Alignment Of Radars With Location Errors And Azimuth Misalignments

Ronald E. Helmick
Naval Surface Warfare Center
Combat System Technologies Branch (Code N35)
Dahlgren, Virginia

Abstract

A mathematical model for the alignment of reference frames of radars with location errors and azimuth misalignments is developed in this paper. The alignment of reference frames in a multiple radar system is necessary to effectively share data between the radars. A non-statistical least-square estimator is constructed to estimate the alignment parameters that define the mathematical model. This algorithm can be employed in situations not requiring extremely accurate sensor alignment, and where the biases in the range, elevation, and time are negligible.

Introduction

The alignment of reference frames for multiple radar systems is examined in this paper. This is an important issue in the integration of multiple radars into a single system because the reference frames of all of the radars must be aligned to effectively share data between the radars. The integration of radars without alignment leads to system performance that is worse than any of the individual radars in the system. Thus, alignment is of paramount importance in the integration process. The alignment problem is often referred to as the radar registration problem or, in Navy applications, the gridlock problem. See Fischer et al. and Dana for a general description of the registration problem.

The source of reference frame errors are bias errors (i.e., systematic errors) in the radars and errors in the locations of the radars. One could consider a range bias, azimuth bias, elevation bias, time bias, and location errors. Factors producing these errors include calibration errors in the radars, errors in the navigation systems on which the radars are located, heading errors in the platforms, etc. The degree of alignment accuracy that is required depends on the application. For example, fire control solutions require very accurate sensor alignment, while the association of tracks from different radars can withstand larger errors in the alignment. The major biases considered in the association of tracks in many Navy applications are azimuth biases (or misalignments) and location errors in the radars.

A mathematical model for the alignment of radars with location errors and azimuth misalignments is developed in this paper. A least-square algorithm is then constructed to estimate the parameters in the mathematical model. The parameters estimated are not the location errors and azimuth misalignments directly, but, instead, functions of the location errors and azimuth misalignments. These functions will be sufficient to align the radar reference frames.

Theoretical Development

A mathematical model for the alignment of radars with location errors and azimuth misalignments is developed in this section. The major result is that one may compensate for the effects of the location errors and azimuth misalignments by a rotation and a translation in one of the radar's reference frames. The rotation and translation are applied only after the spherical Earth transformation between the two frames has been performed.
is the longitude difference between the two radar locations.

Consider the situation where the second radar is tracking an object and measures its coordinates to be $x_2$. If there are random errors in the measurements and an azimuth misalignment in the radar, the (known) measured coordinates $x_2$ will not be the same as $x_0$, the (unknown) true coordinates of the object in the second radar’s local Cartesian frame. Letting $f_2$, $e_2$, and $q_2$ be the (known) measured values of the range, elevation, and azimuth, and letting $f_0$, $e_0$, and $q_0$ denote the (unknown) true values, the measured and true values are related by

$$f_2 = f_0 + e_{f_0}$$
$$e_2 = e_0 + e_{e_0}$$
$$q_2 = q_0 + e_{q_0}$$

where $e_{q_0}$ is the (unknown) azimuth misalignment in the second radar, and $e_{q_0}$ and $e_{e_0}$ are random errors in the measured range, elevation, and azimuth, respectively.

Using (1) and (6), and assuming that $e_{f_0}$, $e_{q_0}$, and $e_{e_0}$ are small quantities, it can be shown that the true and measured Cartesian coordinates are related by:

$$x_2 = x_0 \cos \delta_{q_2} + y_0 \sin \delta_{q_2} + e_{x_2}$$
$$y_2 = -x_0 \sin \delta_{q_2} + y_0 \cos \delta_{q_2} + e_{y_2}$$
$$z_2 = z_0 + e_{z_2}$$

where $e_{x_2}$, $e_{y_2}$, and $e_{z_2}$ are random errors in $x_2$, $y_2$, and $z_2$, respectively. Equations (7) - (9) may be expressed as the following matrix equation:

