How Hard Are Sparse Sets?

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Abstract

We tour topics on the frontier of knowledge regarding the structural properties of sparse sets. Our tour seeks to highlight recent advances and bring outstanding problems to the reader’s attention.

1 Introduction

This paper explores the frontier of knowledge about the structural properties of sparse sets. We survey a collection of topics that are related to the issue of how hard or easy sparse sets are. We present the strongest currently known results, together with the open problems that the results leave.

In 1977, Berman and Hartmanis conjectured that all NP-complete sets are polynomially isomorphic; that is, they conjectured that for any two sets \( A \) and \( B \) that are \( \leq^p \)-complete for NP, there exists a polynomial-time computable bijection \( f \) such that \( f(A) = B \) and \( f(\overline{A}) = \overline{B} \) [BH77]. As evidence of the plausibility of their conjecture, they showed that all then-known NP-complete sets were indeed polynomially isomorphic. All then-known (and, for that matter, all currently known) NP-complete sets are dense; there exists a constant \( c \) such that, for all sufficiently large \( n \), the set contains at least \( 2^n \) elements of length at most \( n \). Thus, if the Berman-Hartmanis Conjecture is true, then sparse sets—sets with polynomially bounded density—cannot be NP-complete. This observation yielded another conjecture: Sparse sets are not NP-complete.

Hartmanis [Har78] proved that some classes, such as PSPACE and EXP, lack sparse \( \leq^p \)-complete sets, and he conjectured that NL and P also lack sparse \( \leq^p \)-complete sets.

These fundamental studies motivated researchers to study broadly the classes of sets whose complete languages could not be reduced to sparse sets unless the classes collapsed. Section 3 discusses the current state of this research stream.

Another motivation for the study of sparse sets is their close relationship to notions of polynomial-time "quasi-solvability." The class of sets having polynomial-size circuits is exactly the class of sets that are polynomial-time Turing reducible to sparse sets (attributed to A. Meyer in [BH77]). Thus, sets that are polynomial-time Turing reducible to sparse sets can be regarded as polynomial-time solvable, give or take a small amount of information. Furthermore, less flexible reducibilities characterize some other notions...
of polynomial-time quasi-solvability [MP79,Sch86a,
reasoned to the appropriate cited papers. For example, given a set $A$, a polynomial-time algorithm that correctly answers the question "$z \in A?$" for all but a sparse set of values $z$ can be considered to be a good polynomial-time approximation of $A$. It is known that the class of sets $S$ so approximable (the P-close sets) are reducible to sparse sets by truth-table reductions that ask only one question per input. Thus, studying the difference between various reducibilities to sparse sets—the topic of Section 4 of this paper—can separate the analogous notions of polynomial-time quasi-solvability. For example, from the fact that $\leq_1^{\text{ TT}}$ reducibility to sparse sets is strictly weaker than $\leq_T$-reducibility to sparse sets [BK88], we can conclude that the P-close sets define a strictly weaker (that is, closer to actual polynomial-time solvability) polynomial-time quasi-solvability notion than polynomial-size circuits.

Note that most discussions of quasi-solvability in some way assume that sparseness (either in the language to which as set is reduced, or in the amount by which an approximation fails) is a "near enough miss" to be satisfying. But how near are sparse sets to being negligible? One of the issues with which this paper is concerned is whether sparse sets are in fact "not hard." The study, discussed earlier, of whether sparse sets can be complete/hard for complexity classes provides one type of evidence that sparse sets are not hard. Evidence of the weakness of sparse sets also comes from many other quarters, perhaps most notably lowness and relativizing with sparse oracles.

The theory of lowness asks which sets yield useful information when used as databases by various classes. Recent work has provided a relatively complete understanding of the lowness of various classes. In Section 5, we study the state of known lowness results for sparse sets, and for sets reducible to sparse sets.

Perhaps the most powerful potential oracle results are ones that would imply results in the unrelativized world; happily, many complexity classes admit to results of the form: separating the classes with any oracle is; this again argues that sparse sets are not more likely to separate the classes than the empty oracle is; this again argues that sparse sets are not likely to separate or collapse complexity classes.

We will skip most of the proofs; interested readers may refer to the appropriate cited papers.

2 Notations

In this section, we will define some notations that will be used throughout this paper (see also [GJ79,
Joh90]). We will be concerned with sets over the alphabet $\Sigma = \{0,1\}$. For $z \in \Sigma^*$, $|z|$ denotes the length of $z$. For a finite set $A$, $\|A\|$ denotes the cardinality of $A$. $N$ denotes the set of nonnegative integers. For $n \in N$, $\Sigma^n$ and $\Sigma^N$ denote, respectively, the set of strings of length $n$ and the set of strings of length at most $n$. For a set $A \subseteq \Sigma^*$, $A^n = A \cap \Sigma^n$ and $A^N = A \cap \Sigma^N$.

Let $f : N \rightarrow N$ be any function; a set $S$ is $f(n)$-sparse if for every $n$, $\|S^n\| \leq f(n)$. A set $S$ is sparse if $S$ is $p(n)$-sparse, for some polynomial $p$. $\text{SPARSE}$ denotes the class of sparse sets. A set is $\text{cosparse}$ if $\overline{S}$ is sparse. $\text{cOSPARSE}$ denotes the class of cosparse sets. A set $T$ is said to be a tally set if $T \subseteq 0^*$. $\text{TALLY}$ denotes the class of tally sets.

Next, we review the reducibility notions we will be concerned with (see [LLS75,LL76,Lon82]). Our reducibilities are defined via relativized computations of Turing machines having various resource bounds and query mechanisms. $\leq_T$ denotes a reducibility with resource bound $t$ and query mechanism $r$. Our resource bounds are $p$, $L$, $NP$, $\text{coNP}$, $RP$, $\text{coRP}$, and $SN$.

The most flexible query mechanism that we will discuss is Turing access, which is denoted by the symbol $T$ and which means that the machine can make adaptive queries.

$$A \leq_T B, A \leq_T B^*, A \leq_T \text{NP} B, A \leq_T \text{NP} B^*,$$

$$A \leq_T \text{RP} B, A \leq_T \text{RP} B^*, A \leq_T \text{SN} B$$ respectively denote $A \in \text{P} B$, $A \in \text{P} B^*$, $A \in \text{NP} B$, $A \in \text{NP} B^*$, $A \in \text{RP} B$, $A \in \text{RP} B^*$, $A \in \text{RP} B^*$, and $A \in \text{NP} B \cap \text{NP} B^*$.

Polynomial-time Turing reductions that make at most $k$ queries on any input, for some constant $k$, are said to be bounded Turing reductions, and are represented by $\leq_k^P$. Polynomial-time Turing reductions that accept exactly when some (respectively, every) query receives the answer "yes" are said to be disjunctive (respectively, conjunctive) Turing reductions, and are represented by $\leq_0^P$ (respectively, $\leq_0^P$). One can combine these notions, obtaining the reducibilities $\leq_{k0}^P$ and $\leq_{k0}^P$.

