Capturing Complexity Classes by Fragments of Second Order Logic

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Abstract

We investigate the expressive power of certain fragments of second order logic on finite structures. The fragments are second order Horn logic, second order Krom logic as well as a symmetric and a deterministic version of the latter. It is shown that all these logics collapse to their existential fragments. In the presence of a successor relation they provide characterizations of polynomial time, deterministic and nondeterministic logspace and of the complement of symmetric logspace. Without successor relation these logics still can express certain problems that are complete in the corresponding complexity classes, but on the other hand they are strictly weaker than previously known logics for these classes and fail to express some very simple properties.

1 Introduction

In this paper we define and investigate fragments of second order logic that capture some of the more important complexity classes.

It is well known that NP can be characterised by existential second order logic in the following sense: A class $L$ of finite structures of some fixed signature is in NP if and only if there exists an existential second order formula $\psi$ of the same signature such that $L$ is precisely the class of finite models of $\psi$. This was proved by Fagin [8] and then extended by Stockmeyer [24] to a similar correspondence between the polynomial time hierarchy and second order logic as a whole. Immerman systematically studied the problem of designing logics that capture other complexity classes [11]-[15] and came up with logical descriptions for all major complexity classes. Most of these logical characterisations require that the underlying structures are ordered (e.g. by a successor relation) and are obtained by augmenting the syntax of first order logic by operators such as the least fixed point operator, various forms of transitive closure operators etc. For instance, the problems solvable in polynomial time are those that are definable by first order logic together with a linear ordering and a least (or inductive) fixed point operator [11, 26]. The most important results in this field are surveyed in [10, 13, 15].

Here we define logical descriptions of complexity classes not by augmenting first order logic but by restricting second order logic: Consider second order formulae

$$(Q_1 P_1) \cdots (Q_r P_r)(\forall \bar{z}) \bigwedge C_i$$

whose first order part is a universal formula over a conjunction of clauses of some special form. In particular we obtain second order Horn logic (SO-HORN) by requiring that every clause is a Horn clause with respect to the relations $P_1, \ldots, P_r$ (but not necessarily with respect to the input relations). Similarly we define second order Krom logic (SO-KROM) by the condition that every clause $C_i$ contains at most two occurrences of the relations $P_1, \ldots, P_r$. We will also define a symmetric and a deterministic variant of SO-KROM.

We prove the following results:

Collapse Theorems. All these logics collapse to their existential fragments, i.e. to every for-
mula in any of these logics there exists an equivalent formula of the form $(\exists P_1) \cdots (\exists P_k) (\forall z) \land C_i$ where the clauses $C_i$ satisfy the same restrictions as in the original formula.

The collapse theorems do not require the presence of a linear ordering and survive in the case where also infinite structures are allowed.

Capturing complexity classes. In the presence of a successor relation

(i) SO-HORN captures P;
(ii) SO-KROM captures NL; the deterministic version SO-DetKROM captures L;
(iii) SO-SymKROM captures Co-SL.

Here, L and NL denote deterministic and nondeterministic log-space; SL is symmetric log-space, a class introduced by Lewis and Papadimitriou [23] which lies between L and NL. A well-known complete problem for this class is UGAP, the undirected graph accessibility problem. In contrast to L and NL, it is not known whether SL is closed under complementation (see [3]).

Note, that (iii) implies that the dual logic to SO-SymKROM captures SL.

To establish these results we prove that the logics SO-HORN, SO-KROM, SO-DetKROM and SO-SymKROM have the same expressive power as, respectively, fixpoint logic and the various forms of transitive closure logics that are known to characterise P, NL, L and Co-SL. The presence of a successor relation is essential for these results. If it is not available, then our second order fragments are strictly weaker than first order logic with least fixpoint or transitive closure, even if a total ordering (instead of the successor relation) is available.

Remark. Different second order characterizations complexity have been obtained by Blass and Gurevich [2] (Henkin quantifiers) and by Leivant [22] (computational formulae).

2 Preliminaries

A vocabulary or signature is a finite set of relation symbols, function symbols and constants.

A formula of vocabulary $\sigma$ is a formula whose free (second order and first order) variables are contained in $\sigma$. A $\sigma$-structure $B$ consists of a universe $|B|$, of predicates and functions defined over $|B|$ and constants in $|B|$ which interpret the corresponding symbols in $\sigma$.

Definition 1 Let $O$ be the class of finite successor structures, i.e. structures $B$ with universe $\{0, \ldots, n-1\}$ (for some $n \in \mathbb{N}$), whose vocabulary contains the two constant symbols 0, $e$ and the binary predicate $S$ whose interpretations are the constants 0, $n-1$ and the successor relation $S = \{(x, x+1) \mid x < n-1\}$. In the sequel, we denote the universe $\{0, \ldots, n-1\}$ by $n$.

