FORCED SYSTEMS WITH MULTIPLE STEADY STATES

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ABSTRACT

When a linear, time invariant system is driven by a sinusoid or other periodic input, the response is periodic with the same period as the input. Furthermore, the response is unique, regardless of how the input got to the periodic steady state if the system is asymptotically stable. For nonlinear systems, this is not the case. There may be two or more different steady state responses, as in the case of jump resonance, or there can be a nonperiodic response as in the case of systems that exhibit chaos.

This paper deals with the application of iteration and Volterra series analysis methods to certain types of problems with interesting solutions. In particular, the jump resonance phenomenon and the initial conditions leading to different harmonic solutions are considered.

I. INTRODUCTION

The design and analysis of electronic systems often require determination of the steady state response of nonlinear circuits. The most frequently used analysis method is harmonic balance, an approximate method that requires a priori assumption concerning the harmonic and intermodulation terms generated by the nonlinearities. Efforts to develop exact methods have used either iterative solutions or Volterra series [1-3].

The purpose of this paper is to reexamine some of the well known facts about nonlinear systems and their analysis, to reconsider the simplest systems that display truly nonlinear behavior, and from these examinations to develop analytic techniques that can be used with confidence to give both qualitative and quantitative analyses of nonlinear systems in the steady state. In this paper the discussion is confined to systems characterized by ordinary differential equations with polynomial nonlinearities.

When an asymptotically stable linear, time invariant system is excited by a single frequency sinusoid, the steady state response is a single frequency sinusoid at the same frequency. If the input consists of a number of sinusoids at different frequencies, the response will contain only those frequencies contained in the input. Superposition can be used to find the response by considering the input sinusoids one at a time.

When an almost linear (weakly nonlinear) system is excited by a single frequency sinusoid, the response will most probably contain components at the input frequency and at the harmonics of that frequency. If the nonlinearities are sufficiently weak, a steady state input will lead to a unique steady state output containing the harmonic frequencies and no other. One of the purposes of this paper is to get more quantitative statements than "sufficiently weak nonlinearities".

The steady state is what happens a long time after the system is assembled and turned on. The approach taken in this paper and in much of the system theory literature is that the system is assembled with no excitation. At $t = 0$ or at some other time $t_0$, the excitation is turned on. At $t = 0$, or $t_0$ there may be initial conditions on the system states. For linear systems, and weakly nonlinear systems, the initial states, and the nature of the excitation when it is "turned on" are irrelevant to the unique steady state response. For nonlinear systems that satisfy a Lipschitz condition, and these include many systems that are not "weakly nonlinear", the response at time $t$ to an excitation that starts at $t = 0$ or $t_0$, is unique [4]. Thus if the steady state is not unique, the different responses must evolve from different starting conditions, the initial state and the phase of the input when turned on.

In this study, a combination of the iteration solution method and Volterra series is used. In spite of rather satisfactory results for the quasi-linear solution case, this modified steady state iteration fails to predict the jump resonance phenomenon. Therefore, a transient iteration is deemed to be necessary in that regard. Such procedure leads to all possible periodic solutions corresponding to different initial condition values. In this analysis, initial conditions are always set to zero and the switching angle of the excitation, i.e., the phase angle at which the oscillation starts, is instead used as the deciding initial state in the attainment of all harmonic system solutions. Computer simulation results, applying to a cubic nonlinearity, are presented to show the regions of initial conditions which lead to two different steady state responses.

II. HARMONIC BALANCE

One approach to finding the steady state response of a nonlinear system is harmonic balance. In this approximate analysis one begins by assuming that the response to one or more sinusoidal inputs is a finite sum of sinusoids. The assumed response with unknown amplitudes for the various frequency terms is substituted into the system equations, and complex amplitudes at each frequency are equated separately. The result is a set of nonlinear algebraic equations relating the various amplitudes. Since these equations are nonlinear, they can have multiple solutions.
The type of equation under consideration is the following

\[ L(p)x + \sum_{k=2}^{N} c_k x^k = u(t) \]  

(1)

where \( p \) is the differential operator \( d/dt \), the linear operator \( L(p) \) is of the form

\[ L(p) = a_2 p^2 + a_1 p + a_0 \]  

(2)

and \( u(t) \) represents a forcing function. For practical purposes, \( u(t) \) is taken as the harmonic variation \( U \cos(\omega t) \).

Equations of the type (1) appear very often in applications [2]. Some important cases arising in the study of nonlinear electrical circuits are described in [5] and [6]. This section is particularly concerned with the existence, appearance, and disappearance of periodic solutions of (1) as the input amplitude \( U \) varies.

