Abstract

This paper presents results on reachability and constrained controllability properties for the linear control system

\[ \dot{x} = Ax - u; u \in U, x(t_f) = p \]

where \( A \in \mathbb{R}^{n \times n}, u : \mathbb{R}^1 \to U \) is a measurable vector valued function, \( U \) is a compact subset of \( \mathbb{R}^n \) containing the origin and \( t_f \) is the terminal time. Existing results are for the particular case when \( p \) is the origin of the state space.

These geometric characterizations are important for many reasons, which include the following. Nonlinear systems, especially bilinear systems, do not have a priori preferred origins to be referred to and one natural way to find solutions of these systems is to linearize the model, but not necessarily at the origin. Reachable set properties are also important for derivation of minimum-time control strategies and constrained controllability properties of dynamic systems.

For the given system, the reachable set to a point \( p \in \mathbb{R}^n \), defined as \( \mathcal{R}(p) \), is the set of all initial points from which \( p \) can be reached in nonnegative time, as the control \( u \) ranges over the set of admissible values in \( U \).

Results presented include properties of \( \mathcal{R}(p) \) such as openness, convexity and inclusion of a reachable set in another reachable set. We show that the boundedness of \( \mathcal{R}(p) \) is directly related to the region where the spectrum of \( A \) lies in the complex plane. We also define constrained controllability of the system in terms of its reachable set to point \( p \) with a new constraint set \( U - Ap \).

We prove that controllability of the system at \( p \) is equivalent to an inclusion property of reachable sets for certain positive times. We also present geometric properties of \( G \), the set of all nonnegative times at which \( p \) is controllable.

1. Introduction

Reachability and constrained controllability properties for linear finite dimensional systems with controls constrained within a compact set containing the origin, are presented in this paper. Existing results emphasized the case when the origin is the terminal point (Ref. 1). We will present new results for any point \( p \) in \( \mathbb{R}^n \), with \( p = 0 \) a particular case of the general results. These results are important for the following reasons: Nonlinear systems do not have a priori defined origins to be referred to. This is especially true for bilinear systems where the origin is the point to be avoided. The question is to find the best set of points about which to linearize the system. Also, some types of nonlinear systems are linear in a wide range of their variables and it is necessary to investigate control theoretic properties at all the possible ranges of such variables. Furthermore, reachable sets are useful for derivation of optimum-time control strategies and constrained controllability properties of dynamic systems.

The geometric properties of the reachable set depends on the terminal point \( p \) in \( \mathbb{R}^n \), the admissible control set \( U \) and the spectrum of the system. We will present some results on the topological properties of reachable sets and constrained controllable points for a given linear system.

In section 2 we define the notion of a reachable set and constrained controllability for a point \( p \) in \( \mathbb{R}^n \) for a given linear time invariant system with controls constrained within a compact admissible set \( U \) containing the origin.

Section 3 presents the addition lemma for reachable sets and provides conditions for which a reachable set is a proper subset of another reachable set for a given system (inclusion properties). In section 4 we show that linear systems, with spectra in the open right half of the complex plane have reachable sets that are bounded proper subsets of \( \mathbb{R}^n \). Also if the spectrum of the system is in the closed left half of the plane, then the reachable set is a convex, open neighborhood of the origin. In section 5 we prove that constrained
controllability of a linear system at a point \( p \) in \( \mathbb{R}^n \) is equivalent to the origin being in the interior of its reachable set to \( p \), with a new admissible control set \( U = A \mathcal{P} \). We also relate this constrained controllability lemma to inclusion results presented on reachable sets in Section 3. Furthermore, we show that \( G \), the set of all nonnegative times at which point \( p \) is constrained controllable, is an open, additive subset of \( \mathbb{R}^+ = \{ t : t \geq 0 \} \) and \( G \) contains a right unbounded interval in \( \mathbb{R}^+ \).

Section 6 presents examples that illustrate the results and Section 7 concludes the paper.

2. Definitions

Consider the linear control system on \( \mathbb{R}^n \)

\[
\dot{x} = Ax - u; u \in U, x(t_f) = p
\]

where \( A \) is a real \( n \times n \) matrix, \( u : \mathbb{R}^1 \rightarrow U \) is a measurable vector-valued function and \( U \) is a compact subset of \( \mathbb{R}^m \) containing the origin. An example of \( U \) is

\[
U = \{ Bv : |v_i| \leq 1 \text{ for all } i = 1, 2, \ldots, m \}
\]

where \( B \) is an \( n \times m \) real matrix.

