A STRUCTURAL EQUIVALENCE BETWEEN NONLINEAR AND LINEAR ADAPTIVE CONTROLLERS

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Abstract

In [1]-[12], a new approach to adaptive control, called Linear Adaptive Control, was introduced and demonstrated. This new approach leads to a new form of adaptive controller that is completely linear and time-invariant. In [7] it was demonstrated by simulation studies of a particular example that Linear Adaptive Controllers generate control signals u(t) that are essentially equivalent to the control signals u(t) generated by conventional nonlinear adaptive controllers.

In this paper, the mathematical structure of a Linear Adaptive Controller is compared with the structure of several conventional types of nonlinear adaptive controllers. A term-by-term comparison of the equations representing each control law, using certain identity relations, enables us to show that the structure of a Linear Adaptive control law is essentially equivalent to the structure of conventional nonlinear adaptive control laws. This result demonstrates that Linear Adaptive Controllers are indeed genuine adaptive controllers, and not just another form of "robust" controller.

Introduction

In [1]-[12] a radically new approach to adaptive control was introduced. This new approach, called Linear Adaptive Control, eliminates the need for plant parameter estimators and adjustable controller gains thereby allowing the adaptive controller to be realized as an all-linear, completely time-invariant controller; i.e., a Linear Adaptive Controller. Historically, all previous adaptive controllers have been highly non-linear devices utilizing non-linear parameter estimators and complex "controller gain-adjustment" schemes. In fact, virtually all textbooks on adaptive control assert, in one way or another, that an adaptive controller must be non-linear by definition. Thus, it is not surprising that some researchers would question the legitimacy of using the term "adaptive" to describe these new Linear Adaptive Controllers.

In [7] it was demonstrated, by simulation studies of a particular example, that a Linear Adaptive Controller generates control signals u(t) that are essentially equivalent to the control signals u(t) generated by contemporary nonlinear adaptive controllers. This equivalence based on comparison of actual control signal plots u(t) vs. t was further demonstrated by the experimental results presented in the various papers comprising the 1988 ACC session on "A Showcase of Adaptive Controller Designs" [13].

In this paper we examine, in detail, the particular mathematical equations defining several of the nonlinear adaptive controllers presented in [13]. A term-by-term comparison of each nonlinear controller equation, with the equation for the linear adaptive controller presented in [6], enables us to show that the mathematical structure of the linear adaptive controller is essentially equivalent to that of several of the nonlinear adaptive controllers presented in [13]. This result provides further evidence that Linear Adaptive Controllers are, indeed, genuine adaptive controllers and not just another form of "robust" controller.

The First-Order Benchmark Example From [13]

The specific adaptive controller equations to be compared in this paper were developed using design approaches presented by various authors who participated in the 1988 ACC session on "A Showcase of Adaptive Controller Designs" [13]. These controller equations are developed for a common "benchmark" example, of first-order, that each author was asked to work. The details of that benchmark example are summarized in this section.

The example plant is first-order and linear, modeled by

\[ \dot{y} + ay = K(u+w) \]  \hspace{1cm} (1)

where \( y \) = plant output, \( u \) = control input, \( w \) = unknown, piecewise-constant disturbance input, and \( (a,K) \) are uncertain constant parameters. The adaptive control task is to steer \( y(t) \) to a given constant set-point \( y_{sp} \) in accordance with the specified ideal-response model

\[ \dot{y}_m + a_m y_m = y_{sp} \] \hspace{1cm} (2)

\[ y_m(t) = \text{ideal plant response } y(t), \] \hspace{1cm} (2)

\[ a_m = \text{specified constant}, \] (2)

and in the face of uncertain \( w(t) \) and perturbations in \( (a,K) \). The allowed inputs to the adaptive controller consist of \( \{y(t),y_{sp}\} \) only. In [13] the value of \( a_m \) was specified as \( a_m = +1 \), but here we will retain \( a_m \) in symbolic form to permit easier association of various terms in the controller equations.

