstable, fast oscillating time-varying mode while maintaining its eigenvalues fixed in the LHP. This point will be demonstrated in the following two case studies.

Case 1: $b = \omega$.

For the class of periodic linear systems described by (1) with $b = \omega$ in (4), it is readily verified that the state-transition matrix $\Phi(t,0)$ is given by:

$$
\Phi(t,0) = \begin{bmatrix}
e^{(a-\omega)t} \cos \omega t & e^{(a-\omega)t} \sin \omega t \\
-e^{(a-\omega)t} \sin \omega t & e^{(a-\omega)t} \cos \omega t
\end{bmatrix}
$$

which can be factored as,

$$
\Phi(t,0) = L(t)Y(t,0)
$$

where,

$$
L(t) = \begin{bmatrix}
\cos \omega t & \sin \omega t \\
-\sin \omega t & \cos \omega t
\end{bmatrix}
$$

is a continuously differentiable, non-singular, periodic matrix and

$$
Y(t,0) = \begin{bmatrix}
e^{(a-\omega)t} & 0 \\
0 & e^{(a-\omega)t}
\end{bmatrix}
$$

is a fundamental solution matrix of an associated linear time-invariant system $\dot{y} = By$ where $B$ is given by:

$$
B = L^{-1}(t) \left[ A(t)L(t) - I(t) \right]
$$

It can be easily verified that $Y(t,0)$ is indeed the fundamental solution of the LTI system $\dot{y} = By$, $y(t_0) = 0$.

We know that (1) will be asymptotically stable if $\Phi(t,0) \to 0$ as $t \to \infty$. From (7) and (8) it can be concluded that asymptotic stability of (1) will be guaranteed by the negative definiteness of the real parts of the eigenvalues of $B$, which are, by definition, the Floquet Characteristic Exponents for (1). Therefore, for the case $b = \omega$ in (4), the Floquet Characteristic Exponents $\mu_{1,2}$ are:

$$
\mu_1 = (a - \omega), \quad \mu_2 = -\omega
$$

which leads to the domain of asymptotic stability, in the $a-\omega$ parameter plane, as

$$
a < \omega, \text{ and } \omega > 0
$$

Note that, for $\omega > 0$ the true domain of stability assessed by the Floquet Characteristic Exponents is properly
contains in the domain of stability $a < 2\omega$ determined by the frozen-time eigenvalues. The intersection of the two domains represents the domain of validity of the "frozen-time LAP stability criterion" and the difference of the two domains represents the region of false stability brought in by the abuse of the LTI criterion to the LTV case. This comparison is depicted in Fig.1. The above analytical result not only reinforces the original Markus-Yamabe example in demonstrating the failure of the sufficiency part of the "frozen-time LAP criterion", it also shows why and how it fails in light of the well-established Floquet stability theory for periodic systems.

\[ F = \begin{bmatrix} x_{11}(t_f) & x_{12}(t_f) \\ x_{21}(t_f) & x_{22}(t_f) \end{bmatrix} \]

which can be shown to be a Floquet Characteristic Matrix for (1). The eigenvalues of $\hat{F}$ are known as the Floquet Characteristic Multipliers $\gamma_i$, from which the Floquet Characteristic Exponents $\mu_i$ can be obtained using the relationship

\[ \mu_i = \frac{\ln \gamma_i}{t_f - t_0} \]  

which is all we need to assess the stability of the system under consideration. It is to be noted here that the steps outlined above can be generalized for any $n \times n$ periodic linear system.

An interesting observation from (10) is that the necessary and sufficient criterion for (1) based on the negative definiteness of the real parts of the Floquet Characteristic Exponents $\mu_i$, translates into an equivalent necessary and sufficient condition in terms of the Floquet Characteristic Multipliers $\gamma_i$ of (1) which can be stated as an alternative theorem:

**Floquet Theorem II**: For a periodic linear system of the form (1) with $A(t+T) = A(t)$ to be asymptotically stable, it is both necessary and sufficient that all Floquet Characteristic Multipliers $\gamma_i$ of that system have modulii less than 1.

Using the technique described above, the stability domain for the class of periodic linear systems described by (1) and (4) have been determined for two illustrative cases viz. $b = 1$ and $b = 1.5$. The results are shown in the following two sub-sections.

**Case 2a**: $b = 1$

In order to determine the domain of validity of the frozen-time LAP stability criterion in the $a-\omega$ parameter plane, it is necessary to plot the variations of the real parts of the Floquet Characteristic Exponents with respect to both $a$ as well as $\omega$. This is illustrated in Fig.2. Here for the sake of brevity, we only consider two cases viz. variation of $\omega$ between 0 and 1 with $a = 1.5$ (Fig.1) and variation of $a$ between 0 and 1.5 with $\omega = 1$ (Fig.3). For a more practical analysis, multiple such cases are to be considered with different values for the constant parameter over their operating range and all such data are then to be interpolated in the $a-\omega$ parameter plane to determine the true domain of stability. The task, though seemingly forbidding, can be very easily achieved through conceptually simple programs.

A few interesting observations can be made from Figs.2 and 3.

a) For very small values of $\omega$ ($\omega < 0.25$) the real parts of the Floquet Exponents are quite close to the frozen-time eigenvalues.

b) For $b = 1$, the domain of stability is $a < 1$, which agrees with the previous analytical case.
In this case we also consider only two illustrative examples viz. variation of \( \omega \) between 0 and 1 with \( a = 2 \) (Fig.4) and variation of \( a \) between 0 and 2 with \( \omega = 1 \) (Fig.5). Like in the previous case, we observe again that for a very small range of \( \omega \) (0.15 < \( \omega \) < 0.5) the real parts of the frozen-time eigenvalues and of the Floquet Characteristic Exponents are very close to each other.

However, as the magnitude \( a \) and/or the frequency \( \omega \) of variation of the parameter increases, the frozen-time eigenvalues depart significantly from the Floquet Characteristic exponents, and consequently fail to predict the stability accurately. This observation agrees quite well with the perturbational analysis - that the frozen-time LHP stability criterion can be used to assess the stability of slowly time-varying linear systems and to study the robustness of LTI systems under small parameter perturbations.

3. SUMMARY AND CONCLUSIONS

In this study, we have presented a generalized Markus-Yamabe equation which has enabled us to take a closer look at the frozen-time LHP stability criterion, in light of the classical Floquet stability theory for periodic linear systems, to show why and how it fails to provide accurate stability assessment in the case of periodic linear systems. The feasibility of a theoretically sound and conceptually simple numerical analysis technique has been demonstrated which is based on the Floquet stability theory for linear periodic systems and which, more importantly, can be applied in practice for any periodic linear system. The results presented in this paper are not only interesting and enlightening but also useful in practical analysis and design of periodic linear control systems, and in robustness analysis of LTI systems.

4. REFERENCES