Tracking Constant Speed Targets
Using A Kinematic Constraint

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ABSTRACT - The tracking of constant speed, maneuvering targets has been studied extensively since the mid 1960s. One of the first works that had a significant and lasting impact on the problem was [1], where a target motion model with an acceleration that is exponentially correlated in time was introduced. In that work, a Kalman filter was utilized to estimate the position, velocities, and accelerations of the target. The rationale for the time-correlated acceleration was derived from inertial systems in that a target accelerating at time $t$ is likely to be accelerating with similar acceleration at time $t + \tau$ for sufficiently small $\tau$. However, the assumptions of exponentially-correlated, zero-mean accelerations of [1] produce a motion model with an acceleration that decreases in magnitude during each state extrapolation. Thus, when the actual target acceleration is constant or increasing, large errors occur in the state estimates. In the early 1970s, decision-directed algorithms were introduced in [2] to respond to the demand for algorithms that could provide better tracking of targets performing high $g$, fast maneuvers. These decision-directed algorithms monitor the tracking errors to detect a maneuver and respond by increasing the process noise covariance and/or the dimension of the target motion model as described in [2,3,4]. While these algorithms provide good tracking performance before and after the maneuver, their performance during and immediately following the maneuver is poor. The problem with these algorithms is that the accelerations during a maneuver are not easily modeled in the sensor reference frame because the accelerations often vary irregularly with time in that frame. In [5], the accelerations were modeled in a target-oriented reference frame to reduce the problems associated with modeling the time-varying accelerations. While the algorithm in [5] provides relatively good tracking performance, the modeling and transformations that must be estimated produce a complicated algorithm. The two-stage Kalman estimator was applied to the tracking of maneuvering targets in [6], and a significant reduction in the convergence time after a maneuver was achieved. However, the two-stage Kalman estimator of [6] is limited to a constant acceleration model with additive white noise. Input estimation and multiple model algorithms presented in [7,8,9,10] were developed to address the problems of time-varying accelerations and convergence time after a maneuver. However, the input estimation and multiple model algorithms require significantly more computations than the previous techniques. A kinematic constraint can be utilized as additional information about the target to improve the tracking of the time-varying accelerations without significantly increasing the computational cost.

When a target's trajectory satisfies a kinematic constraint, the kinematic constraint provides additional information about the target's motion. Using the constraint removes some of the uncertainty of the time-varying accelerations and tends to force the acceleration estimates to change in a manner consistent with the dynamics of the target. However, including the kinematic constraint in the motion model produces a state equation with nonlinear dynamics. The use of a nonlinear state equation significantly increases the computations involved in an extended Kalman filter because the state transition must be computed on-line. Recently, the idea of introducing a kinematic constraint into the tracking process through a pseudomeasurement was proposed in [11]; nonlinearities are easier to accommodate in the measurement equation of the extended Kalman filter. The use of the constraint was shown to improve the performance of a given track filter. The improvement in the tracking performance can be attributed primarily to the reduced errors in the acceleration estimates. The constraint tends to reduce the bias or lag in the acceleration estimates when the actual acceleration is time-varying. However, using the kinematic constraint as formulated in [11] results in marginal improvement in the performance of the track filter, poor transient performance at initialization, and a track filter for which a guarantee of stability may not be achievable. In this paper, a new formulation of the kinematic constraint is proposed for constant speed targets, and the rationale for the new formulation is discussed. The new formulation of the kinematic constraint provides significantly better tracking performance than the one given in [11], and simulation results are given to demonstrate the improvement. Also, a track filter utilizing the new formulation of the kinematic constraint is shown to
be unbiased and observable in [12].

This paper is organized as follows. In section II, the general problem of tracking maneuvering targets is discussed, and the kinematic constraint for constant speed targets is derived. The formulation of the kinematic constraint proposed in [11] is presented in section III. The new formulation for the kinematic constraint and its rationale are given in section IV. In Section V, simulation results are presented to demonstrate the improved tracking performance achieved with the new formulation. In section VI, a summary and conclusions are given.