The most restrictive query mechanism that we will be concerned with is that of many-one reductions, which is denoted by the symbol $m$ and which means that only one question is allowed and the machine accepts if and only if the answer to that query is "yes."
For query mechanisms intermediate between Turing and many-one reductions, we use the following notations. The symbol $tt$ (truth-table) refers to nonadaptive queries. A function $f(n)$ preceding $tt$ means that the number of queries is at most $f(n)$ for inputs of length $n$. A number $k$ preceding $tt$ ($k$-truth-table) stands for $f(n) = k$ in the above. Function classes $F$ preceding $tt$ mean that the number of queries is bounded by some fixed function $f \in F$.

Function classes $F$ preceding $tt$ means that the number of queries is at most $f(n)$ for inputs of length $n$. A number $k$ preceding $tt$ ($k$-truth-table) stands for $f(n) = k$ in the above. One often sees $btt$, $ctt$, and $dtt$ used to refer to nonadaptive notions analogously to similarly-named Turing notions; however, note that in many such cases the similarly-named Turing and truth-table reductions are identical.

Note that we are adopting for our definition of truth-table reducibilities the standard and well-accepted definition of Ladner, Lynch, and Selman [LLS75]. For example, when the resource bound is nondeterministic, the machine doing the truth-table reduction has the following two properties: (1) it can generate different sets of query strings along different computation paths, and (2) even after obtaining all the answers from its oracle, the machine can work nondeterministically, so it can reach different halting states. The reader is cautioned that another notion of reducibility, based instead on single-valued halting states, has also appeared in the literature [BK88], and there are papers in which this different notion co-opts the standard notation.

Let $\leq^1$ be any defined reducibility and $C$ be any class of sets. $R^1(F)(C)$ denotes $\{L | (\exists C \in C)[L \leq^1 C]\}$. $E^1(C)$ denotes $\{L | (\exists C \in C)[L \leq^1 C \text{ and } C \leq^1 L]\}$.

We now define some of the complexity classes we will be concerned with. Deterministic times $n^{O(1)}$, $2^{O(n)}$, and $2^{n^{O(1)}}$ will respectively be represented by $P$, $E$, and $EXP$, and their nondeterministic counterparts will be represented by $NP$, $NE$, and $NEXP$. The levels of the polynomial hierarchy [MS72, Sto77] have their standard definitions (see [Wag90]). Deterministic and nondeterministic space $O(\log n)$ will respectively be represented by $L$ and $NL$.

For a polynomial-time nondeterministic Turing machine $M$ and $x \in \Sigma^*$, $#acc_M(x)$ and $#rej_M(x)$ denote the number of accepting and rejecting computation paths of $M$ on $x$, respectively.

Definition 2.1.

1. [Gil77,Sim75] A set $L$ is in $P$ if there is a polynomial-time nondeterministic Turing machine $M$ such that for every $x \in \Sigma^*$ it holds that $x \in L$ if and only if $#acc_M(x) > #rej_M(x)$.

2. [Sim75, Wag86] A set $L$ is in $C= P$ if there is a polynomial-time nondeterministic Turing machine $M$ such that for every $x \in \Sigma^*$ it holds that $x \in L$ if and only if $#acc_M(x) = #rej_M(x)$.

3. For $k \geq 2$, a set $L$ is in $MOD_kP$ if there is a polynomial-time nondeterministic Turing machine $M$ such that for every $x \in \Sigma^*$ it holds that $x \in L$ if and only if $#acc_M(x)$ is not a multiple of $k$ [CH90].

In particular, $\oplus P$ denotes $MOD_2P$ [PZ83, GP86].

4. [Val76] A set $L$ is in $UP$ if $L \in NP$ is witnessed by a machine having at most one accepting computation path for every input.

5. [AR88] A set $L$ is in $FewP$ if $L \in NP$ is witnessed by a machine $M$ with $#acc_M(x) \leq p(|x|)$ for some polynomial $p$.

6. [CH90] A set $L$ is in $Few$ if there exist a polynomial-time nondeterministic Turing machine $M$, a polynomial $p$, and a polynomial-time predicate $Q$ such that, for every $x$, $#acc_M(x) \leq p(|x|)$ and $(x \in L$ if and only if $Q(x, #acc_M(x)) = true$).

Concerning these classes, the following inclusions are well-known:

**Proposition 2.2.**

1. $UP \subseteq FewP \subseteq NP \subseteq PP$ and $UP \subseteq MOD_kP$ for all $k$.

2. $FewP \subseteq Few \subseteq C=P$.

3. $coNP \subseteq C= P \subseteq PP$.

3 Sparse Complete and Hard Sets

3.1 For $P$

Hartmanis conjectured that both $P$ and $NL$ lack sparse complete sets [Har78]. This fascinating issue defied results of any kind; no major consequences were known even from such outrageous assumptions as the existence of log* $n$-sparse complete sets for these classes. However, recently Hemachandra, Ogiwara and Toda [HOT90] proved that if any logspace self-reducible [Bal90] set $A$ is logspace reducible to
some "very sparse" set, then "A" is recognized by a space-efficient deterministic algorithm. We say that $A$ is logspace self-reducible if $A$ is recognized by a logspace machine $M$ with oracle $A$ via the following mechanism: If $M$ queries $y$ on input $x$ of length $n$, then $|y| = n$, $y < x$, and $y$ and $x$ have the same leftmost $n - \log n$ bits. The classes $P$, $NL$, $NC^k$, and $AC^k$ have such $O(f(p(n))$-complete sets, and so the results of [HOT90] imply that these classes cannot have very sparse hard sets unless implausible class inclusions hold.

**Lemma 3.1.** [HOT90] If a logspace self-reducible set $A$ is $O(\log^k n)$-complete, then $A$ is in $\text{DSPACE}[\log^{k+1} n]$.

**Lemma 3.2.** [HOT90] Let $f$ be a non-decreasing logspace constructible function such that $f(n) = O(q(n))$ for some polynomial $q$. If a logspace self-reducible set $A$ is $O(\log n)$-reducible to an $f(n)$-sparse set $S$, then $A \in \text{DTISP}[(f[p(n)], f(p(n))]$ for some polynomial $p$, where $\text{DTISP}[(f[n], s(n))]$ is the class of sets recognized by a deterministic machine that runs in time $f(n)$ and in space $s(n)$.

**Proof Sketch of Lemma 3.2** Let $f$, $A$, and $S$ be as in the hypothesis. Let $M$ and $N$ be machines witnessing, respectively, that $A$ is logspace self-reducible and $A \leq^f S$. Let $p$ be a polynomial bounding the run-time of $M$. Let $x$ be fixed and let $n = |x|$. Define $W = \{w \mid w < x \text{ and } |w| = n \}$ and $W$ have the same leftmost $n - \log n$ bits, and let $x_0, \ldots, x_m = x$ be an enumeration of all strings in $W$ in increasing order. For $k \in \mathbb{N}$, let $[k] = \{0, \ldots, k\}$. For a set of indices $I \subseteq [m]$, let $T(I) = \{N(x_i) \mid i \in I\}$. For $i \in [m]$, let $C_i = \{j \leq i \mid N(x_j) \in S \text{ and } N(x_j) \notin T([i-1])\}$. For every $i$, it holds that (i) $|C_i| \leq f(p(n))$ and (ii) for every $j \leq i$, $x_j \in A$ if and only if $N(x_j) \in T(C_i)$. Since storing $C_i$ requires only $O(\log n)$ space, given $C_i$, deciding the membership of $x_j$ with $j \leq i$ can be done in polynomial time and in space $O(f(p(n)) \cdot \log n)$. Thus, via self-reducibility, the membership of $x_{i+1}$ can be tested in polynomial time and in space $O(f(p(n)) \cdot \log n)$. Note that $C_0 = \{0\}$ if $x_0$ is accepted by $M$ ($M$ makes no queries on input $x_0$) and $\emptyset$ otherwise, and that $C_{i+1} = C_i \cup \{i+1\}$ if $x_{i+1} \in A$ and $N(x_{i+1}) \notin T(C_i)$ and $C_i$ otherwise. Thus, $C_m$ can be computed inductively in time polynomial in $n$ and space $O(f(p(n)) \cdot \log n)$. Thus, for some polynomial $r$, it holds that $A \in \text{DTISP}[r(n), f(r(n)) \cdot \log n]$.

