Image Enhancement and Processing for Positron Emission Tomography

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ABSTRACT

The quality of Positron Emission Tomography (PET) images is often degraded by radial noise. Because this noise cannot be easily isolated either in the spatial or frequency domain, classical spatial and frequency techniques failed in removing this noise. As an alternative, we propose a new filtering method that works as follows. The image is converted from the rectangular $x$-$y$ coordinate system to the polar $r$-$\theta$ coordinate system. This causes the radial noise to map into horizontal lines. Using the Fourier transform, radial noise is then mapped into and around the vertical axis of the Fourier spectrum in the corresponding $\omega$-$\phi$ space. A class of filters is designed and applied to these images. The filtered images are then converted back to rectangular coordinates system for display and examination. This method has been tested using both synthetic and real data. The corrected images show minor residual noise and practically no loss of information around the heart wall.

1 Motivation

Positron Emission Tomography (PET) is the only imaging modality that provides doctors with early analytic and quantitative biochemical assessment and precise localization of pathology. PET is the newest component in the field of nuclear medicine. In contrast with the two relatively older techniques, Computerized Axial Tomography (CAT) and Nuclear Magnetic Resonance (NMR), PET has truly opened new avenues in both research and clinical ends of medicine by providing the means for precisely localizing and quantitatively assessing the biochemical processes underlying some of the most feared and hard-to-diagnose diseases. Advances in medicine are continuously demonstrating that the earliest and most significant changes underlying most organs' diseases are those that disturb the biochemical processes governing the functions of those organs. Therapeutic procedures--chemotherapy, radiation therapy, and/or hormone therapy--for these disorders attempt to initiate, accelerate, remove, or block the chemical processes that have been disturbed by these diseases [1,2,3].

Unfortunately, the present PET technology does not provide the necessary image quality from which such precise analytic and quantitative measurements can be made. These issues can be addressed directly by the disciplines of image processing and pattern recognition [4,5,6,7].

In PET images, boundary information as well as local pixel intensity are both crucial for manual and/or automated feature tracing, extraction, and identification. The quality of such information is significantly enhanced by the reduction of random noise and artifacts in the image. PET images suffer from significantly high levels of radial noise present in the form of streaks caused by the inexactness of the models used in image reconstruction. This problem pertains to the quality of the tomograph and the exactness of mathematical models used in signal conditioning, one-dimensional signal modeling and correction, and two-dimensional image modeling and reconstruction. Consequently, an image enhancement and restoration phase is necessary before PET data can be correctly interpreted.

The objective of this paper is to model PET noise and remove it without altering dominant features in the image. The ultimate goal here is to enhance these dominant features to allow for automatic computer interpretation and classification of PET images by developing techniques that take into consideration PET signal characteristics, data collection, and data reconstruction. A summary of related work can be found in [8,9,10,11,12].

In this paper, we propose a model for radial noise in PET images and use it to devise a spectral filtering technique combined with rectangular-to-polar and polar-to-rectangular image mappings in order to filter radial noise. A model of radial noise is developed in Section 2. Section 3 is devoted to experimental results illustrating the use of the newly proposed filtering technique and comparison with classical filtering techniques. Conclusions are given in Section 4.

2 Characteristics of Radial Noise

In this section, we model radial noise and study its effect on the frequency characteristics of a digital image. Both a theoretical presentation and a computer simulation of this effect are given. The objective here is to show that, when presented in a rectangular form, radial noise does not map in a specific area of the Fourier spectrum, hence classical low-pass filtering techniques that usually rid an image of random noise would fail. To illustrate this, we apply a first order low-pass Butterworth filter with the following transfer function:

$$H_4(u, v) = \frac{1}{1 + ([u^2 + v^2]/R_4^2)^{\alpha}}. \quad (1)$$
The cut-off frequency, \( R_0 \), is set to a variable fraction of the filtered image radius, \( {R}/2 \). The order of the filter, \( n \), is set to 1 in this experiment. Figure 1-c shows the filtered spectrum for \( R_0 = {R}/20 \); the resulting filtered image is shown in Fig. 1-f. The amount of filtered noise is limited for any reasonable cut-off frequency of the filter. As an alternative, we try to model this noise in a polar representation.