II. Problem Formulation

In this section, the general problem of tracking maneuvering targets is discussed and the kinematic constraint for constant speed targets is derived. The dynamical system model of a maneuvering target in track is given by

\[ X = f(X, u, w) \]  \hspace{1cm} (2.1)

\[ z_k = h(X_k, v_k) \]  \hspace{1cm} (2.2)

where \( X \) is the state vector, \( u \) is the control vector, \( w \) is the process noise vector representing possible deviations in \( f(-) \). \( Z_k \) is the discrete-time measurement vector at time \( k \) and \( v_k \) is the measurement noise vector. The dynamics of the target is a continuous-time process as shown in Equation (2.1) where \( f(-) \) is a dynamic constraint that defines the motion for the target in the form of a differential equation. The dynamic constraint, which is usually unknown to the tracking system, can differ significantly between targets and change for a common target during the tracking process. As indicated by Equation (2.2), the measurement process is a discrete-time process because most sensors used for target tracking record the position and/or radial velocity for a given instant in time.

While \( f(-) \) is usually unknown by the tracking system, the major problem with tracking maneuvering targets is that the control vector is not directly observable by the tracking system. When the target applies a control, a bias or lag develops in the estimates of the target state. The methods proposed in [8] and [9] process the positional tracking errors to estimate the control vector \( u \) which produced the observed bias in the position estimates. However, a significant delay exists between the time that a control input is applied and the time that a bias in the position estimates can be utilized to estimate the input. Thus, for tracking problems, the input must be predicted for the current time from the estimate of a past control input, and this prediction can introduce significant errors. The control can be included as acceleration in the dynamic constraint \( f(-) \), but the acceleration most often varies with time in such a manner that a model cannot be clearly identified during tracking. Thus, the target dynamics are most often modeled as linear in a Cartesian coordinate frame to simplify the filtering and reduce the computations required. Also, for convenience the continuous-time dynamics equation is converted to a discrete-time system. As a result, the dynamics model commonly assumed for a target in track is given by

\[ X_{k+1} = F_k X_k + w_k \]  \hspace{1cm} (2.3)

where \( w_k \sim N(0, Q_k) \) is the process noise and \( F_k \) defines a linear constraint on the dynamics. The target state vector \( X_k \) contains the position \((x, y, z)\), velocity \((\dot{x}, \dot{y}, \dot{z})\), and acceleration \((\ddot{x}, \ddot{y}, \ddot{z})\) of the target at time \( k \) as well as other variables used to model the time-varying acceleration. When the target applies a control and the time-varying acceleration is modeled incorrectly, a bias or lag develops in the estimates of the target state. While multiple dynamics models can be used as proposed in [7] and [10] to identify the best model available in the filter, identifying the exact model is not feasible because the target can apply many different control vectors to evade the tracking system. Also, since Equation (2.3) is a linear function of \( X_k \), the motions in the \( x \), \( y \), and \( z \) coordinates are modeled as independent, while these motions are very seldom independent in the coordinate frame chosen for tracking. Therefore, additional information would be helpful to reduce the modeling error of the time-varying acceleration.

A kinematic constraint can be utilized as additional information about the target to reduce the errors in the estimates of the time-varying accelerations. Using the kinematic constraint tends to force the acceleration estimates to change in a manner that is consistent with the dynamics of the target. A kinematic constraint can be developed for use in tracking constant speed, maneuvering targets. The speed of a target is given by

\[ S = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2} \]  \hspace{1cm} (2.4)

For a target moving at a constant speed,

\[ \frac{dS}{dt} = 0 \]  \hspace{1cm} (2.5)

or

\[ \ddot{x} + \ddot{y} + \ddot{z} = 0 \]  \hspace{1cm} (2.6)

Equation (2.6) can be written as

\[ V \cdot A = 0 \]  \hspace{1cm} (2.7)

where the target velocity \( V \) and acceleration \( A \) are given by

\[ V = [\dot{x} \hspace{0.5cm} \dot{y} \hspace{0.5cm} \dot{z}]^T \]  \hspace{1cm} (2.8)

and

\[ A = [\ddot{x} \hspace{0.5cm} \ddot{y} \hspace{0.5cm} \ddot{z}]^T \]  \hspace{1cm} (2.9)