Lemma 3.2 can be extended to more flexible reducibilities, and we have the following partial solutions to Hartmanis’s conjecture.

**Theorem 3.3.** [HOT90] If $P$ has $O(\log^k n)$-sparse $L$-hard sets, then $P \subseteq \text{DSPACE}[\log^{k+1} n]$. The analogous statements hold for $NL$, $NC^k$ and $AC^k$.

**Theorem 3.4.** [HOT90] If $P$ has $O(\log^k n)$-sparse $L_{tt}$-hard sets (or $L_{tt}$-hard sets), then $P \subseteq \text{SC}$, where $SC = \bigcup_{i,j} \text{DTISP}[n^i, \log^j n]$. The analogous statements hold for $NL$, $NC^k$ and $AC^k$.

**Open Problem 1.** Does Theorem 3.4 hold for $L_{tt}$-reducibility? In particular, does it even hold for $L_{tt}$? We conjecture that it does.

And, of course, Hartmanis’s conjecture still remains open: Does $P$ or $NL$ have sparse $L$-hard sets?

**Open Problem 2.** Can one find structural consequences that would follow were $P$ or $NL$ to have $L$-hard (polynomially) sparse sets?

### 3.2 For NP

#### 3.2.1 Polynomial-Time Reducibilities

Berman and Hartmanis’s conjecture stimulated researchers to work on questions: “Does $NP$ have sparse $L$-hard sets?” and “Does $NP$ have sparse $L_{tt}$-complete sets?” While an absolute answer to this question has proven elusive (note that if $P = NP$ then $NP$ does have sparse complete sets), some conditional answers have been obtained. One early result was that of Karp and Lipton, who showed that the existence of such hard complete sets would imply the collapse of the polynomial hierarchy.

**Theorem 3.5.** [KL80] If $NP$ has sparse $L$-hard sets, then $PH = \Sigma^P_2 \cap \Pi^P_2$.

This immediately yields the following question.

**Open Problem 3.** Does $NP \subseteq R^P_{tt}(\text{SPARSE})$ imply anything stronger than $PH = \Sigma^P_2 \cap \Pi^P_2$? Note that Heller [Hel86] proved that one cannot hope for a relativizable collapse to $\Delta^P_2$. 225
On the other hand, the existence of sparse \( \leq^P \)-complete sets implies a stronger collapse of the polynomial hierarchy. Kadin showed that under that assumption \( \text{PH} = \text{P}^{\text{NP}[\log]} \), where \( \text{P}^{\text{NP}[\log]} \), the \( \Theta^P_2 \) level of the polynomial hierarchy, is the class of sets reducible to an NP set with \( O(\log n) \) adaptive queries. Combined with the fact that NP having sparse \( \leq^P \)-complete sets is equivalent to NP having tally \( \leq^P_t \)-complete sets [Har83], Kadin’s result can be stated as follows.

Theorem 3.6. [Kad89] If NP has sparse \( \leq^P \)-complete sets (or equivalently, tally \( \leq^P_t \)-complete sets), then \( \text{PH} = \text{P}^{\text{NP}[\log n]} \).

Kadin’s result is optimal—the conclusion \( \text{PH} = \text{P}^{\text{NP}[\log n]} \) cannot be replaced with a stronger collapse using relativizable proof techniques, as Kadin showed that for every nice function \( f(n) = o(\log \log n) \) there is a relativized world in which \( f \) cannot replace the \( \log n \) of the above theorem [Kad89].

Concerning reducibilities that are less flexible than Turing reductions, researchers have examined whether a \( P = \text{NP} \) conclusion can be obtained from the assumption of the existence of sparse NP-hard sets. Mahaney’s work, exploiting an observation by Fortune [For79], revealed that \( P = \text{NP} \) if NP has sparse \( \leq^P_{\text{tt}} \)-hard sets [Mah82]. Since then, the same question has been asked for more general reduction types such as \( \leq^P_C \) and \( \leq^P_y \). After much intense effort by many researchers, and various intermediate results [Yap83, Yes83, Ukk83, Wat], Ogiwara and Watanabe successfully extended Mahaney’s result to the challenging \( \leq^P_C \) case, and the \( \leq^P \) case was resolved by [AHH+, RR].

Theorem 3.7. [AHH+, RR]
\( \text{NP} \subseteq \text{R}^{\text{NP}(\text{SPARSE})} \implies P = \text{NP} \).

Theorem 3.8. [OW91]
\( \text{NP} \subseteq \text{R}^{\text{NP}(\text{SPARSE})} \implies P = \text{NP} \).

Though it remains plausible that Theorem 3.8’s constant bound on the number of queries might be raised to \( O(\log n) \), techniques that relativize can raise it no further than that.

Theorem 3.9. ([AHH+, see also [HL91]]) For any polynomial-time computable nondecreasing function \( f \) such that \( f(n) = o(\log n) \), there is a relativized world in which BH does not collapse and NP has a tally \( \leq^P_{f(n)} \)-complete set.

The Ogiwara-Watanabe breakthrough was made possible by their insights into a self-reducibility structure called “left sets.” This self-reducibility has been examined by many researchers, and has yielded a flurry of results.

Theorem 3.10. [HL91]
\( \text{NP} \subseteq \text{R}^{\text{NP}(\text{NP}[\log n])} \implies P = \text{NP} \).

Theorem 3.11. [AKM92]
\( \text{NP} \subseteq \text{R}^{\text{NP}(\text{SPARSE})} \implies P = \text{NP} \).

Note that Theorem 3.11—which relies on the lovely recent result of Buhrman, Longpré, and Spann [BLS92] that \( \text{SPARSE} \subseteq \text{R}^{\text{NP}(\text{TALLY})} \)—unifies Theorems 3.7 and 3.8.

Open Problem 4.

1. \( \text{NP} \subseteq \text{R}^{\text{NP}(\text{SPARSE})} \implies P = \text{NP} \)?

2. \( \text{NP} \subseteq \text{R}^{\text{NP}(\text{SPARSE})} \implies P = \text{NP} \)?

