On Three Dimensional Point Dissipative Systems of Differential Equations with Quadratic Nonlinearity

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Abstract
Sufficient conditions are developed for three dimensional quadratic systems with conservative nonlinearities to be point dissipative. The conditions are motivated geometrically. Lorenz systems are shown to fall within the class of systems studied.

Introduction

We are interested in quadratic dynamical systems
\[ x'(t) = Ax + f(x) \]
where \( f(x) \) is a quadratic vector valued function with the property
\[ x^T f(x) = 0 \]
We refer to this property as the dynamical system having a conservative quadratic term.

We seek conditions on the \( n \times n \) matrix \( A \) and \( f(x) \) so that the system is point dissipative, i.e., there is a bounded region which every trajectory of the system eventually enters and remains. The existence of such a region is guaranteed provided we can construct a Lyapunov function of the form
\[ V(X) = \alpha^T A x - x^T \left( \sum_{i=1}^{n} \alpha_i C_i - \frac{A + A^T}{2} \right) x \geq 0 \]
bounded[2].

For each vector \( \alpha^T = (\alpha_1, \alpha_2, \ldots, \alpha_n) \), we define the matrix \( C(\alpha) \) as follows:
\[ C(\alpha) = \sum_{i=1}^{n} \alpha_i C_i - \frac{A + A^T}{2} \]
It is clear that the set \( S \) will be bounded if we can choose the vector \( \alpha \) so that the matrix \( C(\alpha) \) is positive definite. The following theorem has appeared elsewhere[2].

**Theorem.** If there exists an \( \alpha \) so that \( C(\alpha) \) is positive definite, then the system \( x' = Ax + f(x) \) is point dissipative.

The condition on \( A \) and \( f(x) \) which guarantees the existence of such an \( \alpha \), when \( n=3 \), is the topic of the main theorem. Conditions for the case \( n = 2 \) appeared earlier[1].

Section 1. Geometric Motivation

In order to understand the geometry of the system we introduce the special Lyapunov function \( V_o \) where \( \alpha = 0 \), i.e., \( V_o(x) = x^T x \). Let us denote the region where \( \frac{dV_o(x(t))}{dt} \) is negative by \( V_o^- \). This is the region where the trajectories of the system are attracted toward the origin. We note that \( \frac{dV_o}{dt} = 2x^T Ax = 2x^T A_n x \) where \( A_n = \frac{1}{2}(A + A^T) \) the Hermitian part of \( A \). We also assume for simplicity that \( A_n \) has rank 3. Under a suitable rotation of the dynamical system \( A_n \) can be represented by a real diagonal matrix. There are four cases:
1. All the diagonal elements are positive and the trajectories of the dynamical system have limit infinity as \( t \) goes to infinity since \( \frac{dV_o}{dt} \) is positive for all \( x \).
2. One of the diagonal elements is negative and two are positive. Then \( V_o^- \) is the inside of a cone with its vertex at the origin. The axis of the cone is the eigenvector associated with the negative eigenvalue. See Figure 1.
3. Two of the diagonal elements are negative and one is positive. Then \( V_o^- \) is the outside of a cone with its vertex at the origin. The axis of the cone is the eigenvector associated with the positive eigenvalue. See Figure 1.
4. All the diagonal elements are negative and the trajectories of the dynamical system have the origin as a limit as \( t \) goes to infinity since \( \frac{dV_o}{dt} \) is negative for all \( x \).
We observe that the cone can be wide or narrow but can not contain a plane. However, the outside of the cone unioned with the origin always contains planes. In Section 3 we see that for $||x||$ large $f$ has the effect of forcing the trajectories into $V_0^*$, where the trajectories are attracted toward a neighborhood of the origin.

Section 2: The generators of $f(x)$

We can express $f(x)$ as

$$f(x) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} x^T x$$

where the $a_{ij}$'s, $b_{hk}$'s, and $c_{mn}$'s are real numbers.

If we examine $x^T f(x) = 0$ we see that for all $x$ we have

$$a_{11} x_1^3 + 2a_{12} x_1 x_2 + 2a_{13} x_1 x_3 + a_{22} x_2^2 + 2a_{23} x_2 x_3 + a_{33} x_3^2 +$$
$$b_{11} x_1^2 y + 2b_{12} x_1 x_2 y + 2b_{13} x_1 x_3 y + b_{22} y^2 + 2b_{23} y x_3 y +$$
$$c_{11} x_1^2 z + 2c_{12} x_1 y z + 2c_{13} x_1 x_3 z + c_{22} y^2 z + 2c_{23} y z^2 + c_{33} z^3 = 0$$