The function, \( g(x,y) \), is assumed to vanish outside a disc of radius \( R \). The rectangular-to-polar transformation in the \( x-y \) coordinate system is understood in a polar coordinate system instead of a rectangular space, (a) Simulated noise in a polar representation. (b) Its Fourier spectrum, (c) Low-pass filtered spectrum in the \( u-v \) space, (d) Reconstructed filtered image.

Assume a \( 2R \times 2R \) continuous image \( f(x,y) \), defined for \(-R < x < R \) and \(-R < y < R \), is corrupted with an additive radial noise that results in an image \( g(x,y) \):

\[
g(x,y) = f(x,y) + n(x,y).
\]

The function, \( g(x,y) \), is assumed to vanish outside a disc of radius \( R \). If not, both \( f(x,y) \) and \( n(x,y) \) are multiplied by the unit step function \( u(R^2 - x^2 - y^2) \).

Since the studied noise is radial, its properties are better understood in a polar coordinate system instead of a rectangular one. The rectangular-to-polar transformation in the \( x-y \) coordinate system is given by

\[
\begin{align*}
\rho &= \sqrt{x^2 + y^2} \\
\theta &= \tan^{-1}(y/x),
\end{align*}
\]

where \( 0 \leq \rho \leq R \) and \( 0 \leq \theta < 2\pi \). The inverse transformation, polar-to-rectangular, is hence given in the \( \rho-\theta \) coordinate system by

\[
\begin{align*}
x &= \rho \cos \theta \\
y &= \rho \sin \theta.
\end{align*}
\]

As defined here, both the rectangular-to-polar and polar-to-rectangular transformations are one-to-one mappings. The resulting images from the rectangular-to-polar transformation are \( f(\rho, \theta) \), \( n(\rho, \theta) \), and \( g(\rho, \theta) \). The objective here is to determine where the radial noise \( n(\rho, \theta) \) maps in the frequency domain:

\[
\mathcal{F}
\]

In the \( x-y \) space, the radial noise can be modeled as a series of \( K+1 \) \( \delta \)-functions emanating from the origin of the coordinate system along rays of length \( R \) at \( \theta_0, \theta_0 + \theta_1, \theta_0 + 2\theta_1, \ldots, \theta_0 + \theta_K \). The starting angle, \( \theta_0 \), is random; \( S = \{ \theta_i \} \) is a series of increment angles of the \( \delta \)-functions. Though it is apparent from looking at a PET image, there seem to be some regularity in the noise spatial distribution in PET images. In the following, we examine two radial noise models with the following two properties:

1. The starting angle \( \theta_0 \) is random over the interval \([0, 2\pi]\); the series \( S \) is constant, i.e., \( \theta_0 = \theta_0 = \theta_i = 0, 1, 2, \ldots, K \), with the spacing angle, \( \theta_s \), within the interval \([0, 2\pi]\).

2. The starting angle \( \theta_0 \) is random over the interval \([0, 2\pi]\); the series \( S \) is random but uniformly distributed over the interval \([0, 2\pi]\); i.e., the probability distribution of \( \theta_s \), \( \nu_0(\theta_s) = 0.5\pi^{-1} \).

In both cases, we will show that the spectrum of the noise maps in a local and well defined area of the image spectrum in the \( \omega-\phi \) system.

### 2.1 \( \theta_0 \) random and \( \{ S \} \) constant

If the starting angle \( \theta_0 \) is random and the spacing angle is constant, the radial noise may be modeled as a series of \((K + 1)\) \( \delta \)-functions emanating from the origin along rays of length \( R \) at \( \theta_0, \theta_0 + \theta_1, \theta_0 + 2\theta_1, \ldots, \theta_0 + K\theta_s \). The radial noise is defined as follows:

\[
n_1(x,y) = \sum_{k=0}^{K} \delta(\tan^{-1}(y/x) - \theta_0 - k\theta_s),
\]

where \( x^2 + y^2 < R^2 \), \( 0 \leq \theta_0 < 2\pi \), and \( 0 < \theta_s < 2\pi - \theta_0 \), with \( K = \lceil \frac{2\pi}{\theta_s} \rceil - 1 \). The function \( \lceil x \rceil \) is equal to the largest integer greater or equal to \( x \). In polar representation, \( n(\rho, \theta) \) is defined by

\[
n(\rho, \theta) = \sum_{k=0}^{K} \delta(\theta - \theta_0 - k\theta_s),
\]

with \( 0 \leq \rho < R \) and \( 0 \leq \theta < 2\pi \). The parameters \( \theta_0, \theta_s \), and \( K \) are subject to the conditions mentioned earlier.

