Signal Distortions That Result From Minimum Phase Signal Recovery Using Cepstral Processing

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Abstract

This paper presents a technique that can be used to partially recover the transmitted signal from a received signal that is composed of time delayed versions of the transmitted signal. The recovered signal is expressed analytically in the frequency domain. An approximation for the extracted signal in the time domain is also obtained. Here, the frequency representation of the transmitted signal is restricted to a specific class of functions, namely minimum phase functions. Except in a few cases, the extracted signal is distorted in comparison to the transmitted signal. Consequently, these analytic expressions can be used to determine how the time delays and the form of the transmitted signal affect the amount of distortion observed.

Introduction

The problem of trying to recover a transmitted signal that has been mixed with an unwanted signal that occupies the same frequency band is often encountered in digital signal processing. Of particular interest is the case of multi-path signals that can occur when one or more reflecting surfaces lie between the transmitter and a receiver. In these cases, the received signal can be modeled as the original signal plus time delayed, scaled replicas of the original signal; therefore the unwanted signal has exactly the same frequency signature as the original signal. One method of recovering the desired signal is to use a variant of homomorphic signal processing that operates on the complex cepstrum of the signal. Although this process is conceptually simple, practical considerations quite often make it difficult. It is particularly difficult to predict the distortion of the original signal caused by using these techniques. It is particularly difficult to predict the distortion of the original signal in the frequency domain to be approximated.

In the first part of this paper, an overview of cepstral processing is given. Next, the theory associated with partially recovering the transmitted signal from a received multipath signal is discussed. Afterwards, an example is presented in order to illustrate the application of the theory. Finally, a summary is given and some consequences of the theory are discussed.

Cepstral Processing

The principle of superposition for linear systems requires that if \( T \) is the system transformation, then for any input \( x_1(n) \) and \( x_2(n) \) and any scalar \( c \)

\[
T [x_1(n) + x_2(n)] = T [x_1(n)] + T [x_2(n)]
\]

(1)

and

\[
T [cx_1(n)] = c T [x_1(n)].
\]

(2)

It is difficult to extract the signal \( x_1(n) \) from the total signal \( x_1(n) + x_2(n) \) if \( x_1(n) \) and \( x_2(n) \) overlap in the original space. However, if \( T[x_1(n)] \) and \( T[x_2(n)] \) overlap very little or not at all, then it is possible to filter \( T[x_1(n)] \) from the left hand side of Eq. (1) and apply \( T^{-1} \) to secure \( x_2(n) \). This, of course, assumes that enough is known about \( x_1(n) \) to determine where \( T[x_1(n)] \) is localized in the transform space.

If the system under consideration is non-linear, then the above technique does not hold. To generalize the above technique, consider the following: in the original vector space there is a rule denoted by \( \circ \) for combining inputs with each other (e.g., addition, multiplication, or convolution) and a rule denoted by \( \odot \) for combining inputs with scalars. If there exists a transform, \( D_o \), to a linear space such that

\[
D_o[y_1(n) \circ y_2(n)] = D_o[y_1(n)] + D_o[y_2(n)],
\]

(3)

and

\[
D_o[c \odot y_1(n)] = c D_o[y_1(n)],
\]

(4)

and an inverse \( D_o^{-1} \), then \( D_o[y_1(n)] \) can replace \( x_1(n) \) and \( D_o[y_2(n)] \) can replace \( x_2(n) \) in the previous argument to extract \( y_1(n) \) from the total signal \( y_1(n) \circ y_2(n) \). This is called homomorphic signal processing. Of special interest is homomorphic signal processing when \( \circ \) represents a convolution since multipath signals can be modeled as a
convolution of delayed delta functions with the transmitted signal. In this case, homomorphic signal processing is called cepstral processing [1,2,3].

Consider the case when two time domain functions are convolved together, one a desired function, the transmitted signal, and the other function unwanted, a sum of time delayed delta functions. In the frequency domain, the Fourier transform (FT) of the two functions are multiplied. If the two functions occupy the same frequency band, and for the two functions mentioned above they do, frequency-dependent filtering cannot be used to remove the unwanted function. Taking the logarithm of the multiplied functions and inverse Fourier transforming to the cepstral (time) domain results in functions that now intersect in a region, or regions, where hopefully both of their amplitudes are relatively small. In the cepstral domain, normal filtering can be applied to remove what corresponds to the unwanted function. The desired function, the transmitted signal, can be partially recovered by taking the FT of what remains in the cepstral domain, exponentiating, and then inverse Fourier transforming.

