APPLICATION OF SYMMETRY RESULTS IN 3-D DIGITAL FILTER DESIGN

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ABSTRACT
In this paper a systematic way of designing a three-dimensional (3-D) filter, using the symmetries in the given specifications, is given. In this respect a step by step procedure of an optimization-based design procedure is given. Then to illustrate the advantage of this procedure, two filter design example are taken. The first example is to design a 3-D spherically symmetric lowpass filter. The second example is to design a conic filter, similar to “fan filter” in 2-D.

1. INTRODUCTION
In recent times, the need for three-dimensional (3-D) filters has risen in many applications [1]-[3] such as processing of 3-D data resulting from pictures of moving objects, computer aided tomography, medical diagnosis, etc. In general, an increase in the dimension of the filter results in an increase by many folds of the computations to be performed. This has necessiated the use of high speed computers with large memory capacity. In addition, efforts have been directed to devise ways of simplifying the amount of computations required. In the two-dimensional (2-D) case one method of reducing the complexity of the design and implementation is to make use of the various symmetries that might be present in the frequency response specifications [4]-[6]. To apply similar symmetry properties in the 3-D filter design and implementation schemes is the main objective of our work.

Earlier in [7]-[8], the necessary and sufficient conditions on 3-D analog and digital polynomials and functions such that their frequency responses possess specified symmetries were derived. The derivation was based on the magnitude response viewed in the 3-D frequency space. The various symmetries studied were with respect to the planes perpendicular to the frequency axes. The interrelationships among these symmetries were also presented in these papers. In this paper we discuss the application of these 3-D symmetries in the design of digital filters.

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2. SYMMETRY IN THE DESIGN OF DIGITAL FILTERS
A general 3-D digital filter can be represented by its transfer function as

\[ H(z_1, z_2, z_3) = \frac{P(z_1, z_2, z_3)}{Q(z_1, z_2, z_3)} = \sum_{l=-L}^{L} \sum_{m=-M}^{M} \sum_{n=-N}^{N} p(l, m, n) z_1^l z_2^m z_3^n, \]

The design of a 3-D filter involves the determination of the filter coefficients \( \{p(l, m, n)\} \) and \( \{q(l, m, n)\} \) such that either the filter frequency response or the filter impulse response approximates the specifications within a specified tolerance. In general, the complexity of any design scheme as measured by the time required to calculate the filter coefficients increases exponentially with the number of unknown filter coefficients. This is especially so in schemes using optimization techniques. The design complexity can, however, be reduced if the number of independent filter coefficients are reduced. This can be achieved by making use of the presence of symmetries in the desired frequency or impulse response. The type of symmetries that were investigated in this study are:

(i) symmetries that occur in planes perpendicular to a single axis (single planar symmetries),
(ii) symmetries that are present in two sets of planes which are perpendicular to two specified axes respectively (double planar symmetries), and
(iii) symmetries that occur in three sets of planes which are perpendicular to three axes respectively (triple planar symmetries).

As in 2-D [4]-[6], the symmetries studied in 3-D are quadrant symmetry, diagonal symmetry, 4-fold rotational symmetry, and octagonal symmetry. The definitions of these 3-D symmetries are listed in Table 1.

Suppose that the desired frequency response of a digital filter possesses triple planar quadrant symmetry and if one is interested in designing a FIR filter of the form

\[ H(x_1, x_2, x_3) = \sum_{l=-L}^{L} \sum_{m=-M}^{M} \sum_{n=-N}^{N} h(l, m, n) x_1^l x_2^m x_3^n, \]
then the number of unknown coefficients that are to be determined are \((2L+1)(2M+1)(2N+1)\). But using the triple planar quadrantal symmetry, \(L = M = N\), and \(h(l,m,n) = h(-l,-m,-n) = h(-l,-m,n) = h(l,m,-n) = h(l,-m,-n)\), the only unknowns are, \(h(l,m,n)\) for \(l \leq L, 0 \leq m \leq M, 0 \leq n \leq N\), \((M + 1)^3\) coefficients. In other words, a reduction of approximately 87.5 percent of the original number of coefficients to be determined results. Next the optimization-based design method is briefly gone through to show how the incorporation of symmetry information in the design scheme reduces the design complexity.

