Investigation of a Structure for Continuous-Time Adaptive Signal Processing

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ABSTRACT
The unpredictable and non-stationary characteristics of communication channels has typically demanded the use of adaptive filters for equalization and interference-removal. However, the increased bit rates and modulation complexity of modern communication systems is exceeding the capability of the usual discrete-time adaptive filter implementations. In this paper, an investigation of a continuous-time adaptive filter based on a minimum MSE performance criteria is presented. The filter is implemented using a transversal topology with a basic analog block between tap lines. The characteristics of the MSE performance surface and the effects of the basic analog block are discussed. Finally, the results of a digital computer simulation are presented demonstrating the performance of the filter as an adaptive equalizer.

1. INTRODUCTION
Adaptive signal processing is a very powerful tool which has received much attention in recent years. Telecommunications is one area in which this theory has found several applications. These applications include the equalization of communication channels, echo cancellation, multipath correction, and the removal of time-varying narrowband interferers in wideband spread spectrum systems.

At this time, most adaptive signal processing systems are implemented as digital filters with time-varying or 'adaptive' weighting coefficients. One of the principle reasons for the use of digital technology as opposed to analog implementations, in this area and several others, is the accuracy that it offers. This factor combined with recent improvements in the speed of digital technology, have made digital implementations an attractive solution to many adaptive signal processing problems. However, for signals such as high-capacity PCM signals, which often use complex modulation schemes requiring in excess of 40 MHz bandwidth, the use of high-speed digital technology is prohibited by the excessive size, weight, and power required [1]. For systems with such requirements, analog circuitry, with it's improved size, weight, power, and bandwidth capabilities, provides the only practical alternative. The principle disadvantage of analog implementations, the inaccuracies incurred by non-nominal element parameters, can be overcome by an adaptive system, making an analog implementation even more attractive.

In this paper, an investigation of a continuous-time adaptive filter based on a minimum MSE performance criteria is presented. The filter structure is defined in Section 2. Section 3 presents the characteristics of the MSE performance surface. The effects of the basic analog block used in the structure of the filter are given in Section 4. Section 5 presents a brief discussion of the stochastic gradient algorithm and its application to this problem. Finally, the results of computer simulation demonstrating the use of the adaptive filter as a channel equalizer are presented in Section 6.

2. THE ADAPTIVE FILTER STRUCTURE
Adaptive filter structure considered here, often referred to as an adaptive linear combiner, is shown in Figure 1. The filter consists of a cascade of identical analog blocks, tap lines between each block leading to variable weights and a summer to form the output of the filter. The input and output of the filter is denoted by $x(t)$ and $y(t)$, respectively, and will be assumed to be at least wide-sense-stationary (WSS) signals. The weights of the filter are denoted by the vector $W = [w_0 \ w_1 \ \ldots \ w_M]^T$ and the signals at the taps which are the inputs to the variable weights, are denoted by the vector $X = [x_0(t) \ x_1(t) \ \ldots \ x_M(t)]^T$.

The blocks, denoted by $H$, represent practically any continuous-time linear system which provides some filtering between its input and output. The exact form of $H$ considered here is more clearly defined in Section 4.

3. THE MSE PERFORMANCE SURFACE
The derivation which follows closely parallels that presented in [2] for the discrete-time adaptive linear combiner with much similarity in the resulting formulation.
The goal of the adaption process is to find the filter impulse response which minimizes the mean-square-error (MSE) of the output of the filter. The error, $e(t)$, of the output of the filter is defined as

$$e(t) = d(t) - y(t)$$  \hspace{1cm} (1)$$

where $d(t)$ represents the desired response of the filter. Thus, the quantity to be minimized is defined as

$$MSE = E[e^2(t)] = E[(d(t) - y(t))^2]$$  \hspace{1cm} (2)$$

where $E$ denotes statistical expectation. Since all signals have been assumed to be WSS, Eq. (2) can be expressed in terms of the respective auto- and cross-correlation functions as follows

$$MSE = r_{dd}(0) + r_{yy}(0) - 2r_{dy}(0).$$  \hspace{1cm} (3)$$

The derivation to follow is simplified if performed in the frequency domain. To achieve this, $r_{yy}(t)$ and $r_{dy}(t)$ are Fourier transformed to their corresponding power spectral densities to yield