### Signal Recovery

#### Theory

The received signals that will be considered in this paper have the form

\[
x(z) = \frac{K[1+\beta z^{-N} + \sigma z^{-2N}] \prod_{k=1}^{N_1} [1+d_k z^{-1}]}{\prod_{n=1}^{N_2} [1-2r_n \cos(nN) z^{-1} + (r_n z^{-1})^2]}
\]

in the Z-transform, or frequency, domain. Here \(|\beta| < 1, |\sigma| < 1, |d_k| < 1, (k=1,2,\ldots,N_1), |r_n| < 1, 0 \leq \Theta_n < 2\pi, (n=1,2,\ldots,N_2), |K| > 0 \) and \(N\) is the number of units separating the echoes. The transmitted waveform, \(v(z)\), is given by

\[
v(z) = \frac{K \prod_{k=1}^{N_1} [1+d_k z^{-1}]}{\prod_{n=1}^{N_2} [1-2r_n \cos(nN) z^{-1} + (r_n z^{-1})^2]}
\]

and the term

\[
p(z) = [1 + \beta z^{-N} + \sigma z^{-2N}]
\]

models the reverberations or echoes. Hence,

\[
x(n) = v(n) + \beta x(n-N) + \sigma x(n-2N),
\]

which is the strength of the signal in the time domain. Since this signal is already in the frequency domain, the first step is to apply a logarithmic transform to \(x(z)\). This yields

\[
x(z) = \log(K) + \log[1 + \beta z^{-N}] + \log[1 + \sigma z^{-2N}]
\]

\[
+ \sum_{n=1}^{N_1} \log[1 + d_n z^{-1}] + \sum_{n=1}^{N_2} \log[1 + r_n z^{-1} + \sum_{k=1}^{N_1} \log[1 + r_n z^{-1}],
\]

where \(\tilde{x}(z)\) is used to denote \(\log[x(z)]\) and

\[
a = -2\sigma[\beta + (\beta^2 - 4\sigma)^{1/2}],
\]

and

\[
a' = -2\sigma(\beta - (\beta^2 - 4\sigma)^{1/2}).
\]

Using the expansion

\[
\log[1 + z] = \sum_{n=1}^{\infty} (-1)^n z^n / n,
\]

Eq. (8) can be written as

\[
x(z) = \log(K) + \sum_{m=1}^{N_1} \frac{(-\alpha)^m z^{-mN}}{m} + \sum_{m=1}^{N_2} \frac{(a')^m z^{-mN}}{m} - \sum_{m=1}^{N_1} \frac{2(r)^m \cos(m\theta) z^{-mN}}{m}.
\]

Therefore, the inverse Z-transform of \(\tilde{x}(z)\) is given by

\[
x(m) = 0
\]

for \(m < 0\),

\[
x(m) = \log(K)
\]

for \(m = 0\),

\[
x(m) = \frac{(-\alpha)^m}{m} + \frac{2(r)^m \cos(m\theta)}{m}
\]

for \(m > 0\) and \(m \neq iN\) (where \(i\) is any real, positive integer), and by

\[
x(m) = \frac{(-\alpha)^m}{m} + \frac{2(r)^m \cos(m\theta)}{m}
\]

for \(m > 0\) and \(m = iN\).

The set of Eqs. (12) - (15) represent the signal \(x(n)\) in the cepstral domain. It is clear from these equations that the only dependence of \(\tilde{x}(m)\) on the term responsible for the reverberations (or echoes),

\[
[1 + \beta z^{-N} + \sigma z^{-2N}]
\]
occurs when \( m = \pm N \). To extract these terms from \( \hat{x}(m) \) the filter \( \tilde{E}(m) \),

\[
\tilde{E}(m) = \left[1 - \sum_{i=1}^{N} \delta(m - iN)\right]
\]

(16)
can be applied. When this filter is applied, the resulting function,

\[
\tilde{y}(m) = \tilde{E}(m) \hat{x}(m),
\]

is still given by Eq. (12) for \( m < 0 \), Eq. (13) for \( m = 0 \), and Eq. (14) for \( m > 0 \) and \( m \neq \pm N \). However,

\[
\tilde{y}(m) = 0.
\]