Reachability

The reachable set to point \( p \) at time \( t > 0 \) for system (1), denoted as \( \mathcal{R}_t(p) \), is the set of all points from which a given point \( p \) can be reached at time \( t \). The variation of parameters formula yields

\[
\mathcal{R}_t(p) = \exp(-At)p + \int_0^t \exp(-As)Uds
\]

with

\[
\mathcal{R}_t(0) = \{ \int_0^t \exp(-As)u(s)ds : \text{measurable } u : \mathbb{R}^1 \rightarrow U, t \geq 0 \}
\]

The reachable set to point \( p \), which is the set of all points which can be steered to \( p \) at various times \( t > 0 \), is then defined as

\[
\mathcal{R}(p) = \bigcup_{t \geq 0} \mathcal{R}_t(p)
\]

Attainability

\( \mathcal{A}(p) \), the attainable set from point \( p \), at time \( t \geq 0 \) is the set of all states which can be reached from \( p \) using admissible control functions. Thus

\[
\mathcal{A}(p) = \exp(-At)p - \int_0^t \exp(As)Uds = \exp(At)(p - \mathcal{R}_t(0)).
\]

\( \mathcal{A}(p) \), the attainable set from point \( p \), is defined as

\[
\mathcal{A}(p) = \bigcup_{t \geq 0} \mathcal{A}_t(p).
\]

Constrained Controllability

System (1) is constrained controllable to \( p \) at time \( t > 0 \) if, and only if, \( p \) is in the interior of \( \mathcal{R}_t(p) \). This definition states that all points in some neighborhood of \( p \) can be steered to \( p \), in the same time \( t \), by admissible controls.

System (1) is constrained controllable to \( p \) if it is constrained controllable to \( p \) at some time \( t \geq 0 \). When \( U = \{ Bv : v \in \mathbb{R}^m \} \), constrained controllability to an arbitrary point \( p \) is equivalent to the familiar rank condition on the matrix \( \begin{bmatrix} A^{n-1}B & \ldots & AB & B \end{bmatrix} \).

3. Properties of Reachable Sets

We present lemmas on general properties of the reachable sets to point \( p \) for system (1).

Lemma 3.1 Given system (1) with positive times \( t, s \).

(i) \( \mathcal{L}_t(p) = \mathcal{R}_t(p) - p \)

(ii) \( \mathcal{W}_t(0) = \int_0^t \exp(-A\tau)(U - Ap)d\tau \)

where \( \mathcal{L}_t(p) \) the reachable set to point \( p \) at time \( s + t > 0 \), has the following properties:

(i) \( \mathcal{L}_{s+t}(p) = \mathcal{L}_s(p) + \mathcal{L}_t(p) \)

(ii) \( \mathcal{L}_{s+t}(p) = \mathcal{R}_s(p) + \mathcal{L}_t(p) \)

The addition formula for reachable sets to the origin (Ref. 5) states

\[
\mathcal{R}_{s+t}(0) = \mathcal{R}(0) + \exp(-As)\mathcal{R}(s) \]

Proof: From equation (3),

\[
\mathcal{R}_{s+t}(0) = \mathcal{R}(0) + \exp(-As)\mathcal{R}(s)
\]

The addition formula for reachable sets to the origin (Ref. 5) states

\[
\mathcal{R}_{s+t}(0) = \mathcal{R}(0) + \exp(-As)\mathcal{R}(s)
\]

Further, \( \mathcal{R}_{s+t}(p) = \exp(-As)\exp(-At)p + \mathcal{R}(0) + \exp(-As)\mathcal{R}(s) \)

or

\[
\mathcal{R}_{s+t}(p) = \exp(-As)\exp(-At)p + \mathcal{R}(0) + \exp(-As)\mathcal{R}(s)
\]

or

\[
\mathcal{R}(p) = \bigcup_{t \geq 0} \mathcal{R}_t(p)
\]
which implies
\[ \mathcal{R}_{s+t}(p) = \exp(-As)\mathcal{R}_s(p) + \mathcal{R}_s(0) \]  \[ (10) \]
\[ \mathcal{R}_{s+t}(p) - p = \exp(-As)\mathcal{R}_s(p) + \mathcal{R}_s(0) - p \]
\[ = \exp(-As)(\mathcal{R}_s(p) - p) \]
\[ + (\mathcal{R}_s(p) - p) \]  \[ (11) \]
Hence
\[ L_{s+t}(p) = L_s(p) + \exp(-As)L_s(p) \]