The Fourier transform of \( n(\rho, \theta) \), \( N(\omega, \phi) \), may be computed as follows:

\[
\mathcal{F}[n(\rho, \theta)] = N(\omega, \phi) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} n(\rho, \theta) \exp[-j2\pi(\rho\omega + \theta\phi)] d\rho d\theta.
\]

Since \( n(\rho, \theta) \) is identically zero outside the intervals \([0,R] \) for \( \rho \) and \([0,2\pi] \) for \( \theta \),

\[
\mathcal{F}[n(\rho, \theta)] = \int_{0}^{2\pi} \int_{0}^{R} \delta(\theta - \theta_0 - k\theta_s) \exp[-j2\pi(\rho\omega + \theta\phi)] d\rho d\theta
\]

\[
= \int_{0}^{2\pi} \int_{0}^{R} \exp[-j2\pi(\rho\omega + \theta_0 + k\theta_s)\phi] d\rho d\theta
\]

\[
= \frac{1}{-j2\pi} e^{j2\pi \rho \omega} e^{-j2\pi \theta_0 \phi} \sum_{k=0}^{K} e^{-j2\pi k\theta_s \phi}.
\]

If we let

\[
\mathcal{F}[n(\rho, \theta)] = N(\omega, \phi) = A N_\omega(\phi) N_\theta(\phi),
\]
\[ A = R, \]
\[ N_\omega(\omega) = \frac{1}{-2\pi R_\omega} \sum_{k=0}^{K} e^{-j2\pi R_\omega k}, \]
\[ N_\phi(\phi) = e^{-j2\pi R_\phi \phi} \sum_{k=0}^{K} e^{-j2\pi R_\phi k}. \]

The function \( N_\phi(\phi) \) may be expressed as

\[
N_\phi(\phi) = e^{-j2\pi R_\phi \phi} \sum_{k=0}^{K} e^{-j2\pi R_\phi k}. \]

For \( \phi = \theta_\phi^{-1}, \)

\[
N_\phi(\phi) = (K + 1)e^{-j2\pi R_\phi}, \]

hence,

\[
N(\omega, \phi) = A (K + 1)N_\omega(\omega)e^{-j2\pi R_\phi}. \]

For \( \phi \neq \theta_\phi^{-1}, \)

\[
N_\phi(\phi) = e^{-j2\pi \Phi_\phi \phi} \frac{1 - e^{-j2\pi(K + 1)\Phi_\phi}}{1 - e^{-j2\pi \Phi_\phi}} = e^{-j2\pi \Phi_\phi \phi} \frac{1 - \cos(K'\alpha) - j \sin(K'\alpha)}{1 - \cos \alpha - j \sin \alpha}. \]

The terms \( |1 - \cos(K'\alpha) - j \sin(K'\alpha)| \) and \( |1 - \cos \alpha - j \sin \alpha| \) are both non-zero and finite, consequently, \( |N_\phi(\phi)| \) is finite. (Each expression is bound by \( \sqrt{5} \).

The function \( N_\omega(\omega) \), on the other hand, may be expressed as

\[
N_\omega(\omega) = \frac{e^{-j2\pi R_\omega} - 1}{-j2\pi R_\omega} = \frac{\sin(\pi R_\omega)}{(\pi R_\omega)}. \]

### 2.2 \( \theta_\phi \) random and \( \{S\} \) random

In this case, the starting angle \( \theta_\phi \) is random and the spacing \( \text{random with a uniform probability distribution of the series} S \text{ over} [0, 2\pi] \text{ for the angle} \theta_\phi. \)

The radial noise may be modeled as a series of \( (K + 1) \) functions emanating from the origin of the coordinate system along rays of length \( R \) at \( \theta_0, \theta_0 + \theta_1, \theta_0 + \theta_2, \ldots, \theta_0 + \theta_K. \)

The mathematical derivations shown in the previous section hold true if \( \theta_\phi \) is replaced by \( \theta_\phi \) in all equations until the expression giving \( N_\phi(\phi): \)

\[
N_\phi(\phi) = e^{-j2\pi R_\phi \phi} \sum_{k=0}^{K} e^{-j2\pi R_\phi k}. \]

