Abstract - Motivated by the need to reduce processor dimensionality, increase reliability, and create modularity, the properties of cascaded arrays are investigated. Cascade adaptive array structures can provide these benefits. Cascading is useful when the necessary processing speed is such that a single signal processor would be overwhelmed. An alternative to adapting over all the elements of the array is to select only certain elements of the array to adapt over. This results in a configuration of lower complexity which can be called a cascade array.

I. Introduction

There are numerous reasons to cascade arrays. A time division multiplexed (TDM) processor unit is possible because all the arrays at the same layer have the same optimum weight solution. Smaller subarrays yield processors of lower complexity and these smaller signal processors are less costly. Also there are advantages of using common hardware. The processor may serve several purposes (algorithms) simultaneously and be time division multiplexed. It is necessary to solve for only one subarray per stage. The weights of the subarray computed at each stage may then be copied to the other subarrays at that stage. This greatly lowers computational complexity. In the following the optimum cascade weight values are found using the Wiener-Hopf equation. Thus non-cascade and cascade structures may be compared to one another at convergence. The reliability of the cascade is enhanced through multiple smaller signal processors. If a single processor fails another of a pool of processors may take its place in the array or its output may be suppressed by succeeding signal processors.

II. Cascade Arrays Configuration

In the general cascade configuration the inputs to an antenna array are partitioned into sets of subarrays. The subarrays are then adapted and their outputs are considered as the inputs to the next layer of subarrays. The process is then continued until there is only a single final subarray. Figure 1 portrays this general cascade. Each of the subgroups is treated as one input to the next stage of adaptive combiners. The first group of subarrays implements transformation \( G_1 \), the second group of subarrays implements \( G_2 \), and the last group implements \( G_n \). Using this approach one smaller signal processor can be time division multiplexed between all the subarrays. The computational complexity is lowered. When the initial subarrays have the same structure (interelement phase shift) then their input covariance matrices are equal. The received signal at each of the subarrays in a cascade stage differs only in phase from the initial subarray in the layer. The cross-correlation vectors are also equal, thus all subarrays at a layer have the same optimum weight solution. It is not necessary to adapt all of the subarrays individually but merely to adapt one subarray and copy its weight solution to the other subarrays.

The overall effect of a stage cascade array can be represented as a series of transformations. First there is the \((n \times m)\) subarray input transformation \( G_1 \), which is followed by the \((m \times o)\) second stage transformation. If \( n \) inputs are represented as \( X = [x_1 \ x_2 \ldots \ x_n]^T \) (1) these inputs are transformed by the subarray transformation \( G_1 \) into \( m \) subarray outputs. This is represented by the equation

\[
Y = G_1X
\]  

(2)

where \( Y = [y_1 \ y_2 \ldots \ y_m]^T \) (3)

\( Y \) is a vector of the first stage subarray outputs. The \( m \) subarray outputs are then transformed by the next stage to produce the second stage subarray output vector, \( Z \).
The desired signal steering vector for subarray 2 is
\[ \mathbf{U}_{\text{st}} = e^{j\tau}[1 \exp(-j\phi_{\text{st}}) \ldots \exp(-j(n-1)\phi_{\text{st}})]^T \] (12)

where \( \tau \) is the phase delay between the first element of subarray 1 and 2. When only the signal and Gaussian noise are present the input covariance matrix for the first subarray is
\[ \mathbf{R}_{\text{xx}1} = \mathbf{A}_d \mathbf{U}_{\text{st}} \mathbf{U}_{\text{st}}^H + \sigma^2 \mathbf{I} \] (13)

