ABSTRACT

The optimal solution of the weakly coupled algebraic discrete Riccati equation is obtained in terms of reduced-order continuous type algebraic Riccati equation via the use of the bilinear transformation. The proposed method has the rate of convergence of $O(c)$, where $c$ represents a small coupling parameter, and it is applicable under quite mild assumptions.

I. INTRODUCTION.

The linear weakly coupled continuous systems are studied in different setups by many researchers [1]-[12]. However, the linear weakly coupled discrete systems have not been studied in the literature yet. This is due to the fact that the partitioned form of the main equation of the optimal linear control theory - Riccati equation, has a very complicated form in the discrete time domain. In this paper that problem is overcome by the use of the bilinear transformation, which is applicable under quite mild assumptions, so that the solution of the discrete algebraic Riccati equation of weakly coupled systems is obtained by using known results for the corresponding continuous-time equation.

The algebraic Riccati equation of weakly coupled linear discrete systems is given by

$$P = A^T P A + Q - A^T P B (B^T P B + R)^{-1} B^T P A,$$

$$R > 0, \quad Q > 0 \quad (1)$$

where

$$A = \begin{bmatrix} A_1 & c A_2 \\ c A_3 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & c B_2 \\ c B_3 & B_4 \end{bmatrix}$$

and $c$ is a small coupling parameter. Due to block dominant structure of the problem matrices the required solution $P$ has the form

$$P = \begin{bmatrix} P_1 & c P_2 \\ c^T P_2 & P_3 \end{bmatrix} \quad (2)$$

The main goal in the theory of the weakly coupled systems is to obtain the required solution in terms of the reduced-order problems, namely subsystems. In the case of the algebraic weakly coupled discrete Riccati equation the inversion of the partitioned matrix $(B^T P B + R)$ will produce a lot of terms and make corresponding approach computationally very involved even though one is faced with reduced order numerical problems. In order to solve this problem, we have used the bilinear transformation introduced in [13] to transform the discrete Riccati equation (1) into a continuous-time algebraic Riccati equation of the form

$$A_C P_C + P_C A_C + Q_C - P_C S_C P_C = 0,$$

$$S_C = B_C R_C^{-1} B_C^T \quad (3)$$

such that the solution of (1) is equal to the solution of (3). It is shown that the equation (3) preserves the structure of weakly coupled systems. It can be solved in terms of the reduced order problems very efficiently by using the fixed point
type method developed in [7] which
converges to the required solution with
the rate of convergence of $O(\epsilon^3)$.

II. COMPUTATIONAL ALGORITHM

Since the proposed algorithm for the
discrete algebraic Riccati equation
combines features of the bilinear
transformation [13] and the fixed point
algorithm developed in [7] for weakly
coupled continuous algebraic Riccati
equation, we will briefly summarize main
results from [13] and [7].

The bilinear transformation states that
the equations (1) and (3) will have the
same solution if the following hold [13].

\[
A_C = I - 2D^{-T} \quad (4a)
\]

\[
S_C = 2(I + A)^{-1}S_dD^{-1}, \quad S_d = BR^{-1}B^T \quad (4b)
\]

\[
Q_C = 2D^{-1}Q(I + A)^{-1} \quad (4c)
\]

\[
D = (I + A^T) + Q(I + A)^{-1}S_d \quad (4d)
\]

assuming that $(I + A)^{-1}$ exists. It is
shown in [15] that the matrix $D$ is
invertible. The physical interpretation
of the transformation between the
continuous type and discrete type
algebraic Riccati equation is discussed
in [13].

It can be easily verified that the
weakly coupled structure of matrices
defined in (1) will produce the weakly
coupled structure of transformed matrices
given in (4). Let us introduce compatible
partitions of these matrices

\[
A_C = \begin{bmatrix} A_{11} & cA_{12} \\ cA_{21} & A_{22} \end{bmatrix}, \quad S_C = \begin{bmatrix} S_{11} & cS_{12} \\ cS_{21} & S_{22} \end{bmatrix}
\]

\[
Q_C = \begin{bmatrix} Q_{11} & cQ_{12} \\ cQ_{21} & Q_{22} \end{bmatrix}, \quad P_C = P = \begin{bmatrix} P_1 & cP_2 \\ cP_2 & P_3 \end{bmatrix}
\]

Note that these partitions have to be
performed by computer only in the process
of calculations and there is no need for
the corresponding analytical expressions.

Solution of (3) can be found in terms
of reduced-order problems by imposing
standard stabilizability-detectability
assumptions on subsystems. The efficient
fixed point reduced-order algorithm for
solving (3) is obtained in [7]. It will
be briefly summarized here.