$$X_2 = P_2 X_1 + E_2$$

where the 3 x 3 rotation matrix $P_2$ is defined by

$$P_2 = \begin{bmatrix}
\cos \delta_{q_2} & \sin \delta_{q_2} & 0 \\
-\sin \delta_{q_2} & \cos \delta_{q_2} & 0 \\
0 & 0 & 1
\end{bmatrix}$$

and $E_2 = [e_{x_2}, e_{y_2}, e_{z_2}]$ is the random error in the position of the object in the second radar’s frame. Similarly, the equation relating $x_1$, the coordinates of an object measured by the first radar, and $x_{1t}$, the true coordinates of this object in the first radar’s local Cartesian frame, is given by

$$x_{1t} = P_1 x_1 + E_1$$

where $P_1$ is obtained by replacing $\delta_{q_2}$ in $P_2$ with $\delta_{q_1}$, the azimuth misalignment in the first radar, and $e_{x_1}$ is the random error in the position of the object in the first radar’s frame.

Now consider the case where $x_2$, the coordinates of an object as measured by the second radar, is transformed into the first radar’s local Cartesian frame. The spherical Earth transformation is given by

$$x_2 = M x_{2t} + N$$

where $M$ and $N$ are defined in (3) and (4) except that $x_1$, $y_2$, and $z_2$ are replaced by $x_{1t}$, $y_{2t}$, and $z_{2t}$, respectively, where the subscript $t$ denotes their true values.

Alignment for a Multiple Radar System

Equation (18) aligns the track data measured by the second radar, but expressed in the first radar’s frame. It can also be manipulated to obtain a transformation that aligns track data measured by the first radar in the second radar’s frame. This result, along with (18), can be used to align the frames of a multiple radar system. Solving for $x_{2t}$ in (13) gives

$$x_{2t} = M_{2t} x_2 + N_{2t}$$

This is just the spherical Earth transformation from the first radar’s frame to the second radar’s frame. Solving for $x_{1t}$ in (18) gives

$$x_{1t} = R_{1t} x_1 + R_{1t} T + R_{1t} e$$

where $R_{1t} = P_{1t}^{-1} M_{2t}$, and

$$T = P_{1t}^{-1} \left( N_{2t} - M_{2t} P_1 P_1^T N \right) = P_{1t}^{-1} N \quad (20)$$

$$e = P_{1t}^{-1} \left( M_{2t} e_{x_2} - e_{x_1} \right)$$

Equation (18) is equivalent to (15), and it is expressed entirely in terms of the (known) measured track coordinates. Equation (18) is valid in the first radar’s frame, and it represents a rotation (by $R_1$) followed by a translation (by $T$). It is used to align the second radar’s frame to the first radar’s frame, after the spherical Earth transformation in (13) has been applied to the second radar’s tracks. The quantity $e$ in (18) represents random errors in the track data, and it is a measure of how accurately the tracks are aligned after $R$ and $T$ are applied to the track data. The problem is to determine the matrix $R$ and the vector $T$. Once these have been determined, equation (18) can be used to align any track measured by the second radar in the first radar’s frame.
The first radar can align its reference frame to the second radar's frame by first applying the transformation in (23) to its measured track data \( \mathbf{x}_1 \). This produces \( \mathbf{x}_2' \), which is still in the first radar's frame. Then the spherical Earth transformation in (22) is applied to \( \mathbf{x}_2' \), which aligns \( \mathbf{x}_2' \) to \( \mathbf{x}_2 \), the second radar's measured track data. This procedure aligns the track data measured by the first radar, but expressed in the second radar's frame. The information is required to align the second radar's frame to the first radar's frame as is required to align the first radar's frame to the second radar's frame. That is, \( \mathbf{R} \) and \( \mathbf{T} \) are required to align the second radar's frame to the first radar's frame, and \( \mathbf{R}' \) and \( \mathbf{T'} \) are required to align the first radar's frame to the second radar's frame.