For the second subarray the input covariance matrix is
\[ \mathbf{R}_{\text{xx}2} = \mathbf{A}_d \mathbf{U}_{\text{st}} \mathbf{U}_{\text{st}}^H + \sigma^2 \mathbf{I} \] (14)

substituting (12) into (14) yields
\[ = \mathbf{A}_d e^{j\tau} \mathbf{U}_{\text{st}} \mathbf{U}_{\text{st}}^H + \sigma^2 \mathbf{I} \]
\[ = \mathbf{A}_d \mathbf{U}_{\text{st}} \mathbf{U}_{\text{st}}^H + \sigma^2 \mathbf{I} \]
\[ = \mathbf{R}_{\text{xx}1} \] (16)

Thus the covariance matrices of the two subarrays are equal. Since the desired signal at the first element of each subarray is used as the reference
\[ \mathbf{P}_1 = \mathbf{U}_{\text{st}}^H \mathbf{U}_{\text{st}} = \mathbf{P}_2 \]

So both subarrays 1 and 2 have the same optimal weight solution.

IV. Subarray Covariance Matrices

Subarrays partition the total input covariance matrix \( \mathbf{R}_x \) into submatrices. As an example consider a five element array (Figure 2) treated as two subarrays of four elements each.

\[ \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \end{bmatrix} \]

Figure 2 Four Element Subarrays

The total input covariance matrix is
\[ \mathbf{R}_x = \mathbf{E}(\mathbf{X} \mathbf{X}^H) \] (17)

The covariance matrix for the first stage can be found by using a subarray selection matrix \( \mathbf{Q}_1 \).
\[ \mathbf{R}_{\text{xx}1} = \mathbf{Q}_1 \mathbf{R}_x \mathbf{Q}_1^H \] (18)

where...
The $Q$, matrix picks those elements out of $R_{1}$ that form first subarray input covariance matrix $R_{1x}$. The cross correlation vector for subarray 1, $P_{1}$, is

$$ P_{1} = Q_{1}U_{d} \quad (19) $$

and the optimum weight solution for the first subarray is

$$ W_{1\text{opt}} = R_{xx}^{-1}P_{1} \quad (20) $$

If subarray 2 consists of array elements 2, 3, 4, and 5 then input covariance matrix for subarray 2 is

$$ R_{2x} = Q_{2}R_{xx}Q_{2}^{T} \quad (26) $$

where

$$ Q_{2} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

The $Q$, matrices select the proper partition of the total input covariance matrix $R_{xx}$. In this case the partitions do not cover all of $R_{xx}$. In all cases the subarrays overlap because if the phase centers of the subarrays are too far separated then grating nulls will appear. [3] In general for suboptimal performance this is the case. The correlation vector for subarray 2 is

$$ P_{2} = Q_{2}U_{d} \quad (27) $$

The input covariance matrix for the next stage is

$$ R_{yy} = Q_{y}^{T}R_{xx}Q_{y} \quad (29) $$

where

$$ Q_{y} = \begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} & 0 \\ 0 & w_{11} & w_{12} & w_{13} & w_{14} \end{bmatrix} $$

The second stage cross-correlation is

$$ R_{yy} = Q_{y}^{T}U_{d} \quad (28) $$

The second stage subarray optimum weights solved using

$$ W_{2\text{opt}} = R_{yy}^{-1}P_{y} \quad (30) $$

For Figure 2 the second stage transformation matrix is

$$ Q_{y} = \begin{bmatrix} w_{21} \\ w_{22} \end{bmatrix} $$

and the final cascade gain is

$$ g_{f}(\theta) = |Q_{y}^{T}U_{y}(\theta)| \quad (22) $$

The general formula for the next stage covariance and cross correlation is

$$ R_{xx}^{(i+1)} = Q_{i}^{T}R_{xx}(i)Q_{i}^{T} \quad (23) $$

$$ P_{i}^{(i+1)} = Q_{i}^{T}P_{i}(i) \quad (24) $$

For a particular subarray at stage $i$

$$ R_{xx}^{(i)} = Q_{i}^{T}R_{xx}(i)Q_{i}^{T} \quad (25) $$

$$ P_{i}^{(i)} = Q_{i}^{T}P_{i}(i) \quad (26) $$

The optimum weights for a subarray at stage $i$ are

$$ W_{i\text{opt}} = [R_{xx}^{(i)}]^{-1}P_{i}^{(i)} = (Q_{i}^{T}R_{xx}(i))^{-1}Q_{i}^{T}P_{i}(i) \quad (27) $$