Since \( \theta_\phi \) is uniform over \([0, 2\pi] \), \( N_\phi(\phi) \) may be approximated by the expected value of the right-hand side of the previous equation:

\[
N_\phi(\phi) = e^{-j2\pi R_\phi \phi} \sum_{k=0}^{K} e^{-j2\pi R_\phi k} \]

\[
= \frac{1}{2\pi} \sum_{k=0}^{K} (K + 1)e^{-j2\pi R_\phi k} \]

which is similar to the result found in the previous case when \( \phi = \theta_\phi^{-1} \). Hence,

\[
N(\omega, \phi) = A \frac{1}{2\pi} (K + 1)\overline{N(\omega, \phi)}(1 - e^{-j2\pi R_\phi}) \]

Consequently, in both cases the magnitude of \( N_\phi(\phi) \) behaves like a \( (\sin(\omega)) / \omega \), which achieves its maximum at \( \omega = 0 \) and decays rapidly as \( \omega \) increases.

In summary, since \( A \) and \( N_\phi(\phi) \) are finite,

\[
||N(\omega, \phi)|| = K \frac{\sin(\pi R_\omega)}{\pi R_\omega} \]

behaves like a \( (\sin(\omega)) / \omega \) running along and around the \( \phi \)-axis and is small for other values of \( \omega \) and \( \phi \). Note that the strength of power spectrum is directly proportional to the number of noise streaks in the image, \( K \).

With the rectangular-to-polar mapping, the radial noise maps around the \( \phi \)-axis in the \( \omega \)-\( \phi \) polar image representation. This is easily illustrated through the following experiment. The radial noise pattern shown in Fig. 1-a is converted into the \( \rho-\theta \) representation [Fig. 2-a]. Its Fourier spectrum is shown in Fig. 2-b. Despite the inexact rectangular-to-polar mapping, the spectrum is concentrated on and around the \( \phi \)-axis. This is also true for a noise which is simulated in the \( \rho-\theta \) space [Fig. 2-c] then converted into the \( \omega-\phi \) space [Fig. 2-d]. It is of note that most of the useful image data will map around the origin of the \( \omega-\phi \) space. Complete removal of the radial noise will wipe-out the data. Hence, it is expected that a residual noise will be left around the origin in any filtering scheme.

Figure 2: Radial noise filtering in a polar coordinate system: (a) Simulated noise in \( x-y \) space converted into \( \rho-\theta \) space, (b) Its Fourier spectrum in the \( \omega-\phi \) space, (c) Simulated noise in \( \rho-\theta \) space, and (d) Its Fourier spectrum in the \( \omega-\phi \) space.

### 3 Filtering of Radial Noise

In this section, we show some of the benefits of the mapping and the noise model developed in Section 3. Using properties of this model, we devise a filtering procedure that will rid the image of most of the radial noise.

In order to remove the radial noise present in PET images efficiently, we have to spatially characterize it in the frequency domain. First, a PET image is converted from the \( x-y \) space into the \( \rho-\theta \) space. The spectrum of both images is then computed. Figure 3 shows both images and their spectra.

Neither of the image spectra has distinct features from which the spectral component of the noise can be identified. To probe this issue further, we plot the power spectrum of the original image \( f(x, y) \), \( F(\nu, v) \), and the simulated noise \( n(\rho, \theta) \), \( N(\omega, \phi) \),
in both domains (u-v and ω-ϕ spaces) as a function of \(\sqrt{u^2 + v^2}\), \(\sqrt{\omega^2 + \phi^2}\), \(|\omega|\), and \(|\phi|\) which will be briefly denoted by \(r\). This is summarized in Fig. 4.

This series of plots is typical, i.e., the position of each curve does not vary significantly regardless of the real image data and regardless of the simulated noise data. From the analysis of these plots, we make the following remarks:

- For low values of the radius, \(r\), the power spectrum of the real PET image is higher than that of the simulated noise. The power spectrum in percent for \(r = 10 \approx R/12\) are 51, 54, 64, and 83 for the real PET data versus 8, 9, 11, and 45 for the simulated noise data. This means that towards small radii, on the average, the higher the noise the lower the image content of the image.