$$MSE = r_{dd}(0) + \int [S_{yy}(f) - 2S_{dy}(f)]df$$  \hspace{1cm} (4)$$

where $f$ denotes frequency in hertz. If the frequency response of the filter shown in Figure 1 is denoted as $W(f)$, then

$$W(f) = Y(f)/X(f) = \sum_{k=0}^{K-1} \hat{w}_k H_k(f)$$  \hspace{1cm} (5)$$

where $Y(f)$ and $X(f)$ are the Fourier transforms of the $y(t)$ and $x(t)$, respectively. Eq. (5) and the relationships

$$S_{yy}(f) = W(f) W^*(f) S_{xx}(f)$$  \hspace{1cm} (6)$$

and

$$S_{dy}(f) = W(f) S_{dx}(f)$$  \hspace{1cm} (7)$$

can be substituted in Eq. (4) to obtain

$$MSE = r_{dd}(0) - 2 \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} \hat{w}_k S_{dx}(f) H_k(f) H^*_n(f) df.$$  \hspace{1cm} (8)$$

Eq. (8) can be simplified by the adoption of a matrix notation. The matrix $R$ is defined as

$$R = \begin{bmatrix} r(0,0) & r(0,1) & \ldots & r(0,N-1) \\ r(1,0) & r(1,1) & \ldots & r(1,N-1) \\ \vdots & \vdots & \ddots & \vdots \\ r(N-1,0) & r(N-1,1) & \ldots & r(N-1,N-1) \end{bmatrix}$$  \hspace{1cm} (9)$$

where

$$r(n,m) = \int S_{dx}(f) H^*_n(f) H_m(f) df.$$  \hspace{1cm} (10)$$

and the vector $P$ is defined as

$$P = [p(0) \ p(1) \ \ldots \ p(N-1)]^T.$$  \hspace{1cm} (11)$$

Thus, Eq. (8) can now be rewritten as

$$MSE = r_{dd}(0) + W^* R W - 2P^T W.$$  \hspace{1cm} (12)$$

where $P(MSE, W)$ denotes the partial derivative of the MSE with respect to the weight vector, $W$. The gradient vector is found by taking the derivative of Eq. (13) with respect to the weight vector, thus

$$G = p(MSE, W) = 2RW - 2P.$$  \hspace{1cm} (14)$$

This solution is often referred to as the Wiener solution for the optimum weight vector [1]-[4].

4. THE BASIC ANALOG BUILDING BLOCK

As stated in Section 1 the structure of $H$ in Figure 1 can be comprised of practically any continuous-time linear system which provides some filtering between its input and output. Possible realizations include an integrator or a low-pass filter. The form of $H$ considered here is that of a one-pole, unity d.c. gain, low-pass filter whose
frequency response is given by
\[ H(f) = \frac{a}{a + jf}. \]  
(16)

Converting Eq. (16) to exponential form and substituting into Eqs. (10) and (12) yields
\[ r(m,n) = 2 \int_0^\infty S_{xx}(f) H^M(f) \cos((m-n)\phi(f)) df \]  
(17)

and
\[ p(n) = 2 \int_0^\infty S_{wx}(f) H^M(f) \cos(n\phi(f)) df \]  
(18)

where \( H(f) \) is the magnitude response of \( H(f) \), or
\[ M(f) = \frac{a^2}{a^2 + f^2} \]  
(19)

and \( \phi(f) \) is the phase response of \( H(f) \), or
\[ \phi(f) = \tan^{-1}(-f/a). \]  
(20)

Notice that Eqs. (18) and (19) are purely real. This is due to the even symmetry of the real part and the odd symmetry of the imaginary part of \( H(f) \). Also note that \( r(m,n) = r(n,m) \), thus the matrix \( R \) is symmetric. Due to this symmetry, substitution of Eq. (15) into Eq. (13) yields the following expression for the minimum MSE,
\[ \text{MSE}_{(\text{min})} = r_{dd}(0) - P_W \omega_{\text{opt}}. \]  
(21)