In [6], the linear adaptive controller designed for (1), (2) was presented as a continuous-time controller. However, the non-linear adaptive controllers for (1), (2), derived by some of the other showcase participants, were presented in [13] as discrete-time controllers. Therefore, to facilitate comparison of those non-linear controllers...
with the linear adaptive controller, the plant/ideal-model equations (1), (2) and linear adaptive controller equation will be converted to their discrete-time (difference-equation) counterparts. Thus, the discrete-time version of (1) is

\[ y(k+1) + \alpha y(k) = K[u(k) + w(k)]; \quad k = 0,1,2, \ldots \]

(3)

where it is assumed \( \{u(t), w(t)\} \) remain constant over each sampling-interval, and \((\alpha, K)\) are related to \((a, K)\) by well-known expressions [14]. The discrete-time version of (2) is likewise found to be

\[ y_m(k+1) + \hat{a}_m y_m(k) = (1 + \hat{a}_m) y_{sp} \]

(4)

The Linear Adaptive Controller for (3), (4)

The discrete-time version of the linear adaptive controller for example (3), (4), as developed in [6], is derived as follows. Let the parameter perturbations \( \delta_a, \delta K \) in (1) be defined by

\[ a = a_N + \delta a \]

(5a)

\[ K = K_N + \delta K; \quad (a_N, K_N) = \text{known nominal values.} \]

(5b)

The corresponding perturbations in \( \hat{a}, \hat{K} \) can then be defined as

\[ \hat{a} = \tilde{a}_N + \delta \hat{a} \]

(6a)

\[ \hat{K} = \tilde{K}_N + \delta \hat{K}; \quad \delta \hat{K} = \delta \hat{K}(\delta a, \delta K) \]

(6b)

Substituting (6) into (3) yields

\[ y(k+1) + \tilde{a}_N y(k) = \tilde{K}_N u(k) - (\delta \hat{a} y(k) + (\delta K) u(k) + \hat{K} w(k). \]

(7)

Following the reasoning of Linear Adaptive Control theory [6] one now chooses \( u(k) \) to make the actual plant dynamics (7) consistently mimic the ideal plant model (4). Thus, comparing (4) with (7) it is clear that the linear adaptive control \( u(k) \) in (7) should (ideally) be chosen as

\[ u(k) = \hat{K}^{-1}[\tilde{a}_N y(k) + (1 + \tilde{a}_m) y_{sp} + (\delta \hat{a}) y(k) - (\delta \hat{K}) u(k) - \hat{K} w(k) \]

(8)

A key idea in linear adaptive control theory is recognition that the unknown term \( (\delta \hat{a}) y(k) \) in (8) can be re-written in a way that eliminates the need for on-line estimation of the perturbation \( \delta \hat{a} \). In particular, if we define

\[ e = y_{sp} - y \]

(9)

then

\[ (\delta \hat{a}) y(k) = (\delta \hat{a}) y_{sp} - (\delta \hat{a}) e(k) \]

(10)

and the (ideal) linear adaptive control (8) can thus be written equivalently as

\[ u(k) = \hat{K}^{-1}[\tilde{a}_N y(k) + (1 + \tilde{a}_m) y_{sp} - (\delta \hat{a}) e(k) \]

\[ + (\delta \hat{a}) y_{sp} - (\delta \hat{K}) u(k) - \hat{K} w(k) \]

(11)

The benefit of introducing the decomposition (10) into (11) is that: (i) the new term \( (\delta \hat{a}) y_{sp} \) in (11) is an unknown constant that can be lumped with the other unknown constant term \( \hat{K} w(k) \) in (11), and (ii) the unknown time-varying term \( "(\delta \hat{a}) e(k)" \) in (11) can be shown to obey a completely known constant coefficient, linear homogeneous difference equation [4]. Thus, the product term \( (\delta \hat{a}) u(k) \) in (11) can be physically generated by a simple linear observer with constant gains without necessity of generating the estimate \( \delta \hat{a} \) itself. Moreover, the sum of the last three terms on the right side of (11) can be modeled as an unknown "constant + ramp" time-function (linear spline) [6; p. 2456] and thereby can be physically generated by another simple, constant-gain, linear observer. In Linear Adaptive Control those two observers are consolidated into one composite observer, [6; eq. (51)].

We will now show that the mathematical equations for three non-linear adaptive controllers for example (3), (4), as developed in three of the papers presented in [13], are structurally equivalent to the linear adaptive controller equation (11).

Astrom's Non-Linear Adaptive Controller for (3), (4)

In [15], Astrom presented an adaptive control design procedure for (3), (4) using a methodology called "indirect self-tuning based on adaptive pole placement". The final form of the equation for Astrom's specific adaptive controller for (3), (4) is not explicitly shown in [15]. However, using the general controller design equations presented in [15; eqs. (1)-(13)] and the specific relations relevant to (3), (4) presented in [15; eqs. (14)-(16)] it is found that Astrom's adaptive controller for (3), (4), in fact, has the final form

\[ u(k+1) - u(k) = \hat{K}^{-1}[\tilde{a}_N y(k) + (1 + \tilde{a}_m) y_{sp} \]

\[ \quad + (\delta \hat{a}) y_{sp} - (\delta \hat{K}) u(k) + \hat{K} w(k) \]

(12)

where \( a_o \) is a design parameter, \( |a_o| < 1 \), and the unknown parameters in (12) are to be generated in real-time, using a non-linear, least-squares parameter estimation algorithm.