Definition 2 We say that a logic captures the complexity class $C$ if for every problem $L \subseteq O$, $L$ is in $C$ if and only if there exists a formula $\psi$ in the logic such that $L = \{B \in O \mid B \models \psi\}$.

Note, that every decision problem encodable as a subset $L \subseteq \{0,1\}^*$ can be considered as a subset of $O$: identify a string $w_0 \cdots w_{n-1}$ with the structure $(n, P)$ with the monadic predicate $P = \{i \mid w_i = 1\}$. Decision problems concerning graphs or other first order structures can be treated directly; it is not necessary to encode them as binary strings.

As already mentioned, existential second order logic (SO $\exists$) captures NP and unrestricted second order logic (SO) captures the polynomial time hierarchy. On the other hand, the expressive power of first order logic (FO) is very weak: in the presence of the BIT relation, it captures a uniform version of AC$^0$ [13], a proper subset even of NC$^1$. To characterize complexity classes between AC$^0$ and NP, such as P and NL, one possibility is to increase the power of first order logic with additional operators. The most important ones are the least fixed point (LFP) and the transitive closure (TC).

The least fixed point. Let $\sigma$ be a signature, $P$ an $r$-ary predicate not in $\sigma$ and $\psi(\bar{z})$ be a formula of the signature $\sigma \cup \{P\}$ with only positive occurrences of $P$ and with free variables $\bar{z} = z_1, \ldots, z_r$. Then $\psi$ defines for every $\sigma$-structure $B$ an operator $\psi_B$ on the class of $r$-ary
relations over \(|B|\) by

\[
\psi_B : P \mapsto \{\bar{a} \mid (B, P) \models \psi(\bar{a})\}
\]

Since \(P\) occurs only positively in \(\psi\), this operator is monotone, i.e. \(Q \subseteq P\) implies \(\psi_B(Q) \subseteq \psi_B(P)\). Therefore this operator has a least fixed point which may be constructed inductively. Set \(P^0 := \emptyset, P^{i+1} := \psi_B(P^i)\) and \(P^\omega := \bigcup_{i \in \mathbb{N}} P^i\). If \(B\) is finite then this process will reach the least fixed point \(P^\omega\) in a polynomial number of steps.

The fixed point logic \((FO + LFP)\) is defined by adding to the syntax of first order logic the least fixed point formation rule: If \(\psi(\bar{x})\) is a formula of the signature \(\sigma \cup \{P\}\) with the properties stated above and \(\bar{a}\) is an \(r\)-tuple of terms, then

\[
[LFP_{P, \bar{x}} \psi](\bar{u})
\]

is a formula of vocabulary \(\sigma\) (to be interpreted as \(P^\omega(\bar{u})\)).

Immerman [11] and Vardi [26] proved that in the presence of a linear ordering (or equivalently, a successor relation), \((FO + LFP)\) characterizes precisely the queries that are computable in polynomial time.

Transitive closure. A particularly important special case of a least fixed point is the reflexive and transitive closure of a binary relation. The transitive closure can be turned into an operator \(TC\): Let \(\varphi(\bar{x}, \bar{y})\) be a formula with \(2k\) free variables and let \(\bar{u}\) and \(\bar{v}\) be two \(k\)-tuples of terms. Then

\[
[TC_{x, y} \varphi](\bar{u}, \bar{v})
\]

is a formula, which says that the pair \((\bar{u}, \bar{v})\) is contained in the reflexive, transitive closure of the relation defined by \(\varphi\).

The logic \((FO + TC)\) is obtained by adding this rule to the syntax of first order logic. Immerman proved that this logic captures NL [12]:

**Theorem 1** Let \(L \subseteq O\). The following are equivalent

(i) \(L \in NL\);

(ii) there is a formula \(\psi \in (FO + TC)\) such that \(L\) is the set of finite models of \(\psi\);

(iii) there exists a quantifier-free formula \(\varphi(\bar{x}, \bar{y}) \in FO\) such that \(L\) is the set of finite models of the formula \([TC_{x, y} \varphi](\bar{0}, \bar{e})\).

**Remark.** Immerman's original result was weaker; it said that NL is captured by the logic \((FO + pos TC)\), the restriction of \((FO + TC)\) where the operator TC can occur only positively. However, the closure of NL under complementation [14, 25] implies the equivalence of \((FO + pos TC)\) with \((FO + TC)\). In fact Immerman obtained his proof that Co-NL = NL by showing that the complement of the transitive closure can be expressed in \((FO + pos TC)\) (see [15]). This result strongly depends on the presence of the successