THE CONCEPT OF HYPERSPACE IN DYNAMICAL SYSTEM THEORY

C. D. Johnson and J. Zhu

Electrical and Computer Engineering Department
University of Alabama in Huntsville
Huntsville, AL 35899

Abstract

Thus, at each x the total information needed to determine system behavior x(t) is embodied in the pair of vectors

\[ \{ x, \dot{x} \} = \{ x, F(x,t) \}. \]

This observation leads us to the following definitions.

Definition of Hyper-State

The hyper-state of (1) is a 2n-dimensional vector

\[ h = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \]

Definition of Hyper-State Space (Hyperspace)

The hyper-state space (hyperspace) \( \mathcal{H} \) of (1) is a 2n-dimensional euclidean space whose elements have coordinates \( \{ x_1, \ldots, x_n \}, \{ \dot{x}_1, \ldots, \dot{x}_n \} \).

Definition of the Evolution Manifold in Hyperspace

The evolution manifold \( \mathcal{M}(t) \) of (1) is the manifold of realizable hyperstates \( h \in \mathcal{H} \) defined by

\[ \mathcal{M}(t) = \{ h(t) = (x,F(x,t)) \}. \]

Remarks: The utility of the hyperspace concept derives from the fact that at each point \( h \in \mathcal{M} \), one knows simultaneously, the state \( x \) and the velocity \( \dot{x} = F(x,t) \). This expanded information vector embodies all determinates of state evolution \( x(t) \). Thus, properties of the dynamic evolution of \( x(t) \) can be characterized and analyzed in terms of the "motion" \( h(t) = (x,F(x,t)) \) without explicitly solving the equations of motion (1). This capability is similar to that made possible by using Lyapunov functions to study the stability of (1), and by using "isoclines" to construct state-portraits for (1).

The evolution manifold \( \mathcal{M}(t) \) is an invariant manifold for all motions \( h(t) \), and has dimension less than 2n, because according to (1) the elements \( x_i \) in (3) are functionally related to the elements \( x_i \). Thus, for each system (1) the motion of \( h(t) \) in hyperspace is constrained to a certain manifold \( \mathcal{M}(t) \) in a manner similar to that experienced when "state-overmodeling" [5]-[8] occurs in (1).

Applications of the Hyperspace Concept

The usefulness of the hyperspace concept is

...
demonstrated by the following examples.

Example 1 — Necessary and Sufficient Conditions for Asymptotic Stability of the First-Order System $x = f(x)$.

For this example, we consider the general class of first-order, nonlinear systems represented by

$$\dot{x} = f(x), \quad x = \text{scalar}, \quad f(x) = \text{piecewise continuous} \quad (5)$$

The problem is to find the necessary and sufficient conditions, on the function $f(x)$, to obtain asymptotic stability of the state $x = 0$ of (5). Referring to the generic hyperspace shown in Figure 1 it is immediately obvious from the geometry of $\mathcal{N}$ that the sought conditions are as follows. The evolution manifold $\mathcal{M}$ (i.e., in this case the "graph" of $f(x)$ vs. $x$) must: (i) be continuous in a neighborhood $N$ containing $x = 0$, (ii) pass through the origin $h = 0$ (i.e., $f(0) = 0$), and (ii) lie in the interior of the 2nd and 4th quadrants of $\mathcal{M}$ (i.e., $xf(x) < 0$), for all non-zero $x \in N$; see Figure 2. Moreover, it is also clear from Figures 1, 2 that the necessary and sufficient conditions for global asymptotic stability of $x = 0$ in (5) are: conditions (i) and (ii) above, plus the condition that $\mathcal{M}$ lies in the interior of the 2nd and 4th quadrants of $\mathcal{M}$ for all $x$. Notice that these conditions for asymptotic stability, when viewed in hyperspace, are seen to permit a wide variety of complexity and discontinuities in the function $f(x)$—including complex "hysteresis" effects.

When system (5) is of "time-varying type", i.e. (5) has the form

$$\dot{x} = f(x, t), \quad (6)$$

the necessary and sufficient conditions for asymptotic stability of $x = 0$ are more complicated. However, those conditions are easily determined by examining the time-varying geometry of $\mathcal{M}$ (in Figure 1; see the next example.