This kinematic constraint for constant speed targets is useful information and can be incorporated in the system state in Equation (2.1) or used as a pseudomeasurement in conjunction with Equation (2.2). While both approaches are conceptually feasible, the second approach is more attractive because the first changes the state equation in Equation (2.3) from linear to nonlinear. In the implementation of the extended Kalman filter, including nonlinearities in the measurement equation is computationally less expensive than in the state equation [1]. If the state equation is nonlinear, the transition matrix for propagating the state from time \( k \) to time \( k+1 \) must be computed on-line at each propagation because the transition matrix will be highly dependent on the state of the target. Computing the transition matrix on-line would greatly increase the computational cost of implementing the extended Kalman filter. Thus, incorporating this kinematic constraint into the filter as a pseudomeasurement is the focus of this paper.
In this paper, the measurement process of Equation (2.2) will be assumed to be linear. While most sensors used for target tracking measure the target position in spherical coordinates, the spherical measurements can be transformed into a Cartesian coordinate frame for processing as a linear function of the target state. Thus, the measurement process is then given by

\[ Z_k = H_k X_k + v_k \]  

(2.10)

where \( Z_k \) is the target measurement in the Cartesian coordinate frame and \( v_k \sim N(0, R_k) \) is the measurement error. While the measurement errors in spherical coordinates are usually assumed to be Gaussian and uncorrelated in range, bearing, and elevation, the transformation to the rectangular coordinate frame causes the components of \( v_k \) to become non-Gaussian and correlated. However, \( v_k \) is still assumed to be Gaussian. The \( R_k \) is a full matrix, and the equations for computing \( R_k \) from the spherical error statistics are given in [13].

The measurement update of the Kalman filter can be viewed as a weighted-least-square estimation problem as discussed in [1]. For the system in Equations (2.3) and (2.10), the objective function to be minimized in the least-squares sense with respect to \( X_k \) is

\[ J = \frac{1}{2}((X_k - \hat{X}_{k|k-1})^T P_{k|k-1}^{-1} (X_k - \hat{X}_{k|k-1}) \]

\[ + (Z_k - H_k \hat{X}_k)^T R_k^{-1} (Z_k - H_k \hat{X}_k) \]  

(2.11)

where \( X_{k|k-1} \) is the predicted state estimate for time \( k \) based on measurements through time \( k - 1 \), and \( P_{k|k-1} \) is the covariance of \((X_k - X_{k|k-1})\). The resulting estimate of \( X_k \) that minimizes \( J \) is denoted as \( \hat{X}_{k|k} \). This cost function will be utilized to explain and justify the inclusion of the kinematic constraint as a pseudomeasurement. The formulation for the kinematic constraint used in [11] is presented next.

III. Previous Formulation of Constraint

The kinematic constraint of Equation (2.7) can be used directly in the extended Kalman filter. Since the constraint is a nonlinear function of the target state, it is first linearized about the predicted state \( X_{k|k-1} \) as

\[ V_k \cdot A_k = V_{k|k-1} \cdot A_{k|k-1} + \dot{C}(X_{k|k-1})(X_k - X_{k|k-1}) \]  

(3.1)

where

\[ \dot{C}(X_{k|k-1}) = \begin{bmatrix} \dot{x} & \dot{\dot{x}} & 0 & \dot{y} & \dot{\dot{y}} & 0 & \dot{z} & \dot{\dot{z}} \end{bmatrix}^T \]  

(3.2)

Since

\[ \dot{C}(X_{k|k-1}) X_{k|k-1} = 2V_{k|k-1} \cdot A_{k|k-1} \]  

(3.3)

and \( V_k \cdot A_k = 0 \), Equation (3.1) can be written as

\[ V_{k|k-1} \cdot A_{k|k-1} = \dot{C}(X_{k|k-1}) X_k \]  

(3.4)

Since the constraint has been linearized and the target may deviate slightly from constant speed, the linearized constraint is modeled with an additive error as

\[ V_{k|k-1} \cdot A_{k|k-1} = \dot{C}(X_{k|k-1}) X_k + \mu \]  

(3.5)

where \( \mu \sim N(0, R_k) \) relaxes the constraint. With this final modification, the nonlinear kinematic constraint is in the form of a linear measurement.