In order to examine some of the subtleties of the analysis of a system described by an equation of the form (1), a specific example is discussed. The example chosen is a form of Duffing's equation where \( L(p) \) is a second-order linear operator and the polynomial nonlinear function is assumed to contain the cubic term only.

\[ (p^2 + a_1 p + a_0) x^3 = U \cos(\omega t) \]  

(3)

In the case of harmonic oscillations in which the fundamental component having the frequency \( \omega \) predominates over the higher harmonics, the periodic solution of Eq. (3) may be assumed by the form

\[ x_0 = A_0 \cos(\omega t) + B_0 \sin(\omega t) \]  

(4)

It can be shown that the amplitudes \( A_0 \) and \( B_0 \) of the harmonic balance solution of (3) satisfy the following equation

\[ (G^2 + \omega^2 a_1^2) R^2 = U^2 \]  

(5)

where \( G = \omega^2 - a_0 - 3/4 c_3 R^2 \) and \( R^2 = A_0^2 + B_0^2 \) from which a solution of \( A_0 \) and \( B_0 \) can be determined. Figure 1 shows an amplitude characteristic of the harmonic oscillation, i.e., the relationship between \( U \) and \( R \), in the case where \( a_0 = 0.2 \), \( c_3 = 0.8 \), \( a_1 = 0.2 \), and \( \omega = 1 \).

It should be noticed that this method of solution merely the periodic states of equilibrium, which are not always realized, but are able to exist only so long as they are stable. Following the analysis in [7], the stability of the equilibrium state is investigated and the periodic solutions, sustained in the stable state are found. For the harmonic oscillation to be stable, one can show that the following inequality must be satisfied.

\[ \frac{27}{16} c_3 R^4 + 3 c_0 (a_0 - \omega^2) R^2 + (\omega^2 - a_0)^2 + \omega^2 a_1^2 > 0. \]  

(6)

It should also be noted that if the input amplitude \( U \) is assumed to vary while the frequency \( \omega \) is held constant, one can differentiate \( U^2 \) with respect to \( R^2 \) using Eq. (5). The resulting derivative may be expressed by

\[ \frac{dU^2}{dR^2} = G^2 + \omega^2 a_1^2 - \frac{3}{2} c_3 G R^2 \]  

(7)

which can be shown to be the same as the left side of inequality (6). Therefore, the stability condition may be rewritten as

\[ \frac{dU^2}{dR^2} > 0. \]  

(8)

This shows that the periodic solution is stable under such conditions that the output amplitude \( R \) increases with the increase of the input amplitude \( U \). From the physical point of view, this is a plausible conclusion.

The thick line in Fig. 1 indicates the unstable solutions where \( \frac{dU^2}{dR^2} \) is negative.

II.1 THE JUMP PHENOMENON

For a periodic solution to a nonlinear system the concept of uniqueness generally implies that for a specific input function there exists a single system response which is independent of system initial conditions. That is, given two solutions corresponding to different initial conditions, the solutions will asymptotically approach each other. Referring to Fig. 1, it is clear that there are three possible equilibrium states under certain values of \( U \). Two of them are stable, and the remaining one is unstable. The unstable state can not be sustained and is transferred to one of the two stable states as time increases.

In order to distinguish between the two stable states, the one with larger amplitude is referred to as the resonant state and the other with the smaller amplitude as the nonresonant state. The appearance of discontinuous jumps in amplitude \( R \) as the magnitude of the driving function is varied continuously is called jump resonance.

The curve of Fig. 1 explains the possible appearance of discontinuous jumps in amplitude. Starting from the origin, the amplitude \( R \) increases slowly with the increase of \( U \) while remaining in the nonresonant state. When the equilibrium state comes to the boundary of the thick line, a slight increase in \( U \) will cause a discontinuous jump to the resonant state. Further increase in \( U \) leads to further slow increase in \( R \).

A further reversal of the process, the oscillation jumps down from the resonant to the nonresonant state. This transition takes place at a lower value of \( U \) than before, thus exhibiting a hysteresis phenomenon. The two jumps of the amplitude are marked by the arrows in Fig. 1. Thus, there is a range of amplitudes, located between the arrows of the figure, for which either of two different steady state solutions may exist. Which of the two does exist depends upon the past history of the system.

III. INITIAL CONDITIONS LEADING TO DIFFERENT HARMONIC SOLUTIONS

It was mentioned in the preceding section that two different stable steady state solutions may exist for a given nonlinear system of the form (3), depending on different values of the initial condition. In this section the relationship between the initial conditions and the resulting periodic oscillation will be made clear. But it is usually not possible to solve nonlinear differential equations with arbitrary initial conditions, except for special cases in which exact solutions are known in analytic form. However, one can to some extent succeed in investigating the state of the oscillations by numerical means. The method of solution used in this paper is iteration cooperated with Volterra series.