It is not hard to see that for any reducibility \( r \), if \( \text{NP} \subseteq \text{R}^{\text{NP}(\text{SPARSE})} \) implies \( \text{PH} = \Sigma^P_2 \), it automatically follows that \( \text{NP} \subseteq \text{R}^{\text{NP}(\text{SPARSE})} \implies \text{PH} = \text{NP} \). This observation yields the following corollaries of Theorem 3.5 and Theorem 3.11.

Corollary 3.12. \( \text{NP} \subseteq \text{R}^{\text{NP}(\text{SPARSE})} \implies \text{PH} = \Sigma^P_2 \cap \Pi^P_2 \).

Corollary 3.13. \( \text{NP} \subseteq \text{R}^{\text{NP}(\text{SPARSE})} \implies \text{NP} = \text{coNP} \).

A very nice recent result of Bvhrman and Homer shows that NP is not only unlikely to have (polynomially) sparse complete sets, but is even unlikely to have “almost sparse” complete sets. If it did, then the exponential hierarchy [HIS85], \( \text{NE}^{\text{PH}} \), would collapse to the \( \Delta^P_2 \) level of the strong exponential hierarchy [Hem89].

Theorem 3.14. [BH91] If NP has a Turing-hard set \( S \) satisfying \( (3k)(\forall n)[||S^{\leq^P_n}|| \leq 2^{\log^k n}] \), then \( \text{NE}^{\text{PH}} \) collapses to \( \text{NP} \). If the set \( S \) is in NP, then we can also conclude that \( \text{NE}^{\text{PH}} \) collapses to \( \text{NP} \).

Theorem 3.15. [BH91] If NP has a \( \leq^P \)-hard set \( S \) satisfying \( (3k)(\forall n)[||S^{\leq^P_n}|| \leq 2^{\log^k n}] \), then \( \text{NP} \subseteq \bigcup_{t>0} \text{TIME}[n^{\log^k n}] \) and \( \text{EXP} = \text{NEXP} \).
3.2.2 Other Reducibilities

In this section, we discuss RP, coRP, and coNP-reducibilities. Note that RP ∪ coRP ⊆ BPP and that BPP ⊆ R_T^P(SPARSE) (see [Sch86b]). Thus, we have R_T^P(SPARSE) = R_T^{coRP}(SPARSE) = R_T^{coNP}(SPARSE). So, from Theorem 3.5, we have NP ⊆ R_T^{RP}(SPARSE) ∪ R_T^{coRP}(SPARSE) implies PH = Δ_2^P ∩ Π_2^P. Recently, Ranjan and Rohatgi have studied coRP-reductions and shown the following result.

Theorem 3.16. [RR] If either NP or coNP has a sparse R^hard set, then NP = RP.

Arvind, Köbler, and Mundhenk have strengthened the result of Ranjan and Rohatgi.

Theorem 3.17. [AKM92] NP ⊆ R_m^{coNP}(R_T^P(SPARSE)) → NP = RP.

Open Problem 5.
1. NP ⊆ R_m^{RP}(SPARSE) → NP = RP?
2. NP ⊆ R_m^{RP}(SPARSE) → NP = RP?

On the other hand, for coNP-reducibilities, the known results are as follows.

Theorem 3.18.
1. [AKM92] NP ⊆ R_m^{coNP}(SPARSE) → PH = P.[NP[log n]].
2. [AKM92] Σ_3^P ⊆ R_m^{coNP}(SPARSE) → PH = Δ_3^P.
3. [Yap83] NP ⊆ R_T^{coNP}(SPARSE) → PH = Σ_3^P.

Open Problem 6.
1. NP ⊆ R_T^{coNP}(SPARSE) → PH = P.[NP[log n]]?
2. NP ⊆ R_T^{coNP}(SPARSE) → PH = Δ_3^P?

As a final remark, we mention a result by Lutz and Mayordomo [LM92] that approaches the same problem from a different angle. For two classes C and D, μ(C|D) denotes the resource bounded measure (see [Lut80]) of C ∩ D in D. It is known that μ(P|E) = μ(P|EXP) = 0.

Theorem 3.19. [LM92] If some NP set has a non-dense R^hard set for some α < 1, then μ(NP|E) = μ(NP|EXP) = 0; that is, NP ∩ E and NP ∩ EXP are negligibly small subclasses in E and EXP, respectively.

Notice that this result is incomparable with the open problem "NP ⊆ R_I^{α-μ}(SPARSE) → P = NP?". Theorem 3.19 assumes NP ⊆ R_I^{α-μ}(SPARSE) ≠ ∅ and concludes μ(NP|E) = 0. Thus, the assumption and the conclusion of the open problem are stronger than those in the theorem.

3.3 For Other Classes

Although the issue of sparse hard sets for NP has been considered for many years, similar problems for other classes between P and PSPACE, such as PP, C=P, ⊕P, and so on, have not been frequently addressed. Recently, Ogiwara and Lozano presented general techniques for considering those problems. They studied the notion, due to Lozano and Torán [LT91], of one word-decreasing self-reducible sets. Many classes have complete sets with respect to this type of self-reducibility, yielding broad results, such as the following.

Theorem 3.20. [OL] Let K be a class chosen from {PP, C=P, MOD2P, MOD3P, ...}. Then
1. K ⊆ R^K_p(SPARSE) → K ⊆ NP ∩ coNP and
2. K ⊆ R^K_p(SPARSE) → K ⊆ P.

Very recently, their proof technique has been revisited and yielded, in conjunction with the Buhrman-Longpré–Spaan result [BLS92] in the case of the first part below, the following results.

Theorem 3.21.
1. [AKM92] Let K be a class chosen from {PP, C=P, UP, FewP, Few}. Then, K ⊆ R^K_p(SPARSE) → K ⊆ P.
2. [AHH+] MOD_kP ⊆ R^K_p(SPARSE) → MOD_kP ⊆ P.

NNT [HH91], the class of sets with efficient implicit membership tests, consists of all sets A for which there is a polynomial-time machine M that, given x ≠ ε and its predecessor y, either determines whether x ∈ A or determines whether (x ∈ A ↔ y ∈ A).

Theorem 3.22.
1. [OL] NNT ⊆ R^K_p(SPARSE) → NNT = P.
2. [AHH+] NNT ⊆ R^K_p(SPARSE) → NNT = P.
Moreover, since Karp and Lipton's proof [KL80] works for any reasonable self-reducible sets, we obtain the following.

**Theorem 3.23.** Let $I$ be a class chosen from \{PP, C=P, NNT, MOD2P, MOD3P, \ldots\}. Then $K \subseteq R^{NP}_{\Sigma_2}(\text{SPARSE}) \iff K \subseteq \Sigma_2^p \cap \Pi_2^p$.

**Open Problem 7.**
1. MOD2P $\subseteq R^{P}_{\Sigma_2}(\text{SPARSE}) \iff$ MOD2P $= P$?
2. NNT $\subseteq R^{P}_{\Sigma_2}(\text{SPARSE}) \iff$ NNT $= P$?
3. Let $K$ be a class chosen from \{PP, C=P, UP, FewP, Few\},
   $K \subseteq R^{P}_{\Sigma_2}(\text{SPARSE}) \iff K = P$?