3. OPTIMIZATION-BASED DESIGN PROCEDURE

Methods based on optimization techniques are considered to be one of the most general methods in the design of a digital filter (FIR or IIR) to meet a given frequency response specification. They also have a high potential to provide a substantial saving in computation time by the application of symmetry constraints. First an overall procedure similar to [6] is outlined for the design of 3-D filters using optimization scheme, taking the symmetries in the specified responses into account.

1. Identify both magnitude and phase symmetries present in the specified frequency response and using the interrelations between the defined symmetries choose a set of minimal symmetry conditions to describe the various symmetries present.
2. Choose the type of filter to be designed such as a FIR or an IIR filter. In this chapter, the two examples provided are for the design of 3-D FIR filters. In a similar manner IIR filters can be designed. But the stability of the denominator has to be checked while designing an IIR filter.
3. For the chosen type of filter, here FIR filter, based on the symmetries identified in step 1 write down the polynomials using Tables 2.1 to 2.5.
4. If the phase response is mentioned in the given specification select a suitable polynomial among the various polynomials that satisfy the identified magnitude symmetry. Otherwise when magnitude response alone is specified, then a product of constrained polynomials or any combination of the various factors found in step 3 can be used.
5. Select a suitable order of the filter and using the symmetry relations identify a set of independent variables which will be treated as the variables in the optimization algorithm.
6. Define an error function based on the difference between the transfer function of the designed frequency response and the specified response at a frequency \((\Omega_1, \Omega_2, \Omega_3)\) as

\[
E(\Omega_1, \Omega_2, \Omega_3) = \hat{F}(\Omega_1, \Omega_2, \Omega_3) - \tilde{F}(\Omega_1, \Omega_2, \Omega_3)
\]  

(3)

where \(\hat{F}(\Omega_1, \Omega_2, \Omega_3)\) is the designed frequency response and \(\tilde{F}(\Omega_1, \Omega_2, \Omega_3)\) is the specified response both at \((\Omega_1, \Omega_2, \Omega_3)\).

7. From the identified symmetry in step 1, choose a basic symmetry region \(\Omega_{sym}\) in the frequency plane \([\Omega_1, \Omega_2, \Omega_3] - \pi \leq \Omega_1, \Omega_2, \Omega_3 \leq \pi\). Set up the cost function as

\[
J = \sum_{\Omega_{sym}} \|E(\Omega_1, \Omega_2, \Omega_3)\|^p
\]  

(4)

where \(p \geq 1\). \(p = 2\) corresponds to the least-square error norm and \(p = \infty\) corresponds to the Chebyshev (min-max) error norm [9].

8. Minimize \(J\) using any minimization algorithm such as Fletcher-Powell, Remez exchange etc. and determine the optimized set of coefficients corresponding to the minimum \(J\).

9. If the optimized filter satisfies the specifications within the tolerance specified, proceed to step 10. Otherwise, increase the order of the filter to be designed and repeat steps 5 to 9 until the specifications are met.

10. Complete the design by calculating the dependent coefficients by applying the symmetry relations identified and used in step 5 on the optimized independent coefficients.

The following examples illustrate the design of 3-D FIR digital filters utilizing symmetry properties. The symmetry results listed in [7] are used.

4. DESIGN EXAMPLES

Example 1: Consider the ideal 3-D spherical low-pass filter whose specifications are shown in Figure 1.1 [10]:

\[
H_d(\Omega) = \begin{cases} 
1 & \text{for } \Omega_1 + \Omega_2 + \Omega_3 \leq (\pi/2)^2; \\
0 & \text{otherwise.}
\end{cases}
\]

and the weights \(W(\Omega) = 1\) for all \(\Omega\). The cross-section of the frequency response at \(\Omega_2 = 0\) to be approximated is shown in Figure 1.2.