This result can be used to achieve alignment for a multiple radar system. For example, consider the case of three radars, and let the first and third radars align their reference frames to the second radar. Let \( \mathbf{R}_{12} \) and \( \mathbf{T}_{12} \) be the rotation matrix and translation vector that aligns the second radar's frame to the first radar's frame. Similarly, \( \mathbf{R}_{23} \) and \( \mathbf{T}_{23} \) are the rotation matrix and translation vector that aligns the second radar's frame to the third radar's frame. It is desired to transmit track data from the first radar to the third radar so that the tracks are aligned in the third radar's frame. The tracks measured by the first radar are rotated by \( \mathbf{R}_{12} \) and then translated by \( -\mathbf{R}_{12} \mathbf{T}_{12} \) according to (23). The spherical Earth transformation in (22) is then applied so that the first radar's tracks are aligned in the second radar's frame. Then the tracks are transmitted to the third radar. When these tracks are received by the third radar, they are transformed using the spherical Earth transformation from the second radar to the third radar. Then the tracks are rotated by \( \mathbf{R}_{23} \) and translated by \( \mathbf{T}_{23} \). This procedure aligns the tracks in the third radar's frame. Note that this did not require the second radar to receive the track data from the first radar and transmit it to the third radar. Reversing the above argument shows that track data from the first radar can also be aligned in the first radar's frame. Thus, the first and third radar's reference frames are also aligned. Extensions to more than three radars can easily be made.

The above process is referred to as relative alignment because all of the radars aligned their frames to a common radar, namely, the second radar. The term geodetic alignment refers to the process of aligning the individual radars to some geodetic or absolute standard. Of course, if the second radar's frame is aligned to some geodetic standard, then relative alignment will also produce geodetic alignment.

Approximations

The components of \( \mathbf{R} \) and \( \mathbf{T} \) are trigonometric functions of various quantities (i.e., the reported and true longitude and latitudes of the radar locations, and the azimuth misalignments in the radars). The expressions for \( \mathbf{R} \) and \( \mathbf{T} \) can be simplified by employing small-angle approximations. In particular, it will be assumed that the angular separations between the radars and the location errors are small quantities. For example, a latitude separation of 1° corresponds to a site separation of 60 nmi (nmi = nautical mile), and a longitude separation of 1° corresponds to a site separation of 60 nmi. Although the radars are separated by a fairly large distance, the angular separations are relatively small. Similarly, a latitude error of 0.1° corresponds to a 6 nmi location error, and a longitude error of 0.1° corresponds to a 6000 ft location error. These represent large errors in distance but only small angular errors.

The vector alignment equation presented in (18) may be expressed as the following three scalar equations:

\[
\begin{align*}
x_1 &= R_{11} x_1 + R_{12} y_1 + R_{13} z_1 + T_{11} + e_x \\
y_1 &= R_{21} x_1 + R_{22} y_1 + R_{23} z_1 + T_{21} + e_y \\
z_1 &= R_{31} x_1 + R_{32} y_1 + R_{33} z_1 + T_{31} + e_z
\end{align*}
\]

where the \( R_{ij} \)'s and \( T_{ij} \)'s are the components of \( \mathbf{R} \) and \( \mathbf{T} \), respectively, and \( e_x, e_y \), and \( e_z \) are the components of \( \mathbf{e} \). The quantities \( x_1, y_1, \) and \( z_1 \) are the components of \( \mathbf{x}_1 \), and \( x^2_1, y^2_1 \) and \( z^2_1 \) are the components of \( \mathbf{x}_1' \).

The quantities \( \theta_1, \phi_2 \), and \( z_1 \) have negligible effects in the alignment of the x- and y-coordinates. Thus, the effect incurred by ignoring \( z_1 \) is less than the accuracies of many of the radars employed in applications. The error incurred by ignoring \( z_1 \) is less than 175 ft in the x- and y-coordinates. These error estimates were obtained by assuming a maximum separation of 3° in latitude and longitude between the two radars, location errors of 0.1° in latitude and longitude, and a track with an x-coordinate of 1800 nmi and a y-coordinate of 0.000 nmi. The unit nmi denotes a data mile, which is defined to be a distance of 6000 ft.

The equation for \( z_1 \) in (26) is a complicated expression when the \( R_{ij} \)'s and \( T_{ij} \)'s are substituted into it. Instead of using (26) in the track association process, it is probably better to use the height relative to sea level. It can be shown that the height relative to sea level is the same in both radar's reference frames (i.e., it is an invariant). Moreover, the height relative to sea level is not affected by radar azimuth misalignments or location errors. Thus, only the x- and y-coordinates need to be aligned in this approach.
The alignment equations presented in (31) and (32) represent a rotation by $A_0$ (about the z-axis) followed by translations by $AX$ and $AY$ in the first radar's reference frame. This alignment equation is very similar to the one presented by Bath3m4. Once $AX$, $AY$, and $A_0$ are determined, (31) and (32) can be used to align the x- and y-coordinates of a track reported by the second radar in the first radar's frame. Note that the effects of the location errors and the azimuth misalignments are combined in $AX, AY, and A_0$; that is, the location errors and the azimuth misalignments are not individually separated. Of course, it is not necessary to separate these effects to align the track data.

Least-Square Estimator

In this section, a non-statistical least-square estimator is constructed for the rotation angle $A_0$. Given the estimate of $A_0$, moving average estimators are then used to estimate $AX$ and $AY$. The reason for using a non-statistical approach is that the statistics for the tracking errors (i.e., the statistics of $e_x$ and $e_y$) may not be known. In the statistical approach, the covariance matrix for $e_x$ and $e_y$ is used as a weighting matrix.

Proceeding with the non-statistical approach, we solve for $AX$ and $AY$ in (31) and (32), giving

$$\Delta X = x_i - x_{ii} \cos \Delta \theta - y_{ii} \sin \Delta \theta - \theta_{xi}$$

(36)

$$\Delta Y = y_i + x_{ii} \sin \Delta \theta - y_{ii} \cos \Delta \theta - \theta_{yi}$$

(37)

where the index $i$ represents the $i$th common object tracked by the two radars, also called a mutual track pair. Note that the index $i$ is not placed on $AX, AY, and A_0$ because these quantities depend on the location errors and azimuth misalignments in the radars, not the tracks. Equations (36) and (37) apply to the $i$th mutual track pair.

Eliminating $AX$ from (36) and (38) and $AY$ from (37) and (39) gives the following equations that involve only $\Delta \theta$:

$$c_i = a_i \cos \Delta \theta + b_i \sin \Delta \theta + w_{xi}$$

(40)

$$d_i = -a_i \sin \Delta \theta + b_i \cos \Delta \theta + w_{yi}$$

(41)

where

$$a_i = a_{i1} - x_{i1} \cos \Delta \theta - y_{i1} \sin \Delta \theta - \theta_{x1}$$

(38)

$$b_i = b_{i1} + x_{i1} \sin \Delta \theta - y_{i1} \cos \Delta \theta - \theta_{y1}$$

(39)

Eliminating $\Delta X$ from (38) and (39), and $\Delta Y$ from (37) and (39), gives the following equations that involve only $\Delta \theta$:

$$c_i = a_i \cos \Delta \theta + b_i \sin \Delta \theta + w_{xi}$$

(40)

$$d_i = -a_i \sin \Delta \theta + b_i \cos \Delta \theta + w_{yi}$$

(41)

Equations (40) and (41) may be expressed as the following matrix equation:

$$\begin{bmatrix} c_1 \\ d_1 \\ c_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & \cos \Delta \theta \\ b_1 & -a_1 & \sin \Delta \theta \end{bmatrix} \begin{bmatrix} w_{x1} \\ w_{y1} \end{bmatrix}$$

(43)

Note that this formulation of the problem assumes that the two radars are tracking at least two common objects (i.e., the first and $(i+1)^{th}$ mutual track pairs).