The cost function of Equation (2.11) can be augmented with Equation (3.5) to obtain

\[ J^\mu = \frac{1}{2}((X_k - X_{k|k-1})^T P_{k|k-1}^{-1} (X_k - X_{k|k-1}) \]

\[ + (Z_k - H_k X_k)^T R_k^{-1} (Z_k - H_k X_k) \]

\[ + (V_{k|k-1} \cdot A_{k|k-1} - \dot{C}(X_{k|k-1}) X_k)^T R_k^{-1} \cdot (V_{k|k-1} \cdot A_{k|k-1} - \dot{C}(X_{k|k-1}) X_k) \]  

(3.6)

After, augmenting the measurement with the kinematic constraint, the cost function is given by

\[ J^\mu = \frac{1}{2}((X_k - X_{k|k-1})^T P_{k|k-1}^{-1} (X_k - X_{k|k-1}) \]

\[ + (Z_k^\mu - L_k X_k)^T (R_k^\mu)^{-1} (Z_k^\mu - L_k X_k) \]  

(3.7)

where

\[ Z_k^\mu = \begin{bmatrix} Z_k^\mu_1 \\ \vdots \\ Z_k^\mu_n \end{bmatrix} \]

\[ L_k = \begin{bmatrix} H_k \\ \dot{C}(X_{k|k-1}) \end{bmatrix} \]  

(3.8)

\[ R_k^\mu = \begin{bmatrix} R_k \\ 0 \\ 0 \\ 0 \end{bmatrix} \]  

(3.9)

A Kalman filter can be implemented to minimize \( J^\mu \) in the least-square sense with respect to \( X_k \). For the filter, Equation (2.3) serves as the state model and the measurement model is given by

\[ Z_k^\mu = L_k X_k + v_k^\mu \]  

(3.10)

where \( v_k^\mu = [v_k \mu]^T \). This formulation closely parallels the one given in [11]. A new formulation for including the kinematic constraint in the filter as a pseudomeasurement is given in the next section.

IV. New Formulation of Constraint

Examining the formulation for the kinematic constraint in Section III shows that several modifications can be made to improve the formulation of the kinematic constraint. The kinematic constraint can be linearized about the filtered state estimate \( X_{k|k} \) instead of the predicted state estimate \( X_{k|k-1} \). By processing the measurement \( Z_k \) before linearization, a more accurate linearization can be achieved because the filtering of \( Z_k \) reduces the error in velocity and acceleration estimates that are used in \( \dot{C}(\cdot) \). Also, analyzing Equation (3.4) shows that the acceleration estimates are used in \( \dot{C}(\cdot) \) to observe the target velocities and the velocity estimates are used to observe the target acceleration. Since the acceleration estimates are usually less accurate than the velocity estimates, using the acceleration estimates to observe the velocities may be counterproductive. Thus, from these first two points, the kinematic constraint is written as

\[ V_{k|k} \cdot A_k = 0 \]  

(4.1)

where

\[ V_{k|k} = \begin{bmatrix} \dot{x}_{k|k} & \dot{y}_{k|k} & \dot{z}_{k|k} \end{bmatrix}^T \]  

(4.2)

and

\[ A_k = \begin{bmatrix} \ddot{x}_k & \ddot{y}_k & \ddot{z}_k \end{bmatrix}^T \]  

(4.3)
The $V_{ik}$ is the filtered velocity estimate, and $A_k$ is the actual target acceleration at time $k$. The kinematic constraint of Equation (4.1) is a linear function of the elements of state vector. Since

$$V_{ik} \cdot A_k = |V_{ik}||A_k| \cos \theta$$

and errors in $V_{ik}$ can produce $\theta \neq \pi/2$, $V_{ik} \cdot A_k$ can rather large when the magnitude of the velocity is large. Thus, in order to remove the dependence of constraint equation on the magnitude of the velocity, the constraint is written as

$$\frac{V_{ik}}{S_{ik}} \cdot A_k = 0$$

where

$$S_{ik} = (\dot{z}_{ik}^2 + \dot{y}_{ik}^2 + \dot{z}_{ik}^2)^{\frac{1}{2}}$$

Thus, since the constraint may not be satisfied exactly and $V_{ik}$ is an estimate of the velocity, Equation (4.6) is modeled with additive white noise to relax the rigidity of the constraint. The resulting kinematic constraint equation is

$$\frac{V_{ik-1}}{S_{ik-1}} \cdot A_k + \mu_k = 0$$

where $\mu_k \sim N(0, R_k)$. The $\mu_k$ is a white Gaussian process that accounts for the uncertainty in the $V_{ik}$ and the constraint. Since the initial estimates of $V_{ik}$ may be very poor, $R_k$ is initialized with a large value and allowed to decrease as

$$R_k = r_1(0.75)^k + r_0$$

The filtering equations for the new formulation of the kinematic constraint are given in the following equations where $X_{ik}^k$ denotes the state estimate with after the constraint has been applied, and $P_{ik}^k$ is the associated state error covariance. The filtering equations are