The minimum MSE is not only a function of \( S_{xx}(f) \) and \( S_{wx}(f) \), as expected, but is also a function of the bandwidth, \( \omega \), of the basic analog building block, \( H(f) \). This dependence indicates that a careless selection of \( \omega \) might result in suboptimal performance of the filter in terms of the minimum MSE. To compound the problem, the information necessary to make a determination of a near optimal bandwidth for \( H(f) \), namely the channel characteristics for the case of adaptive equalization, are generally not available. Fortunately, the effect of the low-pass filter (LPF) bandwidth on the minimum MSE was found to be negligible, as will be shown below.

To demonstrate the effect of the LPF bandwidth on the minimum MSE, the specific problem of equalizing a BPSK signal in white noise is presented. For a BPSK signal, the power spectral density of the desired signal is given by [5]
\[ S_{dd}(f) = P_{av} T \left( \sin \frac{n\pi f T}{n\pi T} \right)^2 \]  
(22)

where \( T \) is the bit period, and \( P_{av} \) is the average power in the modulated waveform. Two specific cases of the problem are examined as described below.

4.1. Infinite Bandwidth Signal in Bandlimited Noise

The first case considered is that of equalizing an infinite bandwidth BPSK signal in bandlimited white noise with a 2-tap adaptive equalizer. Recall that the power spectrum density of white noise is a constant for all frequencies. Thus the bandlimiting of the noise was a necessary condition due to infinite power which would result from the evaluation of Eq. (17). In an attempt to reduce the number of variables, the bandwidth, the noise, \( B \), was constrained to satisfy the relationship
\[ B = k/T \quad k = 1, 2, 3, \ldots \]  
(23)

An analytical solution was obtained for the MSE as given by Eq. (21). To determine the bandwidth of the LPF which minimized the MSE, the derivative of the MSE with respect to \( \omega \) was found and set equal to zero. Then using a root-finder routine the equation was solved for the optimum LPF bandwidth, \( \omega_{\text{opt}} \).

Figure 2 shows a plot of the optimum LPF bandwidth versus \( E_b/N_0 \), the bit energy per noise power spectral density ratio, for various values of \( k \). Notice, for a constant value of \( E_b/N_0 \) as \( k \) increases so does the optimum LPF bandwidth. Recall that increasing \( k \) implies increasing the bandwidth of the noise which is equivalent to increasing the power of the noise at the input of the equalizer. Thus the increased LPF bandwidth for increased noise bandwidth is tantamount to allowing more noise power to pass through the equalizer. At first this result might seem incorrect. However, to correctly judge the performance of the equalizer the overall response of the equalizer should be examined, so that the effects of the filter weights can be seen.

In Figure 3, the magnitude response of the equalizer is shown for \( E_b/N_0 = 10 \) dB for three values of \( k \) as indicated in the figure. The three responses were determined with the optimal weight vector, \( \omega_{\text{opt}} \) and LPF bandwidth, \( \omega_{\text{opt}} \), and therefore should be considered the optimal response of the equalizer. The three responses clearly show that the performance of the equalizer was that which would be expected; namely, as the noise bandwidth increases the stopband attenuation also increases.
4.2. Bandlimited Signal in Bandlimited Noise

The second case considered is the equalization of a BPSK signal in white noise where both the signal and the noise are limited to the same bandwidth. The null-to-null bandwidth of the BPSK signal was chosen as the limiting bandwidth due to its wide acceptance as a reliable measure [6]. Again, the effects of the LPF bandwidth on the minimum MSE are desired. However, an analytical solution of Eq. (21) is greatly complicated by the bandwidth restriction on the signal so numerical techniques were employed.

In Figure 4 a plot of the MSE of the output of a 2-tap adaptive filter is shown as a function of the LPF bandwidth for $E_jN_0 = 10 \text{ dB}$. An optimal LPF bandwidth is implied by the minimum value shown in the curve. However, the more important observation is the very small range of values for the minimum MSE to the maximum MSE. Any choice of LPF bandwidth over the range displayed would result in a MSE within $0.1 \text{ dB}$ of the minimum obtainable MSE. It was also found that the range of minimum to maximum MSE decreases for larger values of $E_jN_0$. Similar results were found for longer filters. This nearly independent relationship between the MSE and LPF bandwidth is fortunate and should simplify the design of analog adaptive filters.