Similarly
\[ (\mathcal{R}_s(p) - p) = \exp(-At)p + \mathcal{R}_s(0) - p \]
\[ = (\exp(-At) - I)p + \mathcal{R}_s(0) \]
\[ = \int_0^t \exp(-Aw)(U - Ap)dw \]
\[ = W_t(0) \]  \[ (12) \]

In equation (12) \( W_t(0) \) is the reachable set to the origin at time, \( t > 0 \), for the system
\[ \dot{x} = Ax - v; \quad v \in U - Ap; \]
where
\[ U - Ap = \{ u - Ap : u \in U \} \]

Equation (12) in (11) implies
\[ \mathcal{R}_{s+t}(p) = \mathcal{R}_s(p) + \exp(-As)W_t(0). \]  \[ (14) \]

**Lemma 3.2:** Given \( p, q \in \mathbb{R}^n \), we have \( \mathcal{R}(q) \subset \mathcal{R}(p) \) if, and only if, \( q \in \mathcal{R}(p) \).

**Proof:** \( \mathcal{R}(q) \subset \mathcal{R}(p) \), obviously implies \( q \in \mathcal{R}(q) \), which implies \( q \in \mathcal{R}(p) \). In the converse direction, if \( q \in \mathcal{R}(p) \) then
\[ q \in \exp(-At)p + \int_0^t \exp(-Aw)Udw \]
for some \( t \geq 0 \). Also, for all \( s \geq 0 \), and from Lemma 3.1
\[ \mathcal{R}_s(q) = \exp(-As)q + \mathcal{R}_s(0) \]
\[ \subset \exp(-As)(\exp(-At)p + \int_0^t \exp(-Aw)Udw) + \mathcal{R}_s(0) \]
\[ = \exp(-As)\mathcal{R}_s(p) + \mathcal{R}_s(0) \]
\[ = \mathcal{R}_{s+t}(p). \]  \[ (15) \]

Thus \( \mathcal{R}_s(q) \subset \mathcal{R}_{s+t}(p) \) for all \( s \geq 0 \) and therefore \( \mathcal{R}_s(q) \subset \mathcal{R}(p) \); since \( s \) is arbitrary, we conclude
\[ \mathcal{R}(q) \subset \mathcal{R}(p). \]

**Corollary 3.1:** (Transitivity property) If \( x \in \mathcal{R}(y) \) and \( y \in \mathcal{R}(z) \), then \( x \in \mathcal{R}(z) \).

**Corollary 3.2:** \( \mathcal{R}(0) \subset \mathcal{R}(p) \) if, and only if, \( 0 \in \mathcal{R}(p) \)

**Lemma 3.3:** Given the system (1) with \( U \) compact and convex. Let \( t \geq 0 \) be fixed and define \( W_t(0) = \int_0^t \exp(-Aw)(U - Ap)dw \). Then the following are equivalent geometric results.

(i) \( \mathcal{R}_s(p) \subset \mathcal{R}_{s+t}(p) \) for some \( s \geq 0 \)
(ii) \( \mathcal{R}_s(p) \subset \mathcal{R}_{s+t}(p) \) for all \( s \geq 0 \)
(iii) \( 0 \in \mathcal{R}_s(p) \)

**Proof:** We show (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (i).

(i) implies (ii) trivially, since \( s \) is arbitrary. Assume (ii), from Lemma 3.1
\[ \mathcal{R}_s(p) \subset \mathcal{R}_{s+t}(p). \]

and
\[ \mathcal{R}_{s+t}(p) = \mathcal{R}_s(p) + \exp(-As)W_t(0). \]