Finally, consider one class beyond PSPACE, namely, the class E. Unlike the classes considered in the previous two subsections, the class E is known to lack sparse hard sets under certain reducibilities. Meyer (reported in [BH77]) showed that $E \not\subseteq R^{P}_{m}(\text{SPARSE})$. Watanabe strengthened this separation to $\leq_{PT}^E$ hardness, and his proof technique essentially reveals the following stronger result.

**Theorem 3.24.** [Watt87] $E \not\subseteq R^{P}_{\Sigma_2}(\text{SPARSE})$.

**Open Problem 8.**
1. $E \not\subseteq R^{P}_{\Pi_2}(\text{SPARSE})$?
2. $E \not\subseteq R^{P}_{\Sigma_2}(\text{SPARSE})$?
3. $E \not\subseteq R^{P}_{\Delta_2}(\text{SPARSE})$?

As a final remark in this section, we present a very recent result by Book and Lutz [BL]. They give a nice characterization of $R^{P}_{\Sigma_2}(\text{SPARSE})$ in terms of space-bounded Kolmogorov complexity (see [LV90]). Let HIGH denote the class of sets with almost everywhere high space-bounded Kolmogorov complexity and let ESPACE $= \bigcup_{c>0} \text{DSPACE}[2^{cn}]$. It is known that almost all sets are in HIGH and HIGH $\cap$ ESPACE $= \emptyset$.

**Theorem 3.25.** [BL] Let $A \in$ ESPACE. For any $k$, $A \in R^{P}_{k-\text{tt}}(\text{HIGH}) \iff A \in R^{P}_{k-\text{tt}}(\text{SPARSE})$.

This theorem, combined with Theorem 3.8 and Theorem 3.20, yields the following corollary.

**Corollary 3.26.** Let $K$ be a class chosen from \{NP, PP, C=P, MOD2P, MOD3P, \ldots\}. If $K \neq P$, then $K \not\subseteq R^{P}_{\Delta_2}(\text{HIGH})$.

## 4 Reducibilities and Equivalence to Sparse Sets

### 4.1 Reducibility Classes

One of the most interesting issues in the study of $R^{P}_{\Sigma_2}(\text{SPARSE})$ is in clarifying the power of weak access to sparse sets; that is, in finding the difference between $R^{P}_{\Sigma_2}(\text{SPARSE})$ and $R^{P}_{\Sigma_2}(\text{SPARSE})$ for reducibilities $r$ and $s$. A key study on this subject was that of Book and Ko [BK88]. They proved that $R^{P}_{\Sigma_2}(\text{SPARSE})$ contains a non-collapsing hierarchy with respect to the number of questions asked, and they also showed many other interesting relationships. Following that work, many researchers have studied this subject, and overall the structure of $R^{P}_{\Sigma_2}(\text{SPARSE})$ classes is now quite well understood.

**Theorem 4.1.**
1. [BK88] $R^{P}_{\Sigma_2}(\text{SPARSE}) = R^{P}_{\Pi_2}(\text{SPARSE})$.
2. [BK88] $R^{P}_{m}(\text{SPARSE}) = R^{P}_{k}(\text{SPARSE})$, $R^{P}_{k}(\text{SPARSE}) \subseteq R^{P}_{k-\text{tt}}(\text{SPARSE})$, and $R^{P}_{m}(\text{SPARSE}) \subseteq R^{P}_{\Sigma_2}(\text{SPARSE})$.
3. [BK88] For all $k \geq 1$, $R^{P}_{k-\text{tt}}(\text{SPARSE}) \subseteq R^{P}_{k+1-\text{tt}}(\text{SPARSE})$ and $R^{P}_{\Sigma_2}(\text{SPARSE}) \subseteq R^{P}_{\Sigma_2}(\text{SPARSE})$.
4. [Ko89] For all $k \geq 1$, $R^{P}_{k-\text{tt}}(\text{SPARSE}) \subseteq R^{P}_{k+1-\text{tt}}(\text{SPARSE})$ and $R^{P}_{\Sigma_2}(\text{SPARSE}) \subseteq R^{P}_{\Sigma_2}(\text{SPARSE})$.
5. [AHOW] $R^{P}_{\Sigma_2}(\text{SPARSE}) \subseteq R^{P}_{\Sigma_2}(\text{SPARSE})$.
6. [Ko89] $R^{P}_{d}(\text{SPARSE}) \not\subseteq R^{P}_{\Sigma_2}(\text{SPARSE})$.
7. (From (5) and (6)) $R^{P}_{\Sigma_2}(\text{SPARSE}) \not\subseteq R^{P}_{\Sigma_2}(\text{SPARSE})$.
8. [Ko89] $R^{P}_{k-\text{tt}}(\text{SPARSE}) \not\subseteq R^{P}_{k}(\text{SPARSE})$.
   Therefore, $R^{P}_{\Sigma_2}(\text{SPARSE}) \not\subseteq R^{P}_{\Sigma_2}(\text{SPARSE})$.
9. [BL92] $R^{P}_{b}(\text{SPARSE}) \subseteq R^{P}_{\Sigma_2}(\text{SPARSE})$.
10. [GW] $R^{P}_{d}(\text{SPARSE}) \not\subseteq R^{P}_{d}(\text{SPARSE})$.
11. [Ko89] $R^{P}_{\Sigma_2}(\text{SPARSE}) \not\subseteq R^{P}_{d}(\text{SPARSE})$.
12. [Ko89] $R^{P}_{d}(\text{SPARSE}) \subseteq R^{P}_{d}(\text{SPARSE})$.
13. (From (11)) $R^{P}_{\Sigma_2}(\text{SPARSE}) \subseteq R^{P}_{\Sigma_2}(\text{SPARSE})$.

Unlike $R^{P}_{\Sigma_2}(\text{SPARSE})$, there is no difference between $R^{P}_{\Sigma_2}(\text{SPARSE})$ and $R^{P}_{\Sigma_2}(\text{SPARSE})$; i.e., $R^{P}_{\Sigma_2}(\text{SPARSE})$ consists of only one class.
Theorem 4.2. [AKM92] \( R^P_m(\text{SPARSE}) = R^N_P(\text{SPARSE}) \).

On the other hand, tally sets have a less finely grained hierarchy than sparse sets.

Theorem 4.3.
1. (see [BK88]) \( \mu(P) \) and \( R^P_\mu(TALLY) = R^P_T(TALLY) \) and \( R^P_\mu(TALLY) \subseteq R^P_T(TALLY) \).
2. [Ko89] \( R^P_T(TALLY) \), \( R^P_\mu(TALLY) \), \( R^P_\mu(TALLY) \leq R^P_T(TALLY) \).
3. [Ko89] \( R^P_m(TALLY) \subseteq R^P_T(TALLY) \subseteq R^P_\mu(TALLY) \).
4. [Ko89] \( R^P_m(TALLY) \subseteq R^P_T(TALLY) \subseteq R^P_\mu(TALLY) \).

Open Problem 9.
1. \( R^m_{SN}(\text{SPARSE}) = R^m_{SN}(\text{SPARSE})? \)
2. \( R^m_{SN}(TALLY) = R^m_{SN}(TALLY)? \)
3. \( R^m_{NP}(TALLY) = R^m_{NP}(TALLY)? \)

A recent result of Buhrman, Longpré, and Spaan [BLS92] establishes a wholly unexpected relationship between sparse and tally sets. The following result is a very powerful tool in studying the structure of reductions to sparse sets, and has been exploited as such both in their paper and in the work of Arvind et al. [AKM92].