Example 2 — An Exact Eigenvalue Theory for Time-Varying Linear Systems

In a series of recent papers [9]-[11] an exact eigenvalue theory for a general class of $n$th order scalar time-varying linear systems has been developed. As an illustration of those results, the general solution of the second-order scalar system

$$\ddot{y} + a_2(t)\dot{y} + a_1(t)y = 0; \quad a_1(t) = \text{real piecewise continuous} \quad (7)$$

can be written as the weighted sum of two linearly independent particular solutions of the special form

$$\int_0^t \rho_1(t)dr + \int_0^t \rho_2(t)dr$$

$$y(t) = C_1e^{\rho_1(t)} + C_2e^{\rho_2(t)}; \quad \rho_1(t) \neq \rho_2(t), \quad (8)$$

where $\{\rho_1(t), \rho_2(t)\}$ are the exact time-varying eigenvalues of (7). The "characteristic equation" which determines the eigenvalues $\rho_1(t), \rho_2(t)$ is a differential characteristic equation found by substituting the generic particular solution $y = C \exp(\int \rho(r)dr)$ into (7). The result is the Riccati equation

$$\dot{\rho} + \rho^2 + a_2(t)\rho + a_1(t) = 0, \quad (9)$$

where, for this $n = 2$ example, $\rho_1(t) \neq \rho_2(t)$ is the only requirement needed to generate linearly independent particular solutions in (8). Thus, the time-varying eigenvalues $(\rho_1, \rho_2)$ in (8) are highly non-unique. Note that in the time-invariant case: $a_2 = constant$, $a_1 = constant$, one can set $(\rho_1, \rho_2) = constants$, in which case (8), (9) immediately reduce to the classical eigenvalue results. However, even when $a_1, a_2$ are constant, one can select to let the $\rho_i$ in (8) be time-varying eigenvalues satisfying (9). This latter option is apparently not commonly known in time-invariant linear system theory.

The practical utility of the "time-varying eigenvalue" type of solution structure (8) depends on one's ability to determine behavior of the solutions $\rho(t)$ of (9). An exact analytical solution of (9) is possible in some cases but is not feasible in general. On the other hand, important qualitative information about the general behavior of solutions $\rho(t)$ of (9) can be obtained by examining the hyperspace of (9). To see this, assume for the moment that $\rho(t)$ is real and consider the general hyperspace and evolution manifold $\mathcal{M}$, for (9), as shown in Figure 3. It is immediately evident from Figure 3 that

- Motion of $h(t) \in \mathcal{M}$ is always from right-to-left (left-to-right) for all $h$ corresponding to $\dot{\rho} < 0$, ($\dot{\rho} > 0$). The arrows in Figure 3 indicate motion of $h(t)$ along $\mathcal{M}(t)$ for $|a_2\dot{\rho} + a_1|$ sufficiently small; see (12) below.
- $\mathcal{M}(t)$ crosses (touches) the $\rho$-axis of $\mathcal{M}$ at time $t$ if, and only if,

$$a_2^2(t) - 4a_1(t) > 0. \quad (10)$$

- When condition (10) is satisfied at time $t$, $\dot{\rho}(t) = 0$ at the (real) $\rho$-value(s)

$$\rho_{e1}(t) = -0.5 [a_2 + \sqrt{a_2^2 - 4a_1}] \quad (11.a)$$

$$\rho_{e2}(t) = -0.5 [a_2 - \sqrt{a_2^2 - 4a_1}] \quad (11.b)$$

It will be convenient to refer to the $\rho$-values in (11) as "instantaneous" equilibrium states for the characteristic equation (9). Note that $\rho_{e1}, \rho_{e2}$ are conventional

*If $\rho_{e1} = \rho_{e2} = constant$, in this time-invariant case, one should replace $C_2$ in (8) by $C_2e^{\rho_{e1}t}$ where $C_2$ is a constant.
equilibrium states for (9), and coincide with the conventional (constant) eigenvalues of (7), when \( \lambda_1 \) and \( \lambda_2 \) are constant. Moreover, when \( \lambda_1 \) and/or \( \lambda_2 \) are time-varying, \( \rho_{c1}(t) , \rho_{c2}(t) \) are the so-called (real) "frozen-time" eigenvalues of (7). Although it is well-known that the \( \rho_{c1}(t) \) are not legitimate eigenvalues of (7), when (7) has time-varying coefficients, they are nevertheless often used by engineers to (hopefully) give an indication of the stability of (7).