III.1 STEADY STATE ITERATION

The basic nonlinear system can be represented by the feedback network given in Figure 2 with the nonlinear portion of the system isolated in the feedback loop. This allows the total system response, \( x(t) \), to be described in terms of the nonlinear integral equation (9).

\[ x(t) = \int_{t_0}^{t} [f(t - \tau) u(t) d\tau] - \frac{1}{k} \int_{t_0}^{t} [f(t - \tau) x(t) d\tau]. \]  

(9)
The first integral of (9) represents the response of the linear portion of the system while the second integral gives the nonlinear portion of the system response. In order to simplify the notation of (9), define the convolution operator $H[.]$ as in (10).

$$H[.] = \int [h(t - \tau)z(t)]d\tau$$

Equation (9) may now be written in a more compact form.

$$x(t) = H[.u] - H[.f(x)]$$

In order to solve practical problems, the system nonlinearities must be expressed in a usable format. This paper will assume the nonlinear elements of the system can be represented by a finite polynomial of order greater than one as given below.

$$f(x) = \sum_{k=2}^{N} a_k x^k$$

With this restriction, equation (11) becomes

$$x(t) = H[.u] - \sum_{k=2}^{N} a_k H[.x^k]$$

The iterative approach to solving nonlinear equations has been known for some time [8]. It consists of constructing a sequence of functions which are successive approximations to the actual solution. Each function is itself a solution to the system equation with the accuracy depending upon the number of iterations which have taken place. For equation (13), the sequence of solutions is constructed as follows:

$$x_1 = H[.u]$$

$$x_n = x_1 - \sum_{k=2}^{N} a_k H[.x_{n-1}]$$

It is clear from equation (15) that the complexity of the iterative calculations grows rapidly. Thus, for the sake of simplicity, the polynomial function is assumed to only include terms up to the third power. Furthermore, the allocated memory and the execution time can be reduced by making use of the relationship between the Volterra series and the iteration method [9]. For the $n$-th iterate calculation, the solution can be truncated at the $n$-th order term. The highest frequency component generated after this truncation is the $(2n-1)$ harmonic. This choice is justified by the assumed description of the nonlinearity.

A computer program has been written to demonstrate the viability of this iterative technique. As an example of the program usage, the solution to equation (3) is considered. In order to investigate the relationship between the initial state and the resulting steady state response, the switching angle at which the input oscillation starts, $\theta$, is included in the system equation (3). Using the steady state iteration program, the modified system is solved, for a standard test case where $a_0 = 0.2$, $c_3 = 0.8$, $a_1 = 0.2$, and $\omega = 1$, while varying the values of $\theta$ and keeping $U$ in the jump resonance region. In all cases, only the nonresonant solution is obtained. Numerical integration of the same system equation reveals the existence of both resonant and nonresonant solutions depending upon the prescribed value of $\theta$.

The inapplicability of the steady state iteration to the jump resonance problems raises important questions about the convergence of the steady state iterations. Leon [4] gave a sufficient condition for the convergence of the iterates that is independent of the initial state. By applying this procedure to the cubic nonlinearity case, an upper bound on the input amplitude $U$ is found so that the output is bounded and unique. It can be shown that an estimate of this bound is given by

$$U \leq \frac{2}{9} |H(\omega)| \left(\frac{c_2 Q}{14.47}ight)^\frac{1}{3}$$

where $Q$ is the $L^1$ norm of the linear kernel $h(t)$,

$$Q = \int_0^T |h(t)| dt$$

and $\omega$ is the frequency of the driving function.

Calculation of the upper bound in Eq. (16) requires the computation of the two constants $|H(\omega)|$ and $Q$. The magnitude of the linear transfer function $H(s)$ is easily determined at $s = j\omega$. However, the computation of $Q$ is much more complicated and takes a little more effort. For the test system,

$$h(t) = L^{-1} \left[ \frac{1}{s^2 + a_1 s + a_3} \right]$$

The $L^1$ norm of $h(t)$, computed, is $Q = 14.47$, and the linear transfer function is evaluated at $\omega = 1$ giving, $|H(\omega)| = 1.213$. Substituting the calculated values in (16) gives $U \leq 0.054$. Thus the steady state iterates converge to a unique solution so long as the input amplitude $U$ is less or equal to 0.054. By examining the system amplitude characteristic in Fig. 1, it becomes clear that the convergence requirement is only satisfied for the cases where a unique nonresonant solution may be determined. The maximum allowable excitation that the computations predict is certainly too small to drive the network into the jump resonance region. Hence, the convergence and uniqueness of a steady state iterative solution in a space of periodic functions require the absence of jump resonance phenomenon.