From the hypothesis on \( U, \mathcal{R}(p) \) is nonvoid, convex and compact (Ref. (3), prop. 1, p. 108). Thus
\[ 0 \in \exp(-As)W_t(0) \]
which yields \( 0 \in \mathcal{R}_s(p) \). Conversely, \( 0 \in \mathcal{R}_s(p) \) implies
\[ 0 \in \exp(-As)W_t(0) \]
for all \( s \geq 0 \). Hence \( \mathcal{R}_s(p) \subset \mathcal{R}_{s+t}(p) \) for all \( t \geq 0 \). Hence (iii) implies (ii), and this implies (i) trivially

**Lemma 3.4:** For controllable system (1) if \( p \in \mathcal{A}(0) \cap \mathcal{R}(0) \), then \( \mathcal{R}(p) \) is a convex open neighborhood of the origin.
Proof: From Lemma 3.3, if \( p \in \mathcal{A}(0) \cap \mathcal{R}(0) \), \( \mathcal{R}(0) = \mathcal{R}(0) \). In ([Ref. 1], p. 79), \( \mathcal{R}(0) \) is shown to be a convex open set. Also

\[
0 \in \mathcal{R}(t(0)) \subset \mathcal{R}(t(0) + s(0)) \subset \mathcal{R}(0) \text{ for all } t, s \geq 0
\]

4. Spectrum dependent Properties

For system (1) the geometry of the reachable set \( \mathcal{R}(p) \) depends on the sign of the real part of the eigenvalues of \( A(Re sp A) \), the constraint set \( U \), and the final state \( p \in \mathbb{R}^n \). The following lemmas characterize spectrum dependent properties.

Lemma 4.1: If system (1) has \( Re sp A > 0 \), then, for each \( p \in \mathbb{R}^n \), \( \mathcal{R}(p) \) is a bounded (hence a proper) subset of \( \mathbb{R}^n \).

Proof: Define \( \max \{|u| : u \in U\} = C_2 \) and \( \min Re sp A = \alpha > 0 \). Let \( q \) be any point of \( \mathcal{R}(p) \). Then, for some \( t \geq 0 \) and measurable \( v : [0, t] \rightarrow V \), we have

\[
q = \exp(-At)p + \int_{0}^{t} \exp(-A(t-s))u(s)ds
\]

\[
||q|| \leq ||\exp(-At)|| \cdot |p| + \int_{0}^{t} ||\exp(-A(t-s))|| \cdot |u(s)|ds
\]

\[
\leq C_1 \exp(-\alpha t) \cdot |p| + \int_{0}^{t} C_1 \exp(-\alpha(t-s))C_2 ds
\]

\[
= C_1 \exp(-\alpha t)|p| + C_1C_2 \frac{1}{\alpha} (1 - \exp(-\alpha t))
\]

\[
\leq C_1|p| + C_1C_2 \frac{1}{\alpha} = M;
\]

\[
\alpha > 0, C_1 > 0, C_2 > 0
\]

Thus \( |q| \leq M \) with \( M \) a constant independent of \( q \). Therefore, \( \mathcal{R}(p) \) is a bounded (and hence a proper) subset of \( \mathbb{R}^n \).

Lemma 4.2: Suppose that system (1) is constrained controllable to the origin with \( Re sp A \leq 0 \) and consider any point \( p \in \mathbb{R}^n \). Then \( \mathcal{R}(p) = \mathbb{R}^n \) if, and only if, \( 0 \in \mathcal{R}(p) \).

Proof: The assumptions, constrained controllability of the origin and \( Re sp A \leq 0 \), yield \( \mathcal{R}(0) = \mathbb{R}^n \) ([Ref. 6], Corollary p. 4). From Lemma 3.2, if \( 0 \in \mathcal{R}(p) \) then \( \mathcal{R}(0) \subset \mathcal{R}(p) \) so that \( \mathcal{R}(p) = \mathbb{R}^n \). Conversely, if \( \mathbb{R}^n = \mathcal{R}(p) \), then naturally \( 0 \in \mathcal{R}(p) \).

Corollary 4.1: Given system (1) which is constrained controllable to the origin, then \( \mathcal{R}(p) = \mathbb{R}^n \) if, and only if, \( p \in \mathcal{A}(0) \).