Astrom's adaptive controller (12) appears to be quite different from the linear adaptive controller (11). However, we will now show that (12) is, in fact, essentially equivalent to (11). For this purpose, we first re-arrange the terms in (12) to obtain
\[ u(k+1) = \hat{R}^{-1}\left\{ (\hat{a} - \hat{a}_m)y(k+1) + (1 + \hat{a}_m)y_{sp} - \\
 y(k+1) + ay(k) - Ku(k) + \right. \\
 \left. a_0 \left( (1 + \hat{a}_m)y_{sp} - y(k+1) - \hat{a}_m y(k) \right) \right\}. \]  

Next, notice from (3) that the collection of terms \( y(k+1) + ay(k) - Ku(k) \) in (13) is equivalent to \( \hat{R}w(k). \) Using the latter fact, together with (4) and the definition \( e_m = y_m - y \), we can re-write Astrom's adaptive controller (13) in the equivalent form

\[ u(k+1) = \hat{R}^{-1}\left\{ (\hat{a} - \hat{a}_m)y(k+1) + (1 + \hat{a}_m)y_{sp} - \hat{R}w(k) \\
+ a_0 \left( e_m(k+1) + \hat{a}_m e_m(k) \right) \right\}. \]  

Finally, we substitute the definitions (6.a), (9) into (15), and use (6.b) to write the term \( \hat{R}^{-1} \) in (15) as

\[ \hat{R}^{-1} = \left( \hat{R}_N + \delta K \right)^{-1} \]  

so that the terms in (15) can be re-arranged to yield Astrom's adaptive controller in the equivalent form

\[ u(k+1) = \hat{R}_N^{-1}\left\{ (\hat{a}_N - \hat{a}_m)y(k+1) + (1 + \hat{a}_m)y_{sp} - (\delta a)e(k+1) \\
+ (\delta a)y_{sp} - (\delta K)u(k+1) - \hat{R}w(k) \\
+ a_0 \left( e_m(k+1) + \hat{a}_m e_m(k) \right) \right\}. \]  

In [15, pg. 2425], Astrom elected to choose the design parameter \( a_0 \) in (17) as the "deadbeat" value

\[ a_0 = 0. \]  

In that case, Astrom's non-linear adaptive control law (12), expressed in the alternative form (17), (18), is seen to coincide exactly with the linear adaptive control law (11). The only "difference" between (11) and (17), (18), is in the way they are implemented. Namely, Astrom employs a non-linear parameter estimator to estimate \( \hat{a}, \hat{K} \) in real-time (those estimates are then used in the original form (12) of his adaptive controller). In contrast, in the linear adaptive control equation (11) we do not estimate \( \hat{a}, \hat{K} \) but rather estimate the product \( (\delta a)e(k), \) and the sum of the last three terms in (11), using a simple linear observer with constant gains. Thus when \( a_0 \) is chosen as zero, the two adaptive controllers (11) and (12) only differ in how certain mutually common terms in (11), (12) are grouped together and estimated. Of course, the use of non-linear parameter estimators to implement the adaptive control law (12), rather than constant linear observers, as used to implement the equivalent control law (11), will always yield a larger domain of adaptation in plant-parameter space (and perhaps better adaptation to unmodeled dynamics, etc.). This "degree of robustness" advantage of non-linear adaptive controllers over linear adaptive controllers is intrinsic and has been acknowledged many times in [1]-[12].

When \( a_0 \neq 0 \), Astrom's control law (17) is seen to differ from the linear adaptive control law (11) only by the presence of the one term

\[ a_0 \left[ e_m(k+1) + \hat{a}_m e_m(k) \right]. \]  

However, according to Astrom's design procedure, if \( a_0 \) is chosen to satisfy the stability requirement \(-1 < a_0 < 1\) the actual closed-loop plant response \( y(k) \) will asymptotically approach the ideal-response \( y_m(k) \) defined by (4) with a speed of convergence of \( e_m(k) \rightarrow 0 \), \( e_m \) defined by (14), that increases as \( a_0 \rightarrow 0 \) (maximum speed of convergence of \( e_m(k) \rightarrow 0 \), called "deadbeat", is obtained when \( a_0 = 0 \) as considered above). Thus, the control term (19) will typically be a small term that quickly vanishes as \( e_m(k) \rightarrow 0 \). It is recalled that the terms in the idealized linear adaptive control law (11) represent precisely what is needed to make the actual plant (7) exactly mimic the ideal model (4) for all \( \hat{a}, \hat{K}, \) and \( w \). Thus, the fact that Astrom's controller term (19), with \( a_0 \neq 0 \), does not appear in the ideal linear adaptive controller equation (11) raises a basic question about whether or not the term (19) with \( a_0 \neq 0 \) helps or hinders Astrom's controller (12). In other words, it seems that, technically speaking, the term (19) should be permanently deleted from Astrom's controller equation (12) regardless of the design choice for \( a_0 \).