Hence, we get the following ten equations for the coefficients

1. $a_{11} = 0$
2. $2a_{12} + b_{11} = 0$
3. $2a_{13} + c_{11} = 0$
4. $a_{22} + 2b_{12} = 0$
5. $2a_{23} + 2b_{13} + 2c_{12} = 0$
6. $a_{33} + 2c_{13} = 0$
7. $b_{22} = 0$
8. $b_{23} + 2c_{22} = 0$
9. $b_{33} + 2c_{23} = 0$
10. $c_{33} = 0$

Since there are 18 coefficients in the three symmetric matrices and 10 equations we can express $f(x)$ as

$$x^T C x$$

using only eight coefficients where

$$C_1 = \begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{12} & -2b_{12} & a_{23} \\ a_{13} & a_{23} & -2c_{13} \end{bmatrix}$$

$$C_2 = \begin{bmatrix} -2a_{12} & b_{12} & b_{13} \\ b_{12} & 0 & b_{23} \\ b_{13} & b_{23} & -2c_{23} \end{bmatrix}$$

$$C_3 = \begin{bmatrix} -2a_{13} & -(a_{23} + b_{13}) & c_{13} \\ -(a_{23} + b_{13}) & -2b_{23} & c_{23} \\ c_{13} & c_{23} & 0 \end{bmatrix}$$

Let us choose one of the coefficients equal to 1 and the rest equal to 0. Say, $a_{12} = 1$ and the rest
equal to 0. Then we will let \( M(a_{12}) \) be the 9 x 3 matrix which can represent \( f(x) \),

\[
M(a_{12}) = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
-2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

where the matrix expression denotes

\[
\begin{aligned}
x^T C_1 x \\
x^T C_2 x \\
x^T C_3 x
\end{aligned}
\]

Here is complete list of the \( M \)'s

\[
M(a_{12}) = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
-2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
M(a_{13}) = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
M(b_{12}) = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
M(b_{23}) = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
M(c_{13}) = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
M(c_{23}) = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

It is interesting to note that the first six of quadratic functions are invariant under action of the permutation group on the coordinates \( x, y, \) and \( z \). That is the permutations acting on \( x^T M(a_{12}) x \) give the first six functions. The permutation group acting on \( x^T M(a_{23}) x \) is + or - either one of the last two functions or their difference.

We examine \( M(a_{12}) \) which is the system

\[
\begin{aligned}
\frac{dx}{dt} &= 2xy, \\
\frac{dy}{dt} &= -2x^2, \\
\frac{dz}{dt} &= 0
\end{aligned}
\]

Any trajectory has a fixed value of \( z = z_0 \) and since \( \frac{dy}{dx} = -\frac{x}{y} \) the trajectories lie on circles centered at \((0,0,z_0)\) and go from the positive \( y \) axis to the negative \( y \) axis since \( \frac{dy}{dt} \) is negative. See Figure 2.

\[
M(a_{12}) = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
-2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Figure 2

Hence, the set of limit points as \( t \to \infty \) is the \( y<0 \) half of the \( x=0 \) plane.

Next we examine \( M(a_{23}) \) which is the system

\[
\begin{aligned}
\frac{dx}{dt} &= 2yz, \\
\frac{dy}{dt} &= -x^2, \\
\frac{dz}{dt} &= -2xy
\end{aligned}
\]

Any trajectory has a fixed value of \( y = y_0 \) and since \( \frac{dz}{dx} = -\frac{x}{z} \) the trajectories lie on circles centered at \((0,y_0,0)\). In this case the trajectories do not have a limit as \( t \) goes to infinity but just continue going around a circle. See Figure 3.

The trajectories of an arbitrary \( f(x) \) are more complex than those of the generators. However, the generators do give some insight into the more general \( f(x) \).
Section 3. The influence of f(x)

To see the influence of f(x), the quadratic part of a quadratic dynamical system (1), we use the spherical coordinates

\[ x = r \sin \phi \cos \theta \]
\[ y = r \sin \phi \sin \theta \]
\[ z = r \cos \phi \]

Using \( \theta = \arctan(\frac{y}{x}) \) we find that

\[ \frac{d\theta}{dt} = \frac{x'y' - x'y}{r \sin \phi} \]

Likewise, using \( \phi = \arccos(\frac{z}{r}) \) we find that

\[ \frac{d\phi}{dt} = \frac{-z'}{r^2 - z'^2} \]

Now in (1) if we represent \( A x + f(x) \) as

\[ Ax + f(x) = \begin{bmatrix} A_1 x + f_1(x) \\ A_2 x + f_2(x) \\ A_3 x + f_3(x) \end{bmatrix} \]

in terms of \( r, \theta, \) and \( \phi \) we have

\[ A_j x = r B_j(\theta, \phi) \quad \text{and} \quad f_j(x) = r^2 \delta_j(\theta, \phi); \quad j = 1, 2, 3 \]

Then the derivatives of \( \theta \) and \( \phi \) have the form

\[ \frac{d\theta}{dt} = \frac{1}{r \sin \phi} ( (B_2(\theta, \phi) \cos \theta - B_1(\theta, \phi) \sin \theta) + r (\delta_2(\theta, \phi) \cos \theta - \delta_1(\theta, \phi) \sin \theta) ) \]

Now we can see that for large \( r \) the quadratic term \( f \) dominates the derivatives of \( \theta \) and \( \phi \).