Proof: In the proof of Lemma 4.2, \( \mathcal{R}(p) = \mathbb{R}^n \) is equivalent to \( 0 \in \mathcal{R}(p) \). From Lemma 3.1, \( p \in \mathcal{A}(0) \) is the same as \( 0 \in \mathcal{R}(p) \).

Lemma 4.3: If the system (1) is constrained controllable to the origin and \( Re sp A \leq 0 \), then \( \mathcal{A}(0) \) is a convex open neighborhood of the origin.

Proof: From Section 2, the system is constrained controllable to the origin if, and only if, the origin is in the interior of \( \mathcal{R}(0) \) for all \( t \geq 0 \). This holds also for \( \mathcal{A}(0) \). Also \( \mathcal{R}(0) \) is a proper subset of \( \mathcal{R}(p) \).

Corollary 4.1: Given system (1) which is constrained controllable to the origin, then \( \mathcal{R}(p) = \mathbb{R}^n \) if, and only if, \( p \in \mathcal{A}(0) \).

Proof: In the proof of Lemma 4.2, \( \mathcal{R}(p) = \mathbb{R}^n \) is equivalent to \( 0 \in \mathcal{R}(p) \). From Lemma 3.1, \( p \in \mathcal{A}(0) \) is the same as \( 0 \in \mathcal{R}(p) \).

Corollary 4.2: Given system (1) which is constrained controllable to the origin, then \( \mathcal{R}(p) = \mathbb{R}^n \) if, and only if, \( p \in \mathcal{A}(0) \).

Proof: In the proof of Lemma 4.2, \( \mathcal{R}(p) = \mathbb{R}^n \) is equivalent to \( 0 \in \mathcal{R}(p) \). From Lemma 3.1, \( p \in \mathcal{A}(0) \) is the same as \( 0 \in \mathcal{R}(p) \).

5. Constrained Controllability Properties

The following lemma presents a condition for constrained controllability of a given point \( p \) in \( \mathbb{R}^n \) for system (1).

Lemma 5.1: For the given system (1), \( p \in \mathcal{R}_s(p) \), for some \( s \geq 0 \), if, and only if, the origin is contained in the interior of the reachable set at time \( s \geq 0 \) for the auxiliary control system

\[
\dot{x} = Ax - w; \quad v \in U - Ap
\]

Proof: The condition \( p \in \mathcal{R}_s(p) \) defines the notion of constrained controllability to point \( p \) at time \( s \geq 0 \). Let \( V \) denote the unit ball, centered at the origin in \( \mathbb{R}^n \). Then, \( p \in \mathcal{R}_s(0) \) implies there exists a \( \delta > 0 \) such that

\[
p + \delta V \subset \mathcal{R}_s(p) = \exp(-As)p + \mathcal{R}_s(0)
\]

Thus

\[
\delta V \subset (\exp(-As) - I)p + \mathcal{R}_s(0)
\]

\[
= \int_{0}^{s} \frac{d}{dw} (\exp(-Aw)p + \exp(-Aw)U) dw
\]

and

\[
\delta V \subset \int_{0}^{s} \exp(-Aw)(U - Ap) dw
\]

Hence, the origin is contained in the interior of the reachable set at time \( s \) for the system (17).
Since all the steps are reversible, this completes the proof.

From Lemma 5.1, a necessary and sufficient condition for controllability to point \( p \), at time \( s \geq 0 \), is that

\[
0 \in \text{int} \left( \int_0^s \exp(-A\omega)(U - Ap)d\omega \right) \quad (19)
\]

The lemma relates controllability of system (1) to point \( p \), to controllability of system (3) to the origin. When the origin is a relative boundary point of \( U - Ap \), Brammer's theorem (Ref. 7, Theorem S.3.1, p. 140), developed for linear systems with constrained controls provides conditions for controllability to the origin.

The next lemma establishes the equivalence between the results in Lemma 3.3 and the previous lemma.

Lemma 5.2: Given system (1) with \( U \) convex and compact. Let \( s \geq 0 \) be fixed. Then the following statements are equivalent:

(i) \( R_s(p) \subseteq \text{int} \ R_{s+k}(p) \) for all \( t \geq 0 \)

(ii) \( 0 \in \text{int} \left( \int_0^s \exp(-A\omega)(U - Ap)d\omega \right) \)

(iii) \( p \) is controllable at time \( s \), i.e., \( p \in \text{int} R_s(p) \).