V. Example Cascade Structures

To establish a basis for comparison the weights and pattern for a five element array is calculated. The parameters used to calculate the optimum weights in the examples follow: the interelement to wavelength spacing ratio is $d=0.5$, the variance of the noise is $\sigma^{2}=1$, the desired signal to Gaussian noise ratio $\epsilon_{s}=1$. The amplitude of the desired signal is $A_{s}^{2}=\sigma^{2}$, the interference to Gaussian noise ratio $\epsilon_{i}=10$, the amplitude of the interference is $A_{i}=\sqrt{\sigma^{2}}$, the angle of the desired signal $\theta_{s}=50^{\circ}$, and the angle of interference $\theta_{i}=90^{\circ}$. Example 1. For comparison the weights for a five element array are calculated. From the Wiener-Hopf equation the optimum weights are

$$ W_{\text{opt}} = R_{xx}^{-1}P_{x} \quad (28) $$

$$ w_{1} = 0.206 + 0.114j $$
$$ w_{2} = -0.15 + 0.032j $$
$$ w_{3} = 0.141 + 0.061j $$
$$ w_{4} = -0.225 + 0.071j $$

$$ g_{s}(\theta_{s}) = 0.782 $$
$$ g_{i}(\theta_{i}) = 0.011 $$

The gains at the desired signal and interference angles for the optimum 5 element array pattern are

$$ \text{The voltage at the interference angle is 37.04 dB down from the gain at the desired signal angle. The SNR is found using} $$

$$ g(\theta_{s}) = 0.782 $$
$$ g(\theta_{i}) = 0.011 $$

The voltage at the interference angle is 37.04 dB down from the gain at the desired signal angle. The SNR is found using...
\[ \text{SINR} = \frac{(W^H R_{uu} W)}{(W^H (R_{ii} + R_{nn}) W)} \]  

(29) \[ \text{SINR} = 5.547 \text{ dB} \]

where \( R_{uu} \), \( R_{ii} \), and \( R_{nn} \) are the signal, interference, and noise parts of the covariance matrix respectively.

**Example 2.** The optimum cascade with of a five element array treated as two element subarrays is calculated.

![Figure 4 Two Element Subarrays](image)

The subarray selection matrix for stage one is

\[ Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \]

The first stage subarray weights are

\[ w_{11} = 0.127 + 0.238j \]
\[ w_{12} = 0.065 + 0.262j \]

The first stage transformation matrix is

\[ G_1 = \begin{bmatrix} w_{11} & w_{12} & 0 & 0 \\ 0 & w_{11} & w_{12} & 0 \\ 0 & 0 & w_{11} & w_{12} \end{bmatrix} \]

The second stage subarray selection matrix is

\[ Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \]

The second stage weights are then

\[ w_{21} = 0.831 - 0.131j \]
\[ w_{22} = -0.704 + 0.462j \]

The second stage transformation is

\[ G_2 = \begin{bmatrix} w_{21} & w_{22} & 0 & 0 \\ 0 & w_{21} & w_{22} & 0 \end{bmatrix} \]

The third stage subarray selection matrix is

\[ Q_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \]

The third stage subarray weights are

\[ w_{31} = -0.246 - 0.228j \]
\[ w_{32} = 0.3 - 0.555j \]

The third stage transformation is

\[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \]

\[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \]

The final stage weights are

\[ w_{41} = 0.550 + 0.019j \]
\[ w_{42} = -0.395 + 0.383j \]

The final stage transformation is

\[ G_4 = \begin{bmatrix} w_{41} & w_{42} & 0 \\ 0 & w_{41} & w_{42} \end{bmatrix} \]