Theorem 4.4. [BLS92] \( \text{SPARSE} \subseteq R^P_T(TALLY) \).

4.2 Reducibilities and Equivalences

Another interesting issue in the study of \( R^P_T(\text{SPARSE}) \) is to specify the complexity of sparse sets to which a set \( A \) is polynomial-time reducible. The study of equivalence classes \( E^p_T(\text{SPARSE}) \) to sparse sets is related to this question, because for a given set \( A \), the question "\( A \in E^p_T(\text{SPARSE})? \)" is equivalent to asking whether there is any sparse set \( S \) such that \( A \) is \( \leq^p_T \)-reducible to \( S \) and \( S \) is \( \leq^p_T \)-reducible to \( A \) (i.e., \( S \) is "easy" relative to \( A \)).

Book and Tang [TB91] defined and studied the classes \( E^p_T(\text{SPARSE}) \). Although they solved many problems, they did not address the issue of comparing reduction to equivalence: \( E^p_T(\text{SPARSE}) = R^p_T(\text{SPARSE})? \). This issue was addressed by Allender, Hemachandra, Ogihara, and Watanabe, and for many types of reductions, separations were obtained.

The following is the summary of currently known results.

Theorem 4.5.
1. [TB91] \( E^p_m(\text{SPARSE}) \subseteq E^p_T(\text{SPARSE}) \). For all \( k \geq 1, E^p_k(TALLY) \subseteq E^p_k(TALLY) \).
2. [AHOW,Bin89] Every \( \Sigma^p \) predicates is E solvable if and only if \( E^p_m(TALLY) \cup \{ \Sigma^* \} = E^p_T(TALLY) \). Note that \( \Pi^p_0 = \Sigma^p \).
3. [GWI] For any unbounded polynomial-time computable function \( f, E^p_f(n) \subseteq E^p_T(TALLY) \).
4. [AHOW] For any function \( f(n) = n^{o(1)}, E^p_f(n) \subseteq E^p_T(TALLY) \).

5 Lowness

One way of measuring the information content of a class is to see whether access to that class enhances the power of other complexity classes. For example, it is well-known that \( \Pi^p \cap \text{coNP} = \Pi^p \cap \text{coNP} \) and \( \Pi^p \cap \text{coNP} = \Pi^p \). Thus, making only the reasonable assumption that \( P \neq \Pi^p \cap \text{coNP} \), we know that \( \Pi^p \cap \text{coNP} \) is powerful enough to add power to \( P \), yet is so weak compared to \( P \) that giving \( P \) access to \( \Pi^p \cap \text{coNP} \) is as pointless as giving \( P \) access to the empty set. Results showing that sets or classes give no additional power to a class (usually one from the
polynomial hierarchy) are known as lowness results. Taken from recursive function theory, notion of lowness was introduced in a complexity-theoretic setting by Schöning [Sch83].

**Definition 5.1.** Let \( C \) be any class for which a notion of relativization has been defined, and let \( A \) be any set in \( \text{NP} \). \( A \) is \( C \)-low if \( C^A = C \). For any class of sets \( \mathcal{D} \subseteq \text{NP} \), \( \mathcal{D} \) is \( C \)-low if every \( A \in \mathcal{D} \) is \( C \)-low.

For lowness, the cases where \( C \) is a \( \Sigma \), \( \Delta \), or \( \Theta \) class of the polynomial hierarchy were first studied, respectively, by Schöning [Sch83], Ko and Schöning [KS85], and Long-Sheu/Lozano-Torín [LS91, LT91].

Similarly, the notion of lowness can be extended to sets beyond \( \text{NP} \).

**Definition 5.2.** For any set \( A \) and any \( k \geq 1 \), \( A \) is \( \Sigma^p_k \)-extended-low if \( \Sigma^p_k A \subseteq \text{SAT}^{\text{SAT}} \). For any class of sets \( \mathcal{D} \subseteq \text{NP} \), \( \mathcal{D} \) is \( \Sigma^p_k \)-extended-low if every \( A \in \mathcal{D} \) is \( \Sigma^p_k \)-extended-low. (For the classes \( \Delta^p_k \) and \( \Theta^p_k \), \( k \geq 2 \), extended-lowness is defined similarly.)

For extended lowness, the cases where \( C \) is a \( \Sigma \), \( \Delta \), or \( \Theta \) class of the polynomial hierarchy were first studied, respectively, by Balcázar and Book and Schöning [BBS86], Allender and Hemachandra [AH92], and Long and Sheu [LS91].

Lowness results provide evidence of the "simplicity" of a class, when used as a "database" by some other querying class. One would naturally expect that sparse and tally sets might yield lowness results, and indeed this is the case. Though for many years lowness results have provided only upper bounds on the lowness of classes, Allender and Hemachandra [AH92] recently suggested that lower bounds on the lowness of various classes should be established, and they established essentially optimal results for most known classes, with respect to the \( \Delta \) and \( \Sigma \) levels of the hierarchy. In Tables 1 and 2, we state results at the even finer granularity of \( \Theta \) levels, as has been studied systematically by Long and Sheu (see also [LS91] for a discussion of even more thinly grained refinements, [LT91] for a discussion of the connections between self-reducibility, sparseness, and lowness, [AH92] for a discussion of other lower bounds, and [Sch86b] for a discussion of other upper bounds).

One set of open questions regards the few classes for which the tables reveal a gap between the lower and upper bounds.

**Open Problem 10.** Are all cosparse \( \text{NP} \) sets \( \Sigma^p \)-low? Alternatively, can one construct an oracle relative to which some cosparse \( \text{NP} \) set is not \( \Sigma^p \)-low?

**Open Problem 11.** Since \( \Delta^p_k \)-extended-lowness is identical to \( \Sigma^p_k \)-extended-lowness, it does not make sense to speak of improving the \( \Delta^p_k \)-extended-lowness results of Table 1. However, the \( \Sigma^p_k \)-extended-lowness results are not known to be optimal.

**Open Problem 12.** Are all sets having small circuits (i.e., all sets in \( R^p(T \text{SPARSE}) \) \( \Theta^p_k \)-extended-low or \( \Delta^p_k \)-extended-low? Alternatively, can one show that some set in \( R^p(T \text{SPARSE}) \) is not \( \Theta^p_k \)-extended-low or is not \( \Delta^p_k \)-extended-low? Relatedly, can one find the optimal lowness (with respect to relativizable techniques) of \( R^p(T \text{SPARSE}) \)?

A perhaps more interesting issue is whether lower bounds on lowness can be replaced by structural results.

Allender and Hemachandra developed a theory of absolute lower bounds for the extended low hierarchy. However, for the low hierarchy, they merely presented relativized lower bounds. For each lower bound on lowness, we suggest that the relativized lower bound should be replaced by explicit structural consequences. For example, Ko and Schöning proved that all \( \text{NP} \) sets having small circuits (that is, all sets in \( \text{NP} \cap R^p(T \text{SPARSE}) \)) are \( \Sigma^p_3 \)-low, and Allender and Hemachandra showed a relativized world in which some such sets are not \( \Sigma^p_3 \)-low. Then a question of interest is to show some unlikely structural consequence from the assumption that every set in \( \text{NP} \cap R^p(T \text{SPARSE}) \) is \( \Sigma^p_3 \)-low. In particular, the following questions are interesting.