Proof: The equivalence of (i) and (ii) is proved in Lemma 3.3. From Lemma 5.1, (ii) and (iii) are equivalent.

Next we develop properties of \( G \) the set of all times \( t \geq 0 \) for which \( p \) is controllable.

Lemma 5.3:

Let \( G = \{ t : p \in \text{int} R_t(p) \} \); then \( G \) is an open additive subset of \( R^+ = \{ t : t \geq 0 \} \).

Proof:

From Lemma 5.1, if \( p \) is controllable at time \( s \), then there exists a \( \delta > 0 \) such that \( p + \delta V \subseteq \text{int} R_t(p) \) for some \( t \geq 0 \), with \( V \) the unit ball centered about the origin in \( R^n \). Thus \( \delta V \subseteq \text{int} \left( \exp(-At) - I \right) p + R_t(0) \) for some \( \delta > 0 \). That is

\[
- \left( \exp(-At) - I \right) p \in \text{int} R_t(0) \quad (20)
\]

Thus, there is a ball \( B \), centered at \( - \left( \exp(-At) - I \right) p \), such that \( B \subseteq \text{int} R_t(0) \). Since the boundaries of the reachable sets move continuously with time, there exists an \( \epsilon > 0 \) such that \( B \subseteq \text{int} R_t(0) \), \( |s - t| < \epsilon \). Hence, \( G \) is an open set.

Let \( R_t(0) \) denote the reachable set to the origin at time \( t \geq 0 \) for the auxiliary system \( \dot{x} = Ax - v, v \in U - Ap \). By Lemma 5.1, if \( p \) is controllable at times \( t \) and \( s \), then \( 0 \in \text{int} R_t(0) \) and \( 0 \in \text{int} R_s(0) \); in particular

\[
0 \in \text{int} R_t(0) \text{ and } 0 \in \text{int} \exp(-At) R_s(0) \text{.}
\]

By the addition theorem,

\[
0 \in \text{int} \left( R_t(0) + \exp(-At) R_s(0) \right) \subseteq \text{int} \left( \text{int} R_t(0) + \exp(-At) \text{int} R_s(0) \right) \quad (21)
\]

\[
= \text{int} R_{t+s}(0)
\]

Again, by Lemma 5.1, \( p \) is controllable at time \( t + s \), i.e., \( t, s \in G \) implies that \( t + s \in G \). This completes the proof of Lemma 5.3. A consequence of this lemma is the following result.

Lemma 5.4: Given a controllable point \( p \) of system (1), then \( G \), the set of times \( t \geq 0 \) at which \( p \) is controllable, contains some right unbounded interval \((\theta, \infty) \) in \( R^+ \).

Proof: By assumption, \( G \), the set of times at which \( p \) is controllable, is not empty. By the preceding lemma, \( G \) is open. Thus \( G \) contains some interval \((a, b) \) with \( 0 < a < b < \infty \). Since \( G \) is also additive, it must contain all intervals \((ka, kb) \) for \( k = 1, 2, 3, \ldots \). These intervals begin overlapping for large \( k \); indeed for all \( k > a/b - a \). Thus \( G \) contains an unbounded interval of the form \((\theta, \infty) \).

6. Examples

The following examples of system (1) in \( R^2 \) illustrate the results presented. Simulation results for \( R_t(p) \) at times \( t \geq 0 \), for a specified point \( p \in R^2 \) are shown as reachable and reachable properties are illustrated.

Example 6.1: System with \( R_t \exp A > 0 \)

\[
\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} u; |u| \leq 1
\]

\( x_i \) (final) \( = \mu_i, i = 1, 2 \) \quad (22)