Returning to the case when \( f \) is determined by \( M(a_{23}) \) it seems reasonable to suggest that the dynamical system is point dissipative when the \( y < 0 \) part of the \( x = 0 \) plane is contained in \( V_0^- \). Likewise, when \( f \) is determined by \( M(a_{23}) \) it seems reasonable to suggest that the dynamical system is point dissipative when the \( y \)-axis is contained in \( V_0^- \).

Section 4. The Theorem

We can summarize the previous sections by

1. \( \frac{dV_0(x(t))/dt}{dt} = 2x^TAx \) and hence depends on the Hermitian part of \( A \).
2. \( \frac{d\theta}{dt} \) and \( \frac{d\phi}{dt} \) are dominated by \( f \) for large \( |x| \).

It seems reasonable to suggest that if \( f \) forces the trajectories into \( V_0^- \) as \( |x(t)| \) gets large then the trajectories will turn toward the origin. In the case of \( M(a_{23}) \) for large \( |x(t)| \), \( f \) forces trajectories towards a subset of \( Z(f) \). In the case of \( M(a_{23}) \) for large \( |x(t)| \), \( f \) forces trajectories to circle a fixed distance around a line which is a subset of \( Z(f) \).

Hence, in either case if \( Z(f) \) is contained in \( V_0^- \) then the influence of \( f \) is to force the trajectories into this region when \( |x(t)| \) is large. In this case then the system may well be point dissipative. This is indeed the case as the Theorem of this Section states.

The mapping \( x0y: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) where

\[ x0y = \begin{bmatrix} x^TC_1y \\ \vdots \\ x^TC_ny \end{bmatrix} \]

can be regarded as a commutative multiplication in \( \mathbb{R}^n \). In addition to the standard vector addition and scalar multiplication in \( \mathbb{R}^n \), this multiplication \( x0y \) gives the vector space \( \mathbb{R}^n \) an additional structure of a commutative but generally nonassociative algebra \( \mathcal{A} \). The algebra \( \mathcal{A} \) is determined uniquely by the symmetric \( n \times n \) matrices \( \{C_i\} \). Some algebraic properties of this algebra \( \mathcal{A} \) can be used to investigate the conditions for point dissipativeness of the system \( x' = Ax + f(x) \). We are specially
interested in the concepts of nilpotent and
idempotent elements of this algebra $\mathcal{A}$. A nilpotent 
element $u \neq 0$ satisfies $f(u) = uu = 0$, while an 
idempotent element $v \neq 0$ satisfies $f(v) = vv = v$.
In any such algebra $\mathcal{A}$ generated by any $n$ symmetric 
matrices $\{C_i\}$, there exists at least one or the other 
of these elements[3]. In our case, because of the 
condition $x^Tf(x) = 0$ for all $x$, there cannot exist an 
idempotent element in $\mathcal{A}$. Hence, there must exist at 
least one nontrivial element in $\mathcal{A}$. We denote the set 
of nontrivial zeros of $f$ by $Z(f)$. Since $f_1, f_2,$ and $f_3$ 
are homogeneous polynomials, if $x$ is in $Z(f)$ then $ax$ 
is in $Z(f)$ for any nonzero real number $a$.

Theorem: If $Z(f)$ is contained in $V^0$ then $x'(t) = Ax + f(x)$ is point dissipative.

This theorem is proved by considering the 
various cases of the possible forms of $Z(f)$. That is, 
$Z(f)$ is a one or two dimensional vector space or even 
some union of some collection of these. In each case 
a Lyapunov function of the form $V(x) = (x - \alpha)^T(x - \alpha)$ is shown to exist such that the derivative of 
$V(x(t))$ with respect to time is negative for $x$ 
outside some compact set. Here, $x(t)$ is any solution 
to the dynamical system.

The Lorenz system [4, 5]
\[
\frac{dx}{dt} = \sigma(y-x)
\]
\[
\frac{dy}{dt} = rx - y - xz
\]
\[
\frac{dz}{dt} = xy - bz
\]
where $\sigma, r$ and $b$ are real positive numbers, fits into 
our analysis. $Z(f)$ is the union of the two 
dimensional vector space generated by $e_2$ and $e_3$ and 
the one dimensional space generated by $e_1$ with the 
origin deleted. Note that $(0, y, z)A(0,y,z)^T = -y^2 - 
bz^2 < 0$ for $(0,y,z) \neq 0$ and $(x,0,0)A(x,0,0)^T = -\alpha x^2 < 0$ 
for $x \neq 0$. Our analysis stops here but we conclude 
that all Lorenz systems are point dissipative.

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