**Open Problem 13.**

1. Does some unlikely structural consequence follow from the assumption that every set in \( \text{NP} \cap R^p(T \text{SPARSE}) \) is \( \Sigma^p_3 \)-low?

2. Does some unlikely structural consequence follow from the assumption that \( \text{NP} \cap R^p(T \text{SPARSE}) \) is \( \Theta^p_3 \)-low, or that \( \text{NP} \cap R^p(T \text{SPARSE}) \) is \( \Delta^p_3 \)-low.

3. Does some unlikely structural consequence follow from the assumption that every cosparse \( \text{NP} \) set is \( \Sigma^p_3 \)-low.
### Table 1: Upper and Lower Bounds on Extended Lowness of Sparse and Tally Sets, and of Sets Reducible or Equivalent to Sparse and Tally Sets

<table>
<thead>
<tr>
<th>$\mathcal{D}$</th>
<th>$C$, such that $\mathcal{D}$ is $C$-extended-low</th>
<th>$C$, such that $\mathcal{D}$ is not $C$-extended-low</th>
</tr>
</thead>
<tbody>
<tr>
<td>TALLY</td>
<td>$\Delta^p_2$ [BB86]</td>
<td>doesn’t apply</td>
</tr>
<tr>
<td>SPARSE</td>
<td>$\Theta^p_2$ [LS91]</td>
<td>$\Sigma^p_2$ [AH92]</td>
</tr>
<tr>
<td>coSPARSE</td>
<td>$\Theta^p_2$ [LS91]</td>
<td>$\Sigma^p_2$ [AH92]</td>
</tr>
<tr>
<td>$E^p_f$(TALLY)</td>
<td>$\Delta^p_2$ [BB86]</td>
<td>doesn’t apply</td>
</tr>
<tr>
<td>$R^p_m$(TALLY)</td>
<td>$\Delta^p_2$ [AH92]</td>
<td>doesn’t apply</td>
</tr>
<tr>
<td>$E^p_f$(SPARSE)</td>
<td>$\Theta^p_2$ [LS91]</td>
<td>$\Sigma^p_2$ [AH92]</td>
</tr>
<tr>
<td>$R^p_m$(SPARSE)</td>
<td>$\Theta^p_2$ [LS91]</td>
<td>$\Sigma^p_2$ [AH92]</td>
</tr>
<tr>
<td>$R^p_{1-t_1}$(SPARSE)</td>
<td>$\Theta^p_2$ [LS91]</td>
<td>$\Sigma^p_2$ [AH92]</td>
</tr>
<tr>
<td>$R^p_{2-t_1}$(SPARSE)</td>
<td>$\Sigma^p_2$ [BBS86b]</td>
<td>$\Sigma^p_2$ [AH92]</td>
</tr>
</tbody>
</table>

### Table 2: Upper and Lower Bounds on Lowness of Sparse and Tally Sets, and of Sets Reducible or Equivalent to Sparse and Tally Sets

<table>
<thead>
<tr>
<th>$\mathcal{D}$</th>
<th>$C$, such that $\mathcal{D} \cap \text{NP}$ is $C$-low in some relativized world</th>
</tr>
</thead>
<tbody>
<tr>
<td>TALLY</td>
<td>$\Theta^p_2$ [LS91,LT91]</td>
</tr>
<tr>
<td>SPARSE</td>
<td>$\Theta^p_2$ [LS91,LT91]</td>
</tr>
<tr>
<td>coSPARSE</td>
<td>$\Theta^p_2$ [LS91,LT91]</td>
</tr>
<tr>
<td>$E^p_f$(TALLY)</td>
<td>$\Theta^p_2$ [LS91]</td>
</tr>
<tr>
<td>$R^p_m$(SPARSE)</td>
<td>$\Theta^p_2$ [LS91]</td>
</tr>
<tr>
<td>$E^p_{1-t_1}$(SPARSE)</td>
<td>$\Theta^p_2$ [LS91]</td>
</tr>
<tr>
<td>$E^p_{2-t_1}$(SPARSE)</td>
<td>$\Theta^p_2$ [LS91]</td>
</tr>
<tr>
<td>$R^p_{1-t_1}$(SPARSE)</td>
<td>$\Theta^p_2$ [LS91]</td>
</tr>
<tr>
<td>$R^p_{2-t_1}$(SPARSE)</td>
<td>$\Sigma^p_2$ [KS85]</td>
</tr>
<tr>
<td>$R^p_{1-t_2}$(SPARSE)</td>
<td>$\Sigma^p_2$ [KS85]</td>
</tr>
</tbody>
</table>
6 Relativizing With Sparse Oracles

A most natural question regarding oracle constructions is: Why bother? After all, recent papers have shown that some very natural statements about complexity classes hold in the real world yet fail dramatically in relativized worlds [ShaSO,HJV], and artificial constructs of this sort have been known for many years [Har85].

Perhaps the most potent answer to that pesky "why bother?" question would be to show that constructing an oracle relative to which a certain property holds immediately implies that the property holds in the real world. Though this does not hold for all properties [BGS75], it does hold for many fundamental problems. A particularly crisp example is the result of Rackoff and Seiferas and Simon that if there is any oracle $A$ relative to which $L^A \neq NL$, then $L \neq NL$ [Sim77,RS81].

More commonly, to obtain results of this type, one must restrict the amount, form, or pattern of oracle access [BLS84,BLS85], or restrict the class of oracles used. In this section, we will limit our attention to results where the restriction on the class of oracles used is that it be either SPARSE or TALLY. This research stream, which Long [Lon85] proved distinct from so-called positive relativization [Boo87] notions, was appropriately dubbed "relativizations with sparse oracles" by Long and Selman [LS86]. Throughout, we will attempt to speak directly about what the theorems say about the ability of sparse/tally sets to separate or collapse complexity classes.

6.1 Classes That Tally Sets Have No Special Power to Separate

For many pairs of complexity classes, separating the pair via a tally oracle would imply that the pair truly differ. That is, tally oracles are no more likely to separate such classes than the empty oracle is. Table 3 summarizes most known relations of this type.

A natural strengthening of the results of Table 3 would be to broaden the oracle class from TALLY to SPARSE. Unfortunately, such a generalization is in many cases unlikely: there are sparse oracles relative to which $P^S \neq NP^S \neq coNP^S \neq PSPACE^S$, and thus to generalize, for example, any of the table's first four lines would imply the corresponding unrelativized result [BGS75,BBL084].

Theorems such as those of Table 3 need not be proven ad hoc. Many of these results can be generated via a single "meta"-theorem [HR92]. Finally, as is the case for many of the topics discussed in this paper, one can often generalize a bit beyond sparse sets—including non-sparse sets that are essentially sparse sets in disguise [Har83,HH88].