(17)

One way of obtaining sums for \( \hat{y}_z(z) \) whose functional forms are known is to express \( \hat{y}_z(z) \) as

\[
\hat{y}_z(z) = (-kN) \sum_{i=1}^{N} 2(r^{N} \cos(m\Theta) z^{-N} + \sum_{k=1}^{N} \frac{2(r^{N} \cos(m\Theta) z^{-N}}{mN} - \sum_{k=1}^{N} \frac{2(r^{N} \cos(m\Theta) z^{-N}}{mN} + \sum_{k=1}^{N} \frac{2(r^{N} \cos(m\Theta) z^{-N}}{mN} - \sum_{k=1}^{N} \frac{2(r^{N} \cos(m\Theta) z^{-N}}{mN}.
\]

(18)

which allows the sums to be complete in the Z-domain. Thus, \( \hat{y}_z(z) \) becomes

\[
\hat{y}_z(z) = \log(k) - \sum_{k=1}^{N} \log[m(k - 1) + \sum_{k=1}^{N} \log[m(k - 1) - r^{N} \cos(m\Theta) z^{-N} - \sum_{k=1}^{N} \log[m(k - 1) - r^{N} \cos(m\Theta) z^{-N} + \sum_{k=1}^{N} \log[m(k - 1) - r^{N} \cos(m\Theta) z^{-N} + \sum_{k=1}^{N} \log[m(k - 1) - r^{N} \cos(m\Theta) z^{-N}.
\]

(19)

or

\[
\hat{y}_z(z) = \log(k) + \sum_{k=1}^{N} \log[1 - r^{N} \cos(m\Theta) z^{-N} - \sum_{k=1}^{N} \log[1 - r^{N} \cos(m\Theta) z^{-N} + \sum_{k=1}^{N} \log[1 - r^{N} \cos(m\Theta) z^{-N} + \sum_{k=1}^{N} \log[1 - r^{N} \cos(m\Theta) z^{-N}.
\]

(20)

By taking the inverse logarithm of \( \hat{y}_z(z) \), one obtains the new signal in terms of the original signal:

\[
\hat{y}(z) = \frac{\log[1 - r^{N} \cos(m\Theta) z^{-N}]}{\log(k)}.
\]

(21)

Using this equation it is possible to make a direct comparison of \( \hat{y}_z(z) \) to \( v(z) \).

A comparison between \( v(n) \) and \( \hat{y}_z(n) \) requires that the terms to the \( 1/N^2 \) power in Eq. (21) each undergo a binomial expansion. The result of such an expansion is given by

\[
\hat{y}_z(n) = \sum_{k=0}^{N} a_k n^k \cdot
\]

(22)

which leads to the following representation for \( \hat{y}_z(n) \):

\[
\hat{y}_z(n) = \sum_{k=0}^{N} a_k n^k.
\]

(23)

By computing this sum to a given value of \( k \), one can obtain \( \hat{y}_z(n) \) to arbitrary accuracy.

Example

As an example, this technique will be applied to

\[
\rho(n) = d(n) + \delta(n - N) + \beta^2 \delta(n - 2N),
\]

(24)

\[
v(z) = \frac{1 - z^{-1}}{[1 - 2 \cos(\pi/6) z^{-1} + r^2 z^{-2}]}.
\]

(25)

where \( K = 1, N = 15, r = 0.9, d_1 = r[-\cos(\pi/6) + (3.9)\sin(\pi/6)], \Theta = \pi/6, \) and \( \beta = 0.9. \) In the time domain the signal is given by

\[
v(n) = (0.9)^n \cos(n\pi/6) + (3.9)\sin(n\pi/6),
\]

(26)

and \( x(n) \), the received signal that is corrupted by multipath, is given by

\[
x(n) = (0.9)^n \cos(n\pi/6) + (3.9)\sin(n\pi/6),
\]

(27)

and \( x(n) \), the received signal that is corrupted by multipath, is given by

The graphs of \( v(n) \) and \( x(n) \) are given in Figs. 1 and 2. The graph of the received signal that has been transformed to the cepstral domain is shown in Fig. 3. After filtering \( \tilde{x}(m) \) with the function

\[
\tilde{E}(m) = \left[1 - \sum_{k=1}^{N} \delta(m - 15k)\right]
\]