The $O(\epsilon^2)$ approximation of (3) is
obtained from the following decoupled set
of equations

\[
E_1A_{11} + A_{11}E_1 + Q_{11} - E_1S_{11}E_1 = 0 \quad (5a)
\]

\[
E_3A_{22} + A_{22}E_3 + Q_{22} - E_3S_{22}E_3 = 0 \quad (5b)
\]

\[
E_2(A_{22} - S_{22}E_3) + (A_{11} - S_{11}E_1)E_2 + \\
\quad + E_1A_{12} + A_{12}E_1 + Q_{12} - E_1S_{12}E_3 = 0 \quad (5c)
\]

Unique positive semi-definite
stabilizing solution of (5) exists under
the following assumption:

ASSUMPTION 1. Triples $(A_{ii}, B_{ii}, c_{ii}),
\quad i = 1, 2$ are stabilizable-detectable.

Defining the approximation errors as

\[
P_i = E_i + \epsilon^2 E_i, \quad i = 1, 2, 3. \quad (6)
\]

the fixed point type algorithm, with the
rate of convergence of $O(\epsilon^2)$, is obtained
in [7] in the decoupled form as

\[
E_1^{(j+1)}D_1 + D_1E_1^{(j+1)} = -M_1^{(j)} \quad (7a)
\]

\[
E_3^{(j+1)}D_2 + D_2E_3^{(j+1)} = -M_3^{(j)} \quad (7b)
\]

\[
E_2^{(j+1)}D_2 + D_2E_2^{(j+1)} + E_1D_1E_1^{(j+1)} + \\
\quad + E_1^{(j+1)}D_{12} + D_2E_2^{(j+1)} = -M_2^{(j,j+1)} \quad (7c)
\]
with \( j = 0, 1, 2, \ldots \), and with \( E_1^{(0)} = 0 \), \( E_2^{(0)} = 0 \), where newly defined matrices are given in the Appendix. It is important to point out that \( D_1 \) and \( D_2 \) are stable matrices. The rate of convergence of (7) is \( O(\epsilon^2) \), [7], that is

\[
| P_1 - P_1^{(j)} | = O(\epsilon^2), \quad i = 1, 2, 3; \\
\]

\[
j = 0, 1, 2, \ldots . \quad (8)
\]

where

\[
P_1^{(j)} = E_1 + \epsilon^2 E_1^{(j)}, \quad i = 1, 2, 3; \\
j = 0, 1, 2, \ldots . \quad (9)
\]

Thus, the proposed algorithm for the reduced-order solution of the discrete algebraic Riccati equation has the following form:
1) Transform (1) into (3) by using (4).
2) Solve (3) by using the reduced-order algorithm (5)-(7).

### III. NUMERICAL EXAMPLE

A real world physical example (distillation column [14]) demonstrates the efficiency of the proposed method

\[
A = 10^{-3}
\]

\[
[\begin{array}{cccc}
0.0192 & 6.0733 & 3.9291 & 8.2911 \\
0.0013 & -0.6192 & -13.339 & -18.442 \\
0.0179 & 0.3172 & 1.6974 & 13.298 \\
0.9108 & 17.991 & 183.81 & 668.36
\end{array}]
\]

\[
B = 10^{-3}
\]

\[
[\begin{array}{c}
0.0192 \\
-0.6192 \\
1.6974 \\
183.81
\end{array}]
\]

\[
Q = I_5, \quad R = I_2.
\]

These matrices are obtained from [14] by performing a discretization with the sampling rate \( \Delta T = 0.1 \). The small weakly coupling parameter \( \epsilon \) is built in the problem. It can be roughly estimated from the strongest coupled matrix - in this case matrix \( B \). Apparently the strongest coupling is in the third row, that is

\[
\epsilon = \frac{b_{31}}{b_{32}} = \frac{8.2911}{13.339} = 0.62
\]

The simulation results are presented in Table 1.

Note that the zero-order solution \( j = 0 \) and the required solution \( j = 14 \) are pretty far apart (that means the real \( \epsilon \) built in this problem is either bigger than 0.62 or the constant \( K \) in the definition of \( O(\epsilon^2) < K \epsilon^2 \) is very big) so that it takes 14 iterations to achieve desired accuracy of \( 10^{-7} \).

Simulation results are obtained by using the L-A-S package for computer aided control system [16].