In summary, the result (17) shows that the structure of Astrom's non-linear adaptive controller (12) is exactly the same as the linear adaptive controller (11), under the design parameter choice \( a_0 = 0 \) used by Astrom in [15], and differs only by the one small and quickly vanishing term (19) when other prudent choices of \( a_0 \) are used.

The M'Saad, Landau, et. al Nonlinear Adaptive Controller for (3). The non-linear adaptive controller derived for (3), (4) by M'Saad, Landau, et. al. in [16] is obtained from the (un-numbered) general equations in [16, Section 2.4] using the specific relations relevant to (3), (4) as presented in [16, Sections 3.4]. The explicit expression for that adaptive controller is obtained as follows. From [16, first eq. in Section 2.4] we have \( q^{-1} \) is the backward shift operator; \( d = 0 \) here

\[ S(q^{-1})D(q^{-1})u + R(q^{-1})y = P(q^{-1})\partial y_m \]  

where \( \beta = B^{-1}(1), \) and (5, 6, 7) are found by first choosing
an appropriate P* and then choosing (S,R) to satisfy

\[ P^*(q^{-1}) = A(q^{-1})D(q^{-1})S(q^{-1}) + q^{-1}B(q^{-1})R(q^{-1}) \]  

(21)

The particular choice for P* used by M'Saad, Landau, et. al. in [16] is not stated in [16]. However, the simplest and most natural choice for example (3), (4) is

\[ P^*(q^{-1}) = A_m(q^{-1}) \quad ; \quad A_m = \text{ideal-model operator.} \]  

(22)

Under the choice (22), the lower-order (S,R) that will satisfy (21) for this example are found to be

\[ S(q^{-1}) = 1 \quad ; \quad R(q^{-1}) = \bar{K}^{-1}[(\tilde{\alpha}_m - \tilde{\alpha}_0 + 1) + \tilde{\alpha}q^{-1}] \]  

(23)

Using (22), (23) in (20) we obtain the M'Saad, Landau, et. al. adaptive controller in the form

\[ (1-q^{-1})u(t) + \bar{K}^{-1}[(\tilde{\alpha}_m - \tilde{\alpha}_0 + 1) + \tilde{\alpha}q^{-1}]y(t) = (1 + \tilde{\alpha}y_m^{(1)})\delta y_m \]  

(24)

Since

\[ qy_m = [(1 + \tilde{\alpha}_m)/(1 + \tilde{\alpha}_m q^{-1})]y_m \]  

(25)

and \( \beta = B^{-1}(1) = \bar{K}^{-1} \), we can re-write (24) as

\[ (1-q^{-1})u = \bar{K}^{-1}[(1 + \tilde{\alpha}_m) y_m - (1 + \tilde{\alpha}_m - \tilde{\alpha})y_m] \]  

(26)

Finally, multiplying both sides of (26) by q, the M'Saad, Landau, et. al. nonlinear adaptive controller for (3), (4) can be expressed as

\[ u(k+1) - u(k) = \bar{K}^{-1}[(1 + \tilde{\alpha}_m) y_m - (1 + \tilde{\alpha}_m - \tilde{\alpha})y(k+1) - \tilde{\alpha}y(k)] \]  

(27)

Comparison of (27) with (12) reveals that the M'Saad, Landau, et. al. adaptive control law is precisely the same as Astrom's adaptive control law, under the choice \( a_0 = 0 \). We have already shown that (12) is equal to (11), when \( a_0 = 0 \), it follows that the linear adaptive control law (11) is precisely the same as the M'Saad, Landau, et. al. adaptive control (27).