$$X_{k+1|k} = F_k X_{ik}^k$$

$$P_{k+1|k} = F_k P_{ik}^k F_k^T + Q_k$$

$$X_{ik} = X_{ik-1} + K_k[z_k - H_k X_{ik-1}]$$

$$P_{ik} = [I - K_k H_k] P_{ik-1}$$

$$X_{ik} = [I - K_k^C C_k] X_{ik}$$

$$P_{ik} = [I - K_k^C C_k] P_{ik}$$

where

$$K_k = P_k [I - H_k^T (H_k P_{ik-1} H_k^T + R_k)^{-1}]$$

$$K_k^C = P_k^C C_k^T (C_k P_{ik-1} C_k^T + R_k)^{-1}$$

$$C_k = \begin{bmatrix} 0 & 0 & \dot{z}_{ik} & 0 & \dot{y}_{ik} & 0 & 0 & \dot{z}_{ik} & 0 \end{bmatrix}$$

Simulation results comparing the tracking performance of a filter using the previous formulation of the kinematic constraint and this formulation are in the next section.

V. Simulation Results

In order to evaluate the impact of the kinematic constraint on the tracking performance, three constant speed, maneuvering targets with circular-type trajectories were chosen for use in a simulation study. The horizontal and vertical profiles of the trajectories are shown in Figure 1. The circular trajectory denoted with 2 in Figure 1, turns with a constant rate while undergoing a 2 g maneuver. The other two trajectories in Figure 1 turn with rates that are either linearly increasing or decreasing. The trajectory denoted with 1 has an increasing turning rate which is the result of the acceleration increasing linearly from 2 to 3.5 g, whereas trajectory denoted with 3 has a decreasing turning rate resulting from the acceleration decreasing linearly from 2 to 0.5 g.

A constant acceleration was selected for tracking the targets given in Figure 1. The process noise covariance matrix $Q_k$ was determined by modeling the acceleration errors as a discrete-time process that remains constant between measurement updates. The variance of the acceleration errors was chosen to be $(0.25)^2$ m$^2$s$^{-2}$. The measurement update period was set to 0.20 seconds and the measurement variance errors were chosen to be $(8)^2$ m$^2$ in the range and $(2)^2$ mrad$^2$ in bearing and elevation. For the kinematic constraint formulated in Section III, $R_p = 1$ was suggested in [11]. For the new formulation, $R_k^2$ from equation (4.8) was selected with $r_1 = 100$ and $r_0 = 5$.

A standard Kalman filter, a Kalman filter utilizing the kinematic constraint discussed in Section III, and a Kalman filter utilizing the new formulation of the kinematic constraint were used to track the targets given in Figure 1. The filters used the same state and measurement models as discussed above, however the Kalman filter using the constraint
presented in Section III diverged; the state estimates had extremely large errors. Thus, the 2 g maneuvering target was tracked without measurement errors to test the algorithm and code. The results of that test are given in Figure 2, where the Kalman filter with the new formulation is denoted with 3. The Kalman filter with the new formulation provides the best position and velocity estimates as shown by comparison of the root mean square errors (RMSE). The acceleration estimates are the best at initialization. The value of $R_p$ was increased to help the the Kalman filter with the previous formulation when measurement errors were added, but the errors in the state estimates remained very large. Therefore, the tracking results of that filter are not given in this paper.

In Figure 3, the tracking results of the Kalman filter without the constraint and with the new formulation are given for the 2 g maneuvering target. The tracking results are given for the target with increasing acceleration in Figure 4. The results are an average of 100 experiments of a Monte Carlo simulation. In both cases, the Kalman filter with the constraint provided state estimates with less error. The use of the constraint also improved the settling time of the estimates at initialization. Similar results were obtained for the target with decreasing acceleration.

VI. Summary and Conclusions
The new formulation of the kinematic constraint was found to provide significantly better tracking results than the formulation given in Section III and [11]. Also, the use
The new formulation was shown to improve the tracking of three constant speed, maneuvering targets with a constant acceleration Kalman filter.

While improvement in tracking performance was obtained with the kinematic constraint, the actual benefits gained by using the kinematic constraint as a pseudomeasurement are not clear at this time. The results of this paper are for three specific trajectories. Therefore, additional research is needed to show the benefits and refine the method of implementation. Also, methods for selecting $R_n$ a priori and during the tracking process are also needed.

References