Assuming there are $N+1$ mutual track pairs, there will be $N$ equations of the form presented in (43). Augmenting them into one large system gives

$$y = Ax + w$$

(44)

where $q = [\cos \Delta \theta, \sin \Delta \theta]$, $y = [c_1, d_1, ..., c_N, d_N]$, $w = [w_{x1}, w_{y1}, ..., w_{xN}, w_{yN}]$, and $A$ is the $2N \times 2$ matrix given by

$$A = \begin{bmatrix} a_1 & b_1 \\ b_1 & -a_1 \\ \vdots & \vdots \\ a_N & b_N \\ b_N & -a_N \end{bmatrix}$$

The least-square approximation of $q$ in (44) is the vector $q_{LS}$ that minimizes $w^Tw$. It is given by

$$q_{LS} = (A^TA)^{-1}A^Ty$$

(45)

It is not difficult to show that $A^TA$ is given by

$$A^TA = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$$

(46)

where

$$p = \sum_{i=1}^{N} (a_i^2 + b_i^2)$$

(47)

The estimates of the two components of $q_{LS}$ may be expressed by the following two scalar equations:

$$\cos \Delta \theta_{LS} = \frac{1}{p} \sum_{i=1}^{N} (a_i q_i + b_i d_i)$$

(49)

$$\sin \Delta \theta_{LS} = \frac{1}{p} \sum_{i=1}^{N} (b_i q_i - a_i d_i)$$

(50)

This can also be cast into a recursive form. Let

$$\alpha_i = \frac{1}{p_{i-1}} \left( a_i q_i + b_i d_i \right)$$

(51)

$$\beta_i = \frac{1}{p_{i-1}} \left( b_i q_i - a_i d_i \right)$$

(52)

where

$$\alpha_0 = \frac{1}{p_0} \left( a_0 q_0 + b_0 d_0 \right)$$

(53)

are the least-square estimates of $\cos \Delta \theta$ and $\sin \Delta \theta$ based on the first $i$ mutual track pairs, and

$$p_i = p_{i-1} + (a_i^2 + b_i^2), \quad p_0 = 0$$

(54)

Then, $\alpha_i$ and $\beta_i$ can be calculated recursively by the uncoupled scalar equations:

$$\alpha_i = \alpha_{i-1} + \frac{1}{p_i} \left\{ (a_i q_i + b_i d_i) - (a_{i-1}^2 + b_{i-1}^2) \alpha_{i-1} \right\}$$

(55)

$$\beta_i = \beta_{i-1} + \frac{1}{p_i} \left\{ (b_i q_i - a_i d_i) - (a_{i-1}^2 + b_{i-1}^2) \beta_{i-1} \right\}$$

(56)

This algorithm is initialized by $\alpha_0 = \beta_0 = p_0 = 0$. The estimates of the rotation bias pad can be obtained from

$$\Delta \theta = -\sin^{-1}(\alpha), \quad \Delta \phi = \cos^{-1}(\beta)$$

(57)

Note that this algorithm gives two estimates of $\Delta \theta$. Using (54), it can be shown that $p_i > 0$ for all $i$, provided that $a_i \neq 0$ or $b_i \neq 0$. This is important because the inverse of $p_i$ is
required in this algorithm. From (42), the condition that \(a_i \neq 0 \) or \(b_j \neq 0 \) implies that \(x^1_1 = x^2_1 \) or \(y^1_1 = y^2_1 \). That is, the \(x\)-coordinates of the two tracks measured by the second radar (after being transformed into the first radar’s frame) must be different, or the \(y\)-coordinates must be different.

After the estimate of \(\Delta \theta\) has been found, one can use this estimate of \(\Delta \theta\) in (36) and (37), and then estimate \(\Delta X\) and \(\Delta Y\). The simplest approach is to use the least-square estimates of \(\Delta X\) and \(\Delta Y\). We will first consider the estimate of \(\Delta X\). Assuming there are \(N\) mutual track pairs, there will be \(N\) equations of the form presented in (36). These \(N\) equations can be expressed as the following matrix equation

\[
z = H(AX + e) \tag{58}
\]

where \(H\) is the \(N \times 1\) column vector all of whose entries are one, \(e = [e_1, e_2, \ldots, e_N]^T\), \(z = [y_1, y_2, \ldots, y_N]^T\), where each \(y_j (j = 1, 2, \ldots, N)\) is given by

\[
y_j = x_j - x_1 j \cos \delta_j - y_1 j \sin \delta_j . \tag{59}
\]

In (59), \(\delta_j\) is the least-square estimate of \(\Delta \theta\) based on the first \(j\) mutual track pairs. The least-square approximation of \(\Delta X\) in (58) is the quantity \(\Delta X_{LS}\) that minimizes \(e'e\). It is given by

\[
\Delta X_{LS} = (H'H)^{-1} H'z = \frac{1}{N} \sum_{i=1}^{N} y_i \tag{60}
\]

This can be put into a recursive form by letting

\[
\Delta X_j = \frac{1}{j} \sum_{i=1}^{j} y_i , \tag{61}
\]

where \(\Delta X_j\) is the least-square estimate of \(\Delta X\) based on the first \(j\) mutual track pairs. It can be shown that

\[
\Delta X_j = \frac{j-1}{j} \Delta X_{j-1} + \frac{y_j}{j} \tag{62}
\]

This is the moving average estimator for \(\Delta X\). Proceeding in a similar manner, it can be shown that the least-square estimate of \(\Delta Y\) is given by the moving average estimator

\[
\Delta Y_j = \frac{j-1}{j} \Delta Y_{j-1} + \frac{\xi_j}{j} \tag{63}
\]

where \(\xi_j = y_j - x_j 2 \sin \delta_i - y_1 2 \cos \delta_i \). \(\tag{64}\)

These moving average estimators are initialized by \(\Delta X_0 = \Delta Y_0 = 0\).

For this algorithm to work, the two radars must be tracking at least two common objects. The problem is identifying common objects for starting the algorithm. One possible approach to this problem has been presented by Bath.\(^3\),\(^4\) Assuming that this initialization process has been completed and estimates of \(\Delta \theta\), \(\Delta X\), and \(\Delta Y\) have been obtained, then all of the tracks from the second radar can be aligned in the first radar’s frame. The subsequent track association process should be easier because the reference frame errors due to the azimuth misalignments and location errors have been eliminated. The only remaining errors are random errors in the track data and unmodeled reference frame errors. New associations, or subsequent updates to existing associations, can be used in the algorithm to update the estimates of \(\Delta \theta\), \(\Delta X\), and \(\Delta Y\). Tests with simulated track data indicate that the estimate of \(\Delta \theta\) based upon \(\alpha_i\) (i.e., \(\cos \Delta \theta_i\)) does not converge to its true value in the presence of noisy track data. The reason for this is not known to the author at the present time. Thus, one can dispense with \(\alpha_i\) and only use \(\beta_i\) (i.e., \(\sin \Delta \theta_i\)) to obtain the estimate of \(\Delta \theta\). This algorithm has been tested with simulated track data.\(^5\) Known azimuth biases and location errors, along with noise, were included in the track data. Due to space limitations, these results cannot be presented here. These tests show that the estimates of \(\Delta \theta\) (based upon \(\beta_i\)), \(\Delta X\), and \(\Delta Y\) do converge to their true values. The track data from the second radar was aligned in the first radar’s frame by applying the estimated values of \(\Delta \theta\), \(\Delta X\), and \(\Delta Y\) to the second radar’s track data. As expected, the resulting alignment error was produced by the random errors in the track, which was on the order of several hundred feet. This indicates that this algorithm can be employed in situations not requiring extremely accurate sensor alignment (e.g., track association), and where the biases in range, elevation, and time are negligible.

**Summary**

A mathematical model for the alignment of radars with location errors and azimuth misalignments was developed in this paper. A non-statistical least-square estimator was constructed to estimate the alignment parameters that define the mathematical model. Tests of this algorithm reported elsewhere\(^6\) indicate that this algorithm can be employed in situations not requiring extremely accurate sensor alignment, and where the biases in the range, elevation, and time are negligible.

**References**

6. Helrick, R.E., Alignment of Radars with Location Errors and Azimuth Misalignments, Naval Surface Warfare Center, Dahlgren, Virginia, NAVSWC TR 90-491, September 1990.