6.2 Classes That Tally Sets Have Special Power to Separate

Given the large number of results contained in Table 3, and the ease with which many such results can be obtained, it is tempting to conjecture that such results hold for all standard classes. However, Hemachandra and Rubinstein have recently noted that for "promise classes" such results may fail, and Bovet, Crescenzi, and Silvestri have provided an additional result of this sort.

Table 4 displays known oracles relative to which tally oracles are more likely than the empty oracle to separate complexity classes.

6.3 Classes That Sparse Sets Have No Special Power to Separate

Table 5 lists classes that sparse sets are no more likely than the empty set to separate. (Note: Section 6.5 lists other classes for which even stronger results are known).

We enthusiastically commend to the reader the recent work of Bovet, Crescenzi, and Silvestri, which presents a powerful "sufficient condition" for obtaining such results, and which more generally put the entire study of oracles in a cohesive framework [BCS91a,
Any statement \( A \neq B \), where \( A \in \{L, NL\} \) and \( B \in \{P, NP, coNP\} \), where \( L \) and \( NL \) are relativized with their oracle tape uncumbered by the space bound.

### Table 3: Classes That Tally Sets Have No Special Power to Separate.

If there is a tally set \( T \) such that \( \text{Statement}^T \) holds, then \( \text{Statement} \) holds.

<table>
<thead>
<tr>
<th>Statement</th>
<th>Citation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P \neq NP )</td>
<td>[BLS85,BBL$^+$84]</td>
</tr>
<tr>
<td>( NP \neq coNP )</td>
<td>[BBL$^+$84]</td>
</tr>
<tr>
<td>( P \neq PSPACE )</td>
<td>[BBL$^+$84]</td>
</tr>
<tr>
<td>( NP \neq PSPACE )</td>
<td>[BBL$^+$84]</td>
</tr>
<tr>
<td>( NP \not\subseteq E )</td>
<td>[LS86]</td>
</tr>
</tbody>
</table>

### Table 4: Classes That Tally Sets Have Special Power to Separate.

There is a relativized world \( E \) such that: there is a tally set \( T \) relative to which \( \text{Statement}^E \) holds yet \( \text{Statement}^{E \oplus T} \) fails.

<table>
<thead>
<tr>
<th>Statement(s)</th>
<th>Citation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P \neq UP )</td>
<td>[HR92]</td>
</tr>
<tr>
<td>( P \neq NP \cap coNP )</td>
<td>[HR92]</td>
</tr>
<tr>
<td>( P \neq RP )</td>
<td>[HR92]</td>
</tr>
<tr>
<td>( NP \cap coNP \not\subseteq \oplus P )</td>
<td>[BCS91a]</td>
</tr>
</tbody>
</table>

### Table 5: Classes That Sparse Sets Have No Special Power to Separate.

If there is a sparse set \( S \) relative to which \( \text{Statement}^S \) holds, then \( \text{Statement} \) holds. For simplicity, we sometimes group multiple statements on the same line. Note that the classes in Section 6.5 also admit results of this form.

<table>
<thead>
<tr>
<th>Statement(s)</th>
<th>Citation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Sigma_2^P \not\subseteq \Pi_2^P, \Delta_2^P \not\subseteq \Sigma_2^P )</td>
<td>[LS86,BBS86a]</td>
</tr>
<tr>
<td>( PPP \not\subseteq \Delta_3^P, PPP \not\subseteq \Sigma_2^P, PPP \not\subseteq \Sigma_2^P )</td>
<td>[LS86]</td>
</tr>
<tr>
<td>( PP \not\subseteq \Delta_3^P, PP \not\subseteq \Sigma_2^P, PP \not\subseteq \Sigma_2^P )</td>
<td>[BBS86a,LS86]</td>
</tr>
<tr>
<td>( \Sigma_0^P \not\subseteq PP, \Sigma_1^P \not\subseteq PP )</td>
<td>[Tod92]</td>
</tr>
<tr>
<td>( \Delta_0^P \not\subseteq PP, \Delta_1^P \not\subseteq PP )</td>
<td>[Tod92]</td>
</tr>
<tr>
<td>( \Sigma_2^P \not\subseteq PP, \Sigma_2^P \not\subseteq PP )</td>
<td>[KSTT89]</td>
</tr>
</tbody>
</table>

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Table 6: Classes That Sparse Sets Have No Special Power to Collapse (or Separate).
If there is a sparse set S relative to which Statement holds then Statement holds. If there is a sparse set S relative to which Statement fails then Statement fails.

<table>
<thead>
<tr>
<th>Statement</th>
<th>Citation</th>
</tr>
</thead>
<tbody>
<tr>
<td>PH = PSPACE</td>
<td>[BBL+84]</td>
</tr>
<tr>
<td>PH collapses</td>
<td>[BBL+84]</td>
</tr>
<tr>
<td>$P^{PP} \subseteq PH$</td>
<td>[BBL+84]</td>
</tr>
<tr>
<td>PP $\subseteq PH$</td>
<td>[BBL+84]</td>
</tr>
<tr>
<td>PH = PP</td>
<td>(Book, as cited in [Tod92])</td>
</tr>
</tbody>
</table>

BCSS91b]. As a consequence, they obtain such statements as:

Theorem 6.1. [BCSS91a]
If $C$ is any class from the collection \{($\Sigma^P_k$, $k \geq 0$), ($\Delta^P_k$, $k \geq 0$), $\oplus P$, $P$, $P$, $C=P$\} and $D$ is any class such that for all $X$ it holds that $D^X$ has a complete set, then: $D \subseteq C \iff (\forall$ sparse $S) [D^S \subseteq C^S]$.  

6.4 Classes That Tally Sets Have Special Power to Collapse

It is known that tally sets have (in relativized worlds) the power to collapse complexity classes; for example, there is a recursive oracle $A$ and a tally set $T$ such that for $(B, C)$ chosen to be any of the pairs ($P$, $NP$), ($NP$, $coNP$), ($P$, $PSPACE$), ($NP$, $PSPACE$), it holds that $B^A \neq C^A$, yet $B^A \oplus T = C^A \oplus T$ (see [HR92]).

6.5 Classes That Sparse Sets Have No Special Power to Collapse (or Separate)

There are a few class pairs $A$ and $B$ for which one can show that if $A^S = B^S$ for some sparse set $S$, then $A = B$. However, this "sparse sets have no power to collapse" behavior is a far rarer behavior than the "sparse sets have no power to separate" behavior of Section 6.1. In fact, in all currently known examples of this sort, one of the classes involved is the polynomial hierarchy (which is the union of stacked quantifiers, and thus is perfectly suited to absorb and hide the use of extra quantifiers that is typical—however, see [Tod92]—in such proofs). Indeed, the polynomial hierarchy is so flexible that, for typical pairs involving it, one can show that sparse sets are neither more likely than the empty set to collapse the pairs, nor more likely than the empty set to separate the pairs. Table 6 summarizes such results.

7 Disclaimer and Conclusions

In this paper we have surveyed selected topics, and only certain areas within the scope of those topics. The study of sparse sets is a broad one, and we commend to the reader's attention all its subareas. This being said, we hope that this paper has given some feeling for the current weight of evidence supporting the statement:

**SPARSE SETS ARE EASY.**

References