- When the directions of motion of \( h(t) \) along \( \mathcal{M} \) are as shown in Figure 3, and inequality (10) is satisfied, the solution \( \rho(t) \) of (9) is repelled away from \( \rho_{c2}(t) \) and attracted toward \( \rho_{c1}(t) \) in the manner shown in Figure 3. This mode of behavior occurs if, and only if, the term \( (\hat{\lambda}_2 \rho + \tilde{\lambda}_1) \) does not alter the sign of \( \hat{\rho} = (2\rho - a_0)\rho - (\hat{\lambda}_2 \rho + \tilde{\lambda}_1) \) relative to the case when \( \hat{\lambda}_2 = \hat{\lambda}_1 = 0 \). That is, if and only if

\[
\text{sgn}((2\rho - a_0)\rho - (\hat{\lambda}_2 \rho + \tilde{\lambda}_1)) = \text{sgn}((2\rho - a_0)\rho) \quad \rho \neq 0 \quad (12)
\]

When (12) is not satisfied, the motion of \( h(t) \) can pass back-and-forth through the points \( \rho_{c1}(t), \rho_{c2}(t) \). This can occur, for instance, when \( \rho = 0 \) and \( (\hat{\lambda}_2 \rho + \tilde{\lambda}_1) \) is sufficiently "large." 

- Under condition (12), if \( \rho(t_0) < \rho_{c2}(t_0) \), it follows from Figure 3 that \( \rho(t) \) and \( \rho(t) \) approach \( \approx -\infty \). In fact, it turns out that \( \rho(t) \to -\infty \) and \( \rho(t) \to -\infty \) as \( t \to t_5 \) where \( t_5 = t_5(\rho(t_0), t_0) \) is a finite time (i.e. \( t_5 \) is a "finite escape-time") for the solution \( \rho(t) \). To see this, note that the escape motion of \( \rho(t) \) is eventually governed by the dominating terms in (9) so that

\[
\hat{\rho} = -\rho^2 \quad (13)
\]

for \( |\rho(t)| \) sufficiently large. The existence of a finite \( t_5 \) and the continuability of the solution \( \rho(t) \) across the escape-time \( t_5 \) is established by setting

\[
\xi(t) = \rho^{-1}(t) \quad (14)
\]

in (13) and noting that \( \xi(t) \) obeys the time-invariant linear differential equation

\[
\ddot{\xi}(t) = +1 \quad (15)
\]

Thus, \( \xi(t) \) is a well-defined continuous linear function of \( t \) that passes through zero at a finite time \( t = t_5 \), with a constant slope of +1. It follows that \( \rho(t) \to -\infty \) as \( t \to t_5 \) and, across \( t = t_5 \), \( \rho(t) \) jumps in value from \( -\infty \) to \( +\infty \) with \( \dot{\xi}(t) = -\xi^{-2}(t) \) and \( \dot{\xi}(t) = \xi^{-1}(t) \) taking well-defined values on both sides of \( t_5 \). Thus, across \( t_5 \) the hyperstate \( h(t) \) always "jumps," at "constant" negative infinite velocity \( \dot{\rho} = -\lim_{t \to t_5} \dot{\rho}(t), t = t_5 \) from point \( A = h(B \) as shown in Figure 3. The corresponding plot of \( \rho(t) \) vs. \( t \) is shown in Figure 4.

- When \( \lambda_2^2(t) - 4a_1(t) < 0 \) the real equilibrium states \( \rho_{c1}(t) \) cease to exist. If that condition, and (12), prevail over a sufficiently long interval of time \( [t_1, t_2] \), the hyper-state \( h(t) \) moves along \( \mathcal{M} \), as shown in Figure 5, experiencing a sequence of finite escape times \( t_1, t_2, t_3, \ldots \) of the type just described--until such time that (10) is reestablished.

When real equilibrium states \( \rho_{c2}(t) \) cease to exist it may be useful (but not necessary) to permit the time-varying eigenvalues \( \rho_{c1}(t), \rho_{c2}(t) \) in (8) to be complex-valued--just as in classical time-invariant eigenvalue theory. Setting \( \rho = \sigma(t) + j\beta(t) \) in (9), with \( (\sigma, \beta) = \text{real}, \quad j = \sqrt{-1} \) , the corresponding time-varying "characteristic equations" for \( \sigma(t) \) and \( \beta(t) \) are found to be the coupled set of first-order differential equations

\[
\dot{\sigma} + \dot{\beta}^2 - \alpha_2 \sigma + \alpha_1 = 0 \quad (16.a)
\]

\[
\dot{\beta} + (2\sigma + a_2)\beta = 0 \quad (16.b)
\]