The overall equivalent weights are

\[ w_1 = 0.029 + 0.069j \]
\[ w_2 = -0.186 - 0.005j \]
\[ w_3 = 0.021 - 0.213j \]
\[ w_4 = 0.185 + 0.033j \]
\[ w_5 = -0.042 + 0.062j \]

![Figure 5 Two Element Subarray Pattern](image)

Using (6) and (29) yields

\[ g(\theta_0) = 0.543 \]
\[ g(\theta_i) = 0.11 \]

\[ \text{SINR} = 0.743 \text{ dB} \]

The voltage at the interference angle is 3.87 dB down from the gain at the desired signal angle.

**Example 3.** The optimum weights are calculated for a five element array with three element subarrays.

\[ Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \]

The optimum stage one subarray weights are

\[ w_{11} = 0.202 + 0.267j \]
\[ w_{12} = -0.073 + 0.066j \]
\[ w_{13} = -0.246 - 0.228j \]

The first stage transformation matrix is

\[ G_1 = \begin{bmatrix} w_{11} & w_{12} & w_{13} & 0 \\ 0 & w_{11} & w_{12} & w_{13} \\ 0 & 0 & w_{11} & w_{12} \end{bmatrix} \]

The optimum stage two weights are

\[ w_{21} = -0.582 - 0.243j \]
\[ w_{22} = -0.249 + 0.225j \]
\[ w_{23} = 0.3 - 0.555j \]

The second stage transformation is
The total overall transformation, $W_{eq}$, is $G_1G_2^*$ yielding

$$
\begin{bmatrix}
w_1 \\
 w_2 \\
 w_3 \\
 w_4 \\
 w_5
\end{bmatrix} =
\begin{bmatrix}
0.182 + 0.107j \\
-0.137 + 0.035j \\
0.014 - 0.138j \\
0.127 + 0.062j \\
-0.2 + 0.068j
\end{bmatrix}$$

The voltage at the interference angle is 21.84 dB down from the gain at the desired signal angle.

**Example 4.** A five element array using 4 element subarrays of Figure 2 is evaluated. The optimum weight solution for the first stage is

$$
\begin{bmatrix}
w_{11} \\
w_{12} \\
w_{13} \\
w_{14}
\end{bmatrix} =
\begin{bmatrix}
0.222 + 0.193j \\
-0.14 + 0.022j \\
-0.065 - 0.125j \\
0.287 + 0.065j
\end{bmatrix}$$

The second stage optimum weights are

$$
\begin{bmatrix}
w_{21} \\
w_{22}
\end{bmatrix} =
\begin{bmatrix}
0.552 - 0.117j \\
-0.488 + 0.284j
\end{bmatrix}$$

The equivalent set of weights, $W_{eq}$, is

$$
\begin{bmatrix}
w_1 \\
w_2 \\
w_3 \\
w_4 \\
w_5
\end{bmatrix} =
\begin{bmatrix}
0.012 + 0.086j \\
-0.07 - 0.044j \\
0.029 - 0.046j \\
0.063 + 0.086j \\
-0.041 - 0.003j
\end{bmatrix}$$

Using (6) and (29) yields

$$
g(\theta_d) = 0.754 \\
g(\theta_s) = 0.061$$

SINR = 4.864 dB

The voltage gain at the interference angle is 29.92 dB down from the gain at the desired signal angle. The optimum array pattern is

**Conclusion**

The objective has been reduction in the complexity of calculating the weights for an adaptive antenna array. The advantages of cascaded adaptive arrays were discussed. General formulas for the cascade optimum cascade weights, SINR, and gain were developed. These formulas were applied to a five element array treated as two, three, and four element subarrays.

The following are advantages of a cascade approach:

- Necessary processor complexity is lowered and a time division multiplexed (TDM) processor is possible.
- Only one subarray need be adapted per stage.
- The number of inputs in each subarray is limited and therefore higher complexity algorithms are possible.

**References**