5. APPLICATION OF THE STOCHASTIC GRADIENT ALGORITHM

In Section 3, the adaption process was shown to be analogous to seeking the bottom of the bowl formed by the MSE performance surface. Generally, gradient methods are used for this purpose. One method which has found extensive use in adaptive signal processing is the stochastic gradient algorithm (SGA) [7]. This method forms an instantaneous estimate of the gradient which is then used to adjust the weight vector in the direction of the negative gradient estimate.

Let $X(s)$ and $Y(s)$ represent the Laplace transform of $x(t)$ and $y(t)$, respectively, and let $W(s)$ represent the transfer function of the adaptive filter shown in Figure 1. Then Eq. (2) can be rewritten as

$$\text{MSE} = E\left[(d(t) - L^{-1}\{W(s)X(s)\})^2\right]$$

where $L^{-1}\{\cdot\}$ denotes the inverse Laplace transform. The gradient estimate is then formed by taking the partial derivative of $e^2(t)$ with respect to each of the filter weights. The partial derivative of $e^2(t)$ with respect to $w_k$ is given by

$$p(e^2(t), w_k) = -2(d(t) - y(t))L^{-1}\{p(W(s), w_k)X(s)\}$$

where

$$p(W(s), w_k) = H'(s).$$

Thus the estimate of the gradient vector, denoted by $G'$, is given by

$$G' = -2e(t)X.$$  

Finally, each weight is formed as the integral of the negative gradient. Therefore, the value of $w_k$ at time $t$ is given by

$$w_k(t) = 2u \int_0^t e(s)x_k(s)ds$$

where $u$ is an adaptive gain factor which controls the convergence rate and stability. The effects of $u$ on both the rate of convergence and the stability of discrete-time adaptive filters has received much attention [1]-[4]. Due to the similarity in form of the equations presented here, particularly Eq. (13),
many of the constraints on u for the discrete-time case also apply here.

5. SIMULATION RESULTS

A computer implementation of a two-tap adaptive filter was made using a BFSK signal in white noise as the input. Both the signal and noise were low-pass filtered to the null-to-null bandwidth of the signal to simulate the conditions given in Section 4.2. In order to approximate continuous-time, the input signal was oversampled at a rate of 10 samples/symbol. A one-pole LPF was used as the basic analog block, H, and implemented through a bilinear transform as its discrete-time equivalent system. The output of the equalizer was passed through a threshold device to provide the desired response required by the adaption algorithm described in Section 5. The integral required by the weight update equation, Eq. (28), was performed following each input sample using a simple numerical technique. Finally, the MSE was estimated by passing the squared error of the filter output through a smoothing filter with a memory of 100 samples.

Figure 5 depicts a typical plot of the MSE versus adaption time, generally referred to as a learning curve. The flat line shown in the plot represents the minimum MSE for the conditions simulated. Two specific sources can be cited for the 'noisy' appearance of the curve. First is the error which arises from the numerical integration performed in the weight update equation. Secondly, and more importantly, is the error introduced by the gradient estimate given by Eq. (27). Both of these errors translates to random noise in the weight vector. Although the learning curve does approach the minimum MSE, on the average it will not reach $W_{\text{opt}}$, due to the misadjustment of the weight vector from its optimal value, $W_{\text{opt}}$. However, it is apparent that performance within an acceptable tolerance is obtainable with a continuous-time implementation.

7. CONCLUSIONS

In this paper, a structure suitable for continuous-time adaptive signal processing is presented. The formulation of the MSE performance surface is shown to be very similar to that of the equivalent discrete-time filter, allowing application of previously derived results. The effects of the basic analog block, used in the implementation of the filter, are discussed and an optimum bandwidth is shown to exist if the analog block is realized as a low-pass filter. Finally, an adaption algorithm is developed based on the stochastic gradient algorithm, and the results of a computer implementation of the algorithm are presented.

REFERENCES