Note that \( \dot{\beta}(t) \equiv 0 \) is always a particular solution of (16). The general qualitative behavior of solutions \( \sigma(t), \beta(t) \) of (16), for various combinations of \( a_1(t) \) and \( a_2(t) \) behavior, can be determined just as for (9), by considering the geometry of motion of \( h(t) \) along the associated manifolds \( \mathcal{M}_{\sigma}, \mathcal{M}_\beta \) in the respective hyperspaces of (16.a) and (16.b). In this regard, note that \( \mathcal{M}_\sigma \) always takes the form of a moving line that passes through the origin of \( \sigma_0^2 \) (i.e. \( \sigma_0^2 \) is a time-varying linear subspace of \( \mathcal{M}_\sigma \) and has time-varying slope \( -2\sigma + a_2 \)). The moving equilibrium pairs \( (\sigma_0(t), \beta_0(t)) \) corresponding to \( \dot{\sigma} = \dot{\beta} = 0 \) in (16) constitute typical cases of the illegitimate but popular "frozen-time" eigenvalues of (7).

The "time-varying eigenvalue" type of basis functions \( \exp(\int \rho(t) dt) \) used in the second-order example (7), (8) can be used for the \( n \)th order generalization of (7), (8), in which case the corresponding differential characteristic equation (9) generalizes to an \((n-1)\)th-order nonlinear differential equation; see [9]-[11].

Example 3 - Extension of Example 2 to the Case \( \dot{\mathcal{Z}} = A(t)x \).

In the case of vector linear time-varying systems

\[
\dot{x} = A(t)x, \quad x = (x_1, \ldots, x_n) \quad (17)
\]

the eigenvalue-type functions \( \exp(\int \rho(t) dt) \) used in (7), (8) still apply provided one writes the general solution \( x(t) \) of (17) as
\[
\int_{t_0}^{t} \rho_1(r) dr + \int_{t_0}^{t} \rho_n(r) dr \quad x(t) = C_1 a^{(1)}(t) e^{+...+C_n a^{(n)}(t) e^{+}} \tag{18}
\]

where the \(C_i\) are constants and the \(a^{(i)}(t)\) form a linearly independent set of time-varying, \(n\)-dimensional column eigenvectors to be determined. Substituting (18) into (17) it is found that the eigenvectors \(a^{(i)}(t)\) must satisfy the linear differential equation

\[
1 = [A(t) - \pi(t) I] a^{(i)}(t) ; \quad a^{(i)}(t_0) \neq 0. \tag{19}
\]

Note that (19) reduces to the classical result when \(A(t)\) is constant and one agrees to let \((a^{(i)}, \pi)\) be constants. It is clear from (18), (19) that there will always be considerable latitude (non-uniqueness) in the choice of the pairs of functions \((a^{(i)}(t), \pi(t))\) since each (vector) basis function in (18) consists of the product \(a^{(i)}(t) \exp \left[ \int \pi(r) dr \right]\). If \(A(t)\) in (17) is such that an element \(x_j(t)\) in (18) does not depend on the function \(x_j(t) = \pi(t)\) then the corresponding element \(a^{(i)}_j(t)\) is constrained to be zero. A systematic method for determining sets of functions \(\{a^{(i)}(t)\}, \{\pi(t)\}\) satisfying (19), can be illustrated by considering the general case \(n = 2\) in (17). In that case, (19) yields the two coupled linear differential equations

\[
\begin{align*}
\dot{a}^{(1)}_1 &= a_{11} a^{(1)}_1 + a_{12} a^{(1)}_2 - \pi_1 a^{(1)}_1 \\
\dot{a}^{(2)}_1 &= a_{21} a^{(2)}_1 + a_{22} a^{(2)}_2 - \pi_2 a^{(2)}_1 
\end{align*}
\tag{20}\]

It is clear from (20) that at least one of the unknown elements \(a^{(i)}_j, j = 1,2\), can always be chosen as constant. We will elect to set \(a^{(i)}_1 = \text{constant}, i=1,2\), and agree to choose

\[
a^{(1)}_1 = +1 \tag{21}
\]

whenever \(a^{(1)}_1\) is not otherwise constrained to be zero. As a consequence of the choice (21), expression (20.a) indicates that \(a^{(2)}_{12}\) must satisfy

\[
a^{(2)}_{12} = \pi_1 - a_{11} \tag{22}
\]

Thus, if \(a^{(2)}_{12}\) does not vanish, it follows that \(a^{(2)}_1(t)\) is explicitly defined by (22) as

\[
a^{(2)}_1 = (\pi_1 - a_{11}) a^{(2)}_{12} , \quad i = 1,2 \tag{23}
\]