The eigenvalues of the system are \( 1 \pm j1 \). From the bang-bang theorem, for \( t \in [0, \pi] \), the maximum number of switches is one. (Refs. 1 and 2). Hence

\[
\mathcal{R}_t(p) = \left\{ \begin{pmatrix} \exp(-t)(p_1 \cos t - p_2 \sin t) \\ \exp(-t)(p_1 \sin t + p_2 \cos t) \end{pmatrix} + \int_0^t \exp(-s)(\cos s - \sin s)u(s)ds \right\};
\]

measurable \( u : [0, \pi] \rightarrow [-1, 1], t \in [0, \pi] \)
That is,

\[
\mathcal{R}_t(p) = \left( \exp(-t)(p_1 \cos t - p_2 \sin t) \right) + \int_0^t \left( \exp(-s)(\cos s - \sin s)(1)ds \right) \\
+ \int_0^t \exp(-s)\sin s + \cos s(1)ds \\
+ \int_0^t \exp(-s)(\cos s - \sin s)(-1)ds \\
+ \int_0^t \exp(-s)(\sin s + \cos s)(-1)ds
\]

where \(\sigma \in [0, t], t \in [0, \pi]\). The plots of \(\mathcal{A}(0)\) are shown in Figure 4. Plots of \(\mathcal{R}_t(p)\) for \(p = (\gamma, \zeta, \eta)\) respectively are shown in Figures 5 and 6.

For Example 6.1 with \(R \neq \emptyset\) and \(t = 0\), Figure 1 shows that \(\mathcal{R}(0)\) is convex. Figures 2 and 3 indicate that \(\mathcal{R}(p)\) is non-convex for \(p = (\gamma, \zeta, \eta)\). Thus, convexity of \(\mathcal{R}(p)\) depends on the location of \(p\) in \(R^2\). \(\mathcal{R}(t)\) is approximately the same as \(\mathcal{R}(0)\) for \(t > 2.0\), in all the figures. Thus, for all values of \(p\) shown in the plots, \(\mathcal{R}(t)\) is a proper subset of \(R^2\) with \(\mathcal{R}(p) = \mathcal{A}(t)\). Also \(\mathcal{R}(p) \subset \mathcal{R}(t)\) for all \(t \geq 0, s \geq 0\) which is a condition for constrained controllability of \(p = (\gamma, \zeta, \eta)\) for the example. The system of equation (22) is constrained controllable at \(p = (\gamma, \zeta, \eta)\) for 2.5 \(< t < \pi\) since \(p = (\gamma, \zeta, \eta)\) is contained in the interior of \(\mathcal{R}(p)\) for all \(t\) within that time interval.

Figures 4 to 6 present plots for Example 6.2, a system with \(R \neq \emptyset\) and \(t = 0\). The shape of \(\mathcal{R}(0)\) in Figure 4 is a re-orientation of plots of \(\mathcal{R}(p)\) for Example 6.1 shown in Figure 1. \(\mathcal{R}(0)\) in Example 6.1 has the same properties as \(\mathcal{R}(0)\) in Example 6.2: openness, convexity, boundedness and neighborhood of the origin.

Plots of \(\mathcal{R}_t(p)\) for Example 6.2 with \(p = (\gamma, \zeta, \eta)\) are shown in Figures 5 and 6 respectively. \(\mathcal{R}(p) = R^2\) and the plots of \(\mathcal{R}(p)\) for each \(p\) will enclose the whole space \(R^2\).

Conclusion

We have presented geometric results on reachable sets and constrained controllability of a linear system, with controls constrained within a compact set \(U\) containing the origin, and for any terminal point \(p\) in \(R^n\). Section 3 discussed general properties such as the addition lemma and the inclusion properties of reachable sets (conditions for a reachable set to be a proper subset of another reachable set). Section 4 presented spectrum dependent properties of reachable sets. We proved conditions for boundedness, openness, convexity and neighborhood of the origin. In section 5, we related the inclusion property, presented in section 3 to the notion of constrained controllability of a point \(p\) in \(R^n\). We also characterized the properties of \(G\), the set of all nonnegative times for which the given system is controllable. Section 6 presented examples to illustrate the results.

References


Fig. 1 $\mathcal{R}_t(0)$ for Example 6.1
Fig. 2 $R_c(p)$ for Example 6.1 with $p = (\sigma, \theta)$.
Fig. 3 \( \mathcal{R}_{t}(p) \) for Example 6.1 with \( p = (0.5) \).
Fig. 4 $A_t(0)$ for Example 6.2
Fig. 5 $R_t(0)$ for Example 6.2
Fig. 6 \( R_i(p) \) for Example 6.2 with \( p = \left(\begin{array}{c} 0 \\ -0.5 \end{array}\right) \)