The Goodwin, Salgado, Middleton Nonlinear Adaptive Control for (3), (4)

The non-linear adaptive controller derived for (3), (4) by Goodwin, Salgado, and Middleton in [17] is based on the general \( \delta \)-operator equation \([\delta^p(q^{-1})]\Delta]

\[ L(\delta)u = -P(\delta)y + H(\delta)y_m \]  

(28)

as given in [17; eq. (2.20)]. The design of (L,P) in (28) for the example (3), (4) is obtained by first choosing a suitable operator \( H(\delta) \) and then choosing \( (L,P) \) to satisfy

\[ A(\delta)L(\delta) + B(\delta)P(\delta)y = B(\delta)H(\delta)y_m \]  

(29)

as obtained from [17; eq. (2.25)]. The general \( \delta \)-operator model given in [17; eqs. (2.10)-(2.12)] is related to the more conventional discrete-time models (3), (4) by the following identities

\[ \tilde{\alpha} = \Delta a_0 - 1 \quad ; \quad \bar{K} = \Delta B = \Delta b_0 \quad ; \quad \bar{K}w = \Delta \eta \quad ; \quad \bar{\alpha}_m = \Delta \bar{\alpha}_m - 1 \]  

(30)

where \( a_0 \) in (30) has a different meaning than in (12). The particular choice of \( H(\delta) \) in (20) used for the design in [17] is not explicitly stated in [17]. However, one of the simplest and most attractive choices of \( H(\delta) \) for example (3), (4) is

\[ H(\delta) = b_0^{-1}[\delta^2 + (\tilde{\alpha}_m + \Delta^{-1})\delta + \tilde{\alpha}_m \Delta^{-1}] \]  

(31)

in which case \( (L,P) \) in (29) can be chosen in the simple form [Note: here \( A(\delta) = \delta + \bar{a}_0 \), \( A_m(\delta) = \delta + \bar{a}_m \)]

\[ L(\delta) = \delta \]  

(32)

\[ P(\delta) = b_0^{-1}[\tilde{\alpha}_m \Delta^{-1}] = \delta + \tilde{\alpha}_m \Delta^{-1} \]  

(33)

Substituting the choices (31), (32), (33) into (28), the Goodwin et. al. adaptive controller for (3), (4) is obtained as

\[ u = L^{-1}[\delta y + H y_m] \]  

\[ = b_0^{-1}\delta^{-1}
\begin{Bmatrix}
(a_0 - \bar{\alpha}_m \Delta^{-1})\delta \bar{\alpha}_m \Delta^{-1} \bar{y} + \\
[\delta + (\bar{\alpha}_m + \Delta^{-1})\bar{y} + \tilde{\alpha}_m \Delta^{-1}] \bar{y}_m
\end{Bmatrix} \]  

(35)

The structure of the adaptive controller (35) looks rather different from the linear adaptive controller (11). However, if we now substitute the identities (30) and the plant model (3) into (35), and carefully group the terms, it is found that (35) can be written as

\[ u(k+1) = \bar{K}^{-1}[(\tilde{\alpha} - \bar{\alpha}_m) y(k+1) + (1 + \tilde{\alpha}_m) y_m - \bar{K}w] \]  

(36)

Comparison of (36) with (15) shows that the Goodwin, et. al controller (36) is precisely the same as Astrom's controller, with Astrom's \( \bar{a}_0 \) in (15) set to zero. Since we have already shown that Astrom's controller (15), with \( \bar{a}_0 = 0 \), is the same as (11) we can immediately conclude that the nonlinear adaptive controller (35) based on the Goodwin, et. al. design procedure in [17] is precisely the same as the linear adaptive controller (11).

Summary and Conclusions

In this paper, the mathematical expressions for three non-linear adaptive controllers developed for the example (3), (4) by Goodwin in [15; with \( \bar{a}_0 = 0 \), M'Saad, Landau, et. al. in [16], and Goodwin et. al. in [17] have been shown to be precisely equivalent to the mathematical equation of the linear adaptive controller developed in [6]. The only "difference" is in the way certain terms in the controller equations are grouped together and estimated. In particular, the unique grouping of terms in
the linear adaptive control laws allows one to replace the traditional non-linear plant parameter estimators and complicated "gain adjustment" schemes by a simple constant coefficient, linear observer. Of course, the use of non-linear parameter estimators to implement the adaptive control laws (12), (24), (35) rather than constant linear observers, as used to implement the equivalent control law (11), will always yield a larger domain of adaptation in plant-parameter space (and perhaps better adaptation to unmodeled dynamics, etc.). This "degree of robustness" advantage of non-linear adaptive controllers over linear adaptive controllers is intrinsic and has been acknowledged many times in [1]-[12].

The results obtained here shed some interesting light on the nature of contemporary non-linear adaptive controllers and should remove any doubt that the linear adaptive controllers described in [1]-[12] are legitimate adaptive controllers. From the practical point of view, the lessened degree of robustness that accrues from the unique way linear adaptive controllers are implemented is (hopefully) offset by the easy and exact stability analysis made possible by the all-linear, constant coefficient nature of closed-loop systems using linear adaptive controllers.

References Cited