At this point, \(\rho_1(t)\) in (20), (23) is still unspecified. To specify the required \(\rho_1(t)\) associated with the particular choice of eigenvectors (21), (23) we substitute (21), (23) into (20.b) to obtain the following explicit differential characteristic equation for \(\rho_1(t)\)

\[
\dot{\rho} + \rho^2 + \rho \left( -a_{11} a_{11} - a_{22} a^{(-1)}_{12} \right) + (a_{11} a_{22} + a_{12} a^{(-1)}_{11} a_{11} + a_{11} a_{12} a^{(-1)}_{12}) = 0. \tag{24}
\]

Thus, if \(a_{12} \neq 0\) it is now only necessary to choose as time-varying eigenvalues any two solutions \(\rho_1(t), \rho_2(t)\) of (24) that will yield eigenvectors \((a^{(1)}(t), a^{(2)}(t))\) in (21, 23) that are linearly independent for all \(t \geq t_0\). It is readily verified that this latter condition is realized if, and only if \(\rho_1(t_0) \neq \rho_2(t_0)\). The general behavior of solutions \(\rho(t)\) of (24) can be studied by the same hyperspace methodology used in Example 2.

If \(a_{12} \equiv 0\), the equation for \(\dot{x}_1\) in (17) is uncoupled from the \(x_2(t)\) motion. Associating \(\rho_1(t)\) with \(x_1\)-motion, it follows that one must then set \(a^{(2)}_1 \equiv 0\) in (21) and therefore (18) becomes

\[
\begin{align*}
\int_{t_0}^{t} \rho_1(r) dr \\
\int_{t_0}^{t} \rho_2(r) dr
\end{align*}
\tag{25.a}
\]

\[
\begin{align*}
x_1 &= C_1 a^{(1)}_1 e^{+ (\pi_1 t)} + a^{(2)}_1, \quad a^{(2)}_1 \equiv 0 \tag{25.a}
\end{align*}
\]

\[
\begin{align*}
x_2 &= C_1 a^{(1)}_2 e^{+ (\pi_1 t)} + C_2 a^{(2)}_2 e^{+ (\pi_2 t)} . \tag{25.b}
\end{align*}
\]

It is clear from (17), (25.a) that one must now set

\[\rho_1(t) = a^{(1)}_1(t)\]

and therefore (20.b) yields the two expressions

\[
\begin{align*}
a^{(1)}_2 &= a_{21} + a^{(1)}_2(a_{22} - \pi_1) \tag{26.a}
\end{align*}
\]

\[
\begin{align*}
a^{(2)}_2 &= a^{(2)}_2(a_{22} - \pi_2) \tag{26.b}
\end{align*}
\]

If, for instance, one now elects to choose

\[
\rho_2 = a^{(2)}_2(t) \tag{27}
\]

any solutions of equations (26) can be used to complete the eigenvectors \((a^{(1)}(t), a^{(2)}(t))\) in (26), provided \(a^{(1)}_2(t) \equiv 0\). If \(a^{(2)}_2(t) \equiv 0\), \(x_2\) in (17) is uncoupled from \(x_1(t)\) motion and we are then obliged to set \(a^{(1)}_2(t) \equiv 0\).

Note that (27) permits the special choice \(a^{(2)}_2(t) \equiv +1\). As an alternative to the choice (27), when \(a^{(2)}_2 \equiv 0\), one can choose any \(a^{(2)}_2(t)\) such that \(a^{(2)}_2(t) \neq 0\) so that \(\rho_2(t)\) is then defined by (26.b) to be

\[
\rho_2(t) = a^{(2)}_2(t) - (a^{(2)}_2 / a^{(2)}_2) \tag{28}
\]

An alternative approach to problems of this type has been proposed by Wu [12].
Summary and Conclusions

The traditional state-space for an n-dimensional dynamical system \( \dot{x} = F(x,t) \) does not embody sufficient information to determine the evolution motion \( x(t) \) at each state \( x \). In this paper we have introduced an extension of state-space, called hyperspace \( \mathcal{H} \), that consists of the set of 2n-dimensional hyper-state vectors \( h = (x|\dot{x}) \). The motion of \( h(t) \) in hyperspace is confined to a certain manifold \( \mathcal{M}(t) \) defined by \( h = (x|F(x,t)) \). The geometry of the motion \( h(t) \) \( \mathcal{M}(t) \) can be determined by inspection, without solving the original differential equation \( \dot{x} = F(x,t) \). This feature enables one to resolve complex questions about solutions \( x(t) \) by simple inspection of \( h(t) \) motions.

The practical utility of the hyperspace concept has been illustrated here by considering three examples.

References Cited