- The image mapping from the \(x-y\) space raises the image content for the real data with higher increments than for the simulated data for low values of \(r\). This is clear by comparing curves A to B and E to F, respectively. This effect is significantly masked by the averaging effect inherent in the mapping from the \(x-y\) space into \(\rho-\theta\) space which tends to "push" the image power spectrum towards the origin.

- By comparing curve B and F (both in the \(\omega-\phi\) space), say for \(r \leq 30\), we see that if filtered beyond this radius, 87% of the image is retained, in contrast to 21% of the noise content.

- By comparing the relative position of curve G with respect to curve F versus the curve H to F, we see that the \(\omega-\phi\)-axis is more dominant than the \(\rho-\theta\)-axis. This is evident in the surge in the curve H, particularly for small radii, which remains significantly higher not only to E, F, and G but also higher than all other curves for small values of the radius \(r\) (see table also). This fact was verified theoretically in Section 3 when the noise was modeled by a series of impulses along rays emanating from the origin running radially towards the edges of the image. This is also expressed in Fig. 2.

In summary, the image mapping and noise model proposed in Section 3 show that the power spectrum of the noise (radial artifacts) present in PET images which is spread randomly over the spectral domain in the rectangular representation [Fig. 1] is mapped locally around the \(\phi\)-axis in the spectral representation of the image after the above mapping is applied [Fig 2 and 4]. Using this fact we designed a notch filter that attenuates the image spectral content along the \(\phi\)-axis and leaves the rest of the image untouched. Understandably, to preserve the "ideal image", this filter cannot remove the image content around the origin. This will allow a residual radial noise to remain in the image. This residual noise can be controlled by the radius around the origin: \(R_o\), which, for practical purposes, will be the only parameter in this notch filter:

\[
H_n(\omega, \phi) = [u(\omega^2 - R_o^2) - u(R_o^2 - \omega^2 - \phi^2)]G_n(\omega, \phi) . \tag{3}
\]

The weighting term \(G_n(\omega, \phi)\) is a Gaussian smoothing function (zero mean and \(\sigma^2\) variance) used to avoid the ringing effects that often result from discontinuities in the filter transfer function. The degree of smoothing is controlled by the parameter \(\sigma^2\) [6]. In this experiment, \(R_o = 12\) and \(\sigma^2 = 9\). It is worth emphasizing again that the performance of this filter is unchanged for a wide variety of images. This filter is applied to the PET image; the results are shown in Figs. 3 and 5.

Visual inspection and comparison between the original image and filtered version reveal a significant reduction in artifact noise and an overall improvement in image quality. The artifacts have been removed without severe degradation of the edge information present in the image. This can be further seen by comparing the outcome of this filtering procedure against that obtained by classical noise removal techniques based on spectral low-pass filtering. In Fig. 6, we show (a) the original PET image, (b) the resulting filtered image using the technique presented in the previous section, (c) the resulting filtered image.
using spectral low-pass filtering with a cut-off in the radius of \( R_h = R/10 \), and (d) the resulting filtered image using spectral low-pass filtering with a cut-off radius of \( R_h = R/20 \). As mentioned earlier, a reasonable trade-off cut-off radius cannot be achieved. It can be easily seen in Fig. 6-(c) and (d). However, using the transfer function of the notch filter given in Eq. 3, we were able to remove most of the artifacts without significant degradation in the dominant features of the image.

4 Conclusions

The quality of boundary information and local pixel intensity in PET images is significantly enhanced by the reduction of random noise and artifacts. Noise characteristics in PET do not allow for attenuation or elimination of these artifacts using simple noise removal techniques. A method based on the spectral characteristics of the targeted noise and mapping of the image data from rectangular-to-polar and vice versa was developed and tested using synthetic and real data. Specifically, we have modeled the noise steaks in PET images in both rectangular and polar domains and have shown both analytically and through computer simulation that it exhibits consistent mapping patterns. A class of filters was designed and applied successfully. Visual inspection of the filtered images show clear enhancement over the original images.

Future work will involve testing of this technique on a large set of heart and brain images and the investigation of the statistical characteristics of this method and its degree of dependence on the noise level in the image data. The notch filter given in Eq. 3 is uniquely specified through the parameter \( R_h \) which depends largely on the ratio of the ideal image data to noise data. In this paper, the radius \( R_h \) was selected by trial-and-error; future work will address the automatic selection of this parameter as a function of some quantitative information extracted from the image.

References