### III.2 TRANSIENT ITERATION

The failure of the steady state iteration method to predict the existence of multiple solutions does suggest some interesting experiments. Using a transient iteration approach to solve system (3), is it possible to determine all solution states? This subsection addresses this question and examines the solutions of the test system in particular.

Basically, the transient iteration differs from the steady state one in the implementation of the convolution operation $H[.]$ in Eq. (15). The evaluation of the transient iterates uses the time domain integration where the convolution operands are integrated from some initial time, $t_0$, to some final time $t_f$. When an equilibrium state is sought, the value of $t_f$ chosen is usually large to insure that all transients have been eliminated. On the other hand, the steady state iterates are computed in the frequency domain by conventional steady state analysis.

As an indication to the transient iteration capabilities, the cubic nonlinearity case is examined. In particular the equilibrium states of the test system, described in subsection III.1, are found by this method. In order to obtain all possible solutions, the initial state prescribed with the switching angle $\theta$ is allowed to vary between 0 and $2\pi$. The results are presented graphically by means of a diagram showing the resonant and nonresonant regions. The diagram applying to the test system, driven by $u(t) = 0.33 \cos(t + \theta)$, is shown in Fig. 3. The diagonal axes of the diagram are the boundary lines separating initial conditions which lead to one stable state from those which lead to the other. Also in the figure the shaded area is the region of initial conditions which give rise to the resonant

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state, and the blank area corresponds to the nonresonant state. Fig. 4 shows the same results in the R-0 plane. Here R is the amplitude of the harmonic solution taking on the two possible stable values depending upon the prescribed switching angle. These values show good agreement with the theoretical harmonic analysis of section II. However for the values of U near the lower or upper bound of the jump resonance region in Fig. 1, the transient iteration results do not compare well with the harmonic balance solutions. This discrepancy is attributed to the fact that the theoretical method assumed solution is only a first-order approximation whereas iteration yields the exact solution within the computer accuracy.

It is obvious from Fig. 3 that a switching angle of \( \theta + \pi \) radians leads to the same harmonic state as the one attained with \( \theta \). This fact is due to the symmetrical characteristic of the nonlinearity. In fact, the test system governing equation (3) is unchanged if the sign of \( x \) is reversed and \( t \) is shifted by \( \pi / \omega_0 \) radians. Hence this property remains valid provided that the nonlinearity function is odd.

In order to illustrate how the transient response drives the system to one of the different equilibrium states, the complete test system responses corresponding to two different initial states are shown in Fig. 5. The system is driven by a sinusoidal function of amplitude 0.33. In this figure, the nonresonant case where \( \theta = 0 \) is indicated by 'a'. The curve indicated by 'b' shows a switching angle of \( \pi / 2 \) radians leading the system to the resonant state.

Similar computer simulations are carried out for several amplitudes of the forcing functions. The results pertaining to two particular cases where \( U = 0.35 \) and \( U = 0.31 \) respectively are plotted in Fig. 6. As may be expected from the preceding analysis (Fig. 1), the blank regions related to the nonresonant state are contracted with the increase of the forcing amplitude. On the other hand, the shaded regions will disappear if the applied amplitude tends toward the jump resonance threshold determined from Fig. 1. It is important to note that this approach will also show the cases in which the steady state iteration and Volterra series methods will not yield meaningful results.

IV. CONCLUSIONS

The above discussions have dealt with the harmonic oscillations in forced nonlinear systems. A detailed analysis of a specific second-order nonlinear system with a cubic nonlinear characteristic was performed. The harmonic balance technique provided the approximate closed form harmonic solutions and predicted the existence of a multiple-valued equilibrium state. The possibility of determining all equilibrium states was also investigated using both a steady state iteration and a transient iteration. The initial state was described by the switching angle of the forcing function. In this analysis, it has been found that the convergence of the steady state iteration requires the uniqueness of the system solution in a space of periodic functions. Hence, this type of iteration fails to predict the jump resonance phenomenon. However, the transient iteration produced the actual system solution even for driving amplitude values within the jump resonance region.

REFERENCES

Figure 2. Basic nonlinear feedback loop.

Figure 3. Regions of initial conditions leading to the resonant and nonresonant oscillations with $U = 0.33$.

Figure 4. Amplitude of the harmonic oscillation of the test system driven with $U = 0.33$.

Figure 5. Transient oscillations leading to 2 different states of equilibrium. (a) $U = 0.33$, $\theta = 0$, and (b) $U = 0.33$, and $\theta = \frac{\pi}{2}$.

Figure 6. Regions in which the harmonic oscillations (resonant and nonresonant) are sustained. (a) $U = 0.35$, and (b) $U = 0.31$. 

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