It can be observed that \(H_d(\Omega)\) possesses triple planar octagonal symmetry. Therefore to design a linear phase filter, in addition to the triple planar octagonal symmetry or double planar diagonal symmetry result in [7], the imaginary part of the frequency response is assumed to be zero. It is required to design a 2 \(\times\) 2 \(\times\) 2 order FIR digital filter which approximates the desired specification in an optimal sense. In this case mean square error will be taken as the objective function for minimization. The number of unknown parameters without using symmetry is 27. Using triple planar octagonal symmetry the number of unknown parameters is reduced from 27 to 4. That is

\[
H(\Omega_1, \Omega_2, \Omega_3) = A(1) + A(2) \ast (c_1 + c_2 + c_3) + A(3) \ast (c_1 \ast c_2 + c_2 \ast c_3 + c_3 \ast c_1) + A(4) \ast (c_1 \ast c_2 c_3)
\]

where \(c_i = \cos \Omega_i\) for \(i = 1\) to 3

\[
A(1) = -0.88624E-02, A(2) = -9.9047E-02
\]

\[
A(3) = -0.142836, A(4) = -0.205951
\]

The designed filter response in the various constant \(\Omega_3\) planes are given in Figures 1.3.1 - 1.3.3.
Consider a conic filter, similar to a “fan filter” in 2-D. A conic filter can be represented as [10]:

$$H_d(\Omega) = \begin{cases} 1, & \text{for } \Omega_1^2 + \Omega_2^2 \leq \Omega_3^2 \\
0, & \text{otherwise.} \end{cases}$$

The specifications are illustrated in Figure 2.1. A cross-section of the frequency response is shown in Figure 2.2. It can be observed that $H_d(\Omega)$ possesses triple planar quadrantal symmetry. It is required to design a $2 \times 2 \times 2$ order FIR digital filter which approximates the desired specification in an optimal sense. In this case mean square error is taken as the objective function for minimization. The number of unknown parameters without using symmetry is 27. Using triple planar quadrantal symmetry polynomial $P_3(z_1 + z_1^{-1}, z_2 + z_2^{-1}, z_3)$ in [7], the number of unknown parameters is reduced from 27 to 8. That is

$$H(\Omega_1, \Omega_2, \Omega_3) = A(1) + A(2) \cdot c_1 + A(3) \cdot c_2 + e^{-j\Omega_3} \cdot (A(4) + A(5) \cdot c_1 + A(6) \cdot c_2) + c_1 \cdot c_2 + e^{-j\Omega_3} \cdot (A(7) + A(8) \cdot e^{-j\Omega_3})$$

where $c_i = \cos \Omega_i$ for $i = 1, 2$.

$$A(1) = -0.2218663, A(2) = -0.2388956,$$
$$A(3) = -0.2388956,$$
$$A(5) = 0.68255868E-02, A(6) = 0.7164139E-02,$$
$$A(7) = 3.5062145E-02, A(8) = 2.03415553E-02.$$

The designed filter response in the various $\Omega_3$ planes are given in Figures 2.3.1 - 2.3.3.

5. CONCLUSION

The examples presented in this paper illustrate the effect of applying symmetry in the digital filter design. In Example 1 without using symmetry the number of filter coefficients was 27 and the design time (CPU time) in VAX 8800 was found to be 55 min 16.8 sec. When symmetry was applied the number of unknown coefficients was reduced to 4 and the CPU time was 1.03 sec. In Example 2 the number of coefficients was reduced from 27 to 8 and the CPU time taken for the design was reduced from 88 min 34.1 sec to 19.22 sec. The above examples possessed triple planar octagonal symmetry and triple planar quadrantal symmetry respectively. Thus the study made on applying the symmetry results in these designs explicates the effect of symmetry in the design of digital filters.