<table>
<thead>
<tr>
<th>( j )</th>
<th>( P_1 )</th>
<th>( P_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>95.254 &amp; 4.1053</td>
<td>18.491 &amp; 14.867 &amp; 198.35 &amp; 3.3032 &amp; 1.3547 &amp; 6.0305</td>
</tr>
<tr>
<td>1</td>
<td>177.70 &amp; 11.344</td>
<td>35.194 &amp; 25.521 &amp; 317.25 &amp; 4.3012 &amp; 1.8746 &amp; 9.9835</td>
</tr>
<tr>
<td>2</td>
<td>236.17 &amp; 16.851</td>
<td>45.620 &amp; 31.422 &amp; 372.20 &amp; 4.7748 &amp; 2.1239 &amp; 12.113</td>
</tr>
<tr>
<td>3</td>
<td>285.96 &amp; 22.098</td>
<td>54.651 &amp; 35.840 &amp; 415.05 &amp; 5.2120 &amp; 2.3434 &amp; 13.836</td>
</tr>
<tr>
<td>4</td>
<td>301.55 &amp; 23.782</td>
<td>58.690 &amp; 36.884 &amp; 415.38 &amp; 5.2160 &amp; 2.3452 &amp; 13.847</td>
</tr>
<tr>
<td>6</td>
<td>302.40 &amp; 23.874</td>
<td>58.690 &amp; 36.884 &amp; 415.45 &amp; 5.2160 &amp; 2.3452 &amp; 13.847</td>
</tr>
<tr>
<td>7</td>
<td>302.43 &amp; 23.878</td>
<td>58.690 &amp; 36.884 &amp; 415.45 &amp; 5.2160 &amp; 2.3452 &amp; 13.847</td>
</tr>
<tr>
<td>8</td>
<td>302.43 &amp; 23.878</td>
<td>58.690 &amp; 36.884 &amp; 415.45 &amp; 5.2160 &amp; 2.3452 &amp; 13.847</td>
</tr>
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<td>9</td>
<td>302.43 &amp; 23.878</td>
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<td>58.690 &amp; 36.884 &amp; 415.45 &amp; 5.2160 &amp; 2.3452 &amp; 13.847</td>
</tr>
<tr>
<td>14</td>
<td>302.43 &amp; 23.878</td>
<td>58.690 &amp; 36.884 &amp; 415.45 &amp; 5.2160 &amp; 2.3452 &amp; 13.847</td>
</tr>
</tbody>
</table>

Table 1
IV. CONCLUSION

Reduced order optimal solution of the algebraic discrete weakly coupled Riccati equation is obtained. This result will play the important role in the design procedure of the optimal and near-optimal controllers and filters for weakly coupled discrete systems since it reduces off-line computational requirements.

APPENDIX

\[
D_1 = A_{11} - S_{11}E_1, \quad D_2 = A_{22} - S_{22}E_3 \\
D_{12} = A_{12} - S_{12}E_2, \quad D_{21} = A_{21} - S_{21}E_1 \\
D_{12}^{(j)} = A_{12} - S_{12}E_{12}^{(j)}, \quad D_{21}^{(j)} = A_{21} - S_{21}E_{21}^{(j)} \\
M_1^{(j)} = P_2 \left( D_{21}^{(j)} + D_{21}^{(j)} P_2 \right)^T - P_2 S_{22} P_2 - \epsilon_2 E_1 S_{11} E_1 \\
M_2^{(j)} = P_2 \left( D_{12}^{(j)} + D_{12}^{(j)} P_2 \right)^T - P_2 S_{12} P_2 - \epsilon_2 E_2 S_{22} E_3 \\
M_{2, j+1}^{(j)} = P_2 S_{12} E_2^{(j+1)} + E_1 S_{11} P_2^{(j+1)} + P_2 S_{22} E_3^{(j+1)} - \epsilon_2 E_1 S_{11} E_1^{(j+1)} \\
S_{12}^{(j)} = S_{12}^{(j)} + \epsilon_2 E_1 S_{11} E_1^{(j+1)} \\
S_{22}^{(j)} = S_{22}^{(j)} + \epsilon_2 E_2 S_{22} E_3^{(j+1)} \\
E_1^{(j+1)} = E_1^{(j+1)} + \epsilon_2 E_1 S_{11} E_1^{(j+1)} \\
E_2^{(j+1)} = E_2^{(j+1)} + \epsilon_2 E_2 S_{22} E_3^{(j+1)} \\
E_{11}^{(j+1)} = E_{11}^{(j+1)} + \epsilon_2 E_1 S_{11} E_1^{(j+1)} \\
E_{22}^{(j+1)} = E_{22}^{(j+1)} + \epsilon_2 E_2 S_{22} E_3^{(j+1)} \\
E_{12}^{(j+1)} = E_{12}^{(j+1)} + \epsilon_2 E_1 S_{11} E_1^{(j+1)} \\
E_{21}^{(j+1)} = E_{21}^{(j+1)} + \epsilon_2 E_2 S_{22} E_3^{(j+1)}
\]

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