382
one obtains \( \hat{y}(m) \) and subsequently \( Y(z) \), which is given by

\[
Y(z) = \frac{\gamma(z)(1 - (2.5)z^{-1} + 2.5z^{-2} - z^{-3})}{1 + (2.5)z^{-1} - (3.9)z^{-2} + (2.9)z^{-3}}.
\]

Expanding the denominator and numerator [excluding \( \gamma(z) \)], it is possible to secure the first five terms of the infinite series in Eq. (22). \( Y(z) \) is then approximated by

\[
Y(z) = (1 + a_1 z^{-15} + a_2 z^{-30} + a_3 z^{-45} + a_4 z^{-60} + a_5 z^{-75})
\]

where

\[
a_1 = \frac{(d_i)_{15}}{15}, \quad a_2 = \frac{(d_i)_{30}}{225}, \quad a_3 = \frac{(d_i)_{45}}{3375}, \quad a_4 = \frac{(d_i)_{60}}{18,900(15)^4}, \quad a_5 = \frac{(d_i)_{75}}{(9(15)^4)}.
\]

Therefore, \( Y_n(n) \) can be approximated by

\[
Y_n(n) = \gamma(n) + a_1 y(n-15) + a_2 y(n-30) + a_3 y(n-45) + a_4 y(n-60).
\]

The graph of \( Y_n(n) \) is presented in Fig. 4 and the difference between this approximation and \( \gamma(n) \) is graphed in Fig. 5. This polynomial approximation of \( Y_n(n) \) only contains five terms; however, many more terms can be secured using commercial software such as DERIVE [4] or MACSYMA [5].
Results and Conclusions

In this paper, cepstral processing has been demonstrated as a method for extracting an analytical expression in the frequency domain that approximates the transmitted waveform when the received waveform is given by Eq. (5). The accuracy to which this extracted term, \( Y_r(z) \), approximates the frequency representation of the transmitted waveform depends on two quantities. The first quantity is the delay time of the echoes and the second is the rate at which the magnitude of the cepstral representation of the transmitted waveform, \( \Phi(n) \), decreases as the quefrency (or time), \( \xi \), in the cepstral domain increases (\( \xi > 0 \)). These quantities affect \( Y_r(z) \) in the following way. First, for a given \( N \), the more peaked \( \Phi(n) \) is about the origin the smaller are the contributions of \( \Phi(n), i=1,2,\ldots \) to the total function \( \Phi(n), n=1,2,\ldots \). Second, for a given \( \Phi(n) \) that is assumed to go to zero as \( n \to \infty \), the contribution of \( \Phi(n), i=1,2,\ldots \) to \( \Phi(n), n=1,2,\ldots \), decreases as \( N \) tends to infinity. Once the extracted term \( Y_r(z) \) has been obtained it is possible to secure an expression for the recovered signal in the time domain, Eq. (23), by expanding \( Y_r(z) \) in the form of Eq. (22).

The technique developed in this paper has been applied to received signals having the form of Eq. (5), which has no poles or zeros outside the unit circle. There are other forms that \( v(z) \) can take such as

\[
v(z) = \prod_{k=1}^{N_1} \frac{1+e_k z^{-1}}{1+e_k z^{-1}} \prod_{m=1}^{N_2} \frac{1+f_m z^{-1}}{1+f_m z^{-1}},
\]

which is a combination having poles and zeros both inside and outside of the unit circle. This technique should be directly applicable to transmitted functions having these forms with echoes in the received signal. Additionally, it may be possible to apply this technique to the analog form of signals as well.

The example used was simple and does not necessarily represent a signal of interest. However, it appears that the generation of more realistic (complex) signals is possible with the general set of functions given in Eq. (36). If it is possible to generate more realistic signals having analytical representations in the frequency domain, the cepstral processing technique that has been developed in this paper could be used to analyze problems that arise in real systems. For example, if it is known that a system will receive multipath signal having certain time delays, it may be desirable to search for a waveform that will minimize the distortion observed in the extracted waveform. As another example, suppose that a system has a given waveform; then the effect that different time delays in multipath signals have on the recovery of the transmitted waveform can be investigated using the results of this technique.

References

5. MACSYMA, produced by Symbolics, Inc, 8 New England Executive Park, East, Burlington, Massachusetts 01803.