6. REFERENCES

Fig. 10. Magnitude Response of the Designed 3-D Spherical Lowpass Filter - Cross Section $\Omega_3 = 0 \text{ r/s}$

Fig. 3. Magnitude Response of the Designed 3-D Spherical Lowpass Filter - Cross Section $\Omega_3 = \frac{\pi}{4} \text{ r/s}$

Fig. 11. Magnitude Response of the Designed 3-D Spherical Lowpass Filter - Cross Section $\Omega_3 = \frac{\pi}{2} \text{ r/s}$

Fig. 12. Response at $\Omega_3 = 0 \text{ r/s}$ Plane of the Conical Filter to be Approximated

Fig. 1. 3-D Conically Symmetric Fan Filter

Fig. 2. Magnitude Response of the Designed 3-D Conical Filter - Cross Section $\Omega_3 = 0 \text{ r/s}$
Fig 2.2. Magnitude Response of the Designed 3-D Conical Filter - Cross Section $\Omega_2 = \frac{4\pi}{5}$ r/s

Fig 2.3. Magnitude Response of the Designed 3-D Conical Filter - Cross Section $\Omega_3 = \frac{4\pi}{5}$ r/s

Table 1 Definitions of 3-D Symmetries

<table>
<thead>
<tr>
<th>Symmetry</th>
<th>Definition</th>
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<tbody>
<tr>
<td>Spatial Centro SCS</td>
<td>$f(\omega_1, \omega_2, \omega_3) = f(-\omega_1, -\omega_2, -\omega_3)$</td>
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<tr>
<td>Single Planar SPQS$_{w_3}$</td>
<td>$f(\omega_1, \omega_2, \omega_3) = f(\omega_1, -\omega_2, \omega_3) = f(-\omega_1, -\omega_2, \omega_3) = f(-\omega_1, \omega_2, \omega_3)$</td>
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<tr>
<td>SPDS$_{w_3}$</td>
<td>$f(\omega_1, \omega_2, \omega_3) = f(\omega_1, \omega_2, \omega_3) = f(-\omega_1, -\omega_2, \omega_3) = f(-\omega_1, \omega_2, \omega_3)$</td>
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<tr>
<td>SPRS$_{w_3}$</td>
<td>$f(\omega_1, \omega_2, \omega_3) = f(\omega_1, -\omega_2, \omega_3) = f(-\omega_1, -\omega_2, \omega_3) = f(-\omega_1, \omega_2, \omega_3)$</td>
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<tr>
<td>SPOS$_{w_3}$</td>
<td>$f(\omega_1, \omega_2, \omega_3) = f(\omega_1, -\omega_2, \omega_3) = f(-\omega_1, -\omega_2, \omega_3) = f(-\omega_1, \omega_2, \omega_3)$</td>
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<tr>
<td>Double Planar DPQS$_{w_3}$</td>
<td>SPQS$<em>{w_3}$, SPQS$</em>{w_3}$</td>
</tr>
<tr>
<td>DPDS$_{w_3}$</td>
<td>SPDS$<em>{w_3}$, SPDS$</em>{w_3}$</td>
</tr>
<tr>
<td>DPRS$_{w_3}$</td>
<td>SPDS$<em>{w_3}$, SPDS$</em>{w_3}$</td>
</tr>
<tr>
<td>DPQ$_{w_3}$</td>
<td>DPQ$_{w_3}$</td>
</tr>
<tr>
<td>Triple Planar TPQS</td>
<td>SPQS$<em>{w_3}$, SPQS$</em>{w_3}$, SPQS$_{w_3}$</td>
</tr>
<tr>
<td>TPDS</td>
<td>SPDS$<em>{w_3}$, SPDS$</em>{w_3}$, SPDS$_{w_3}$</td>
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<tr>
<td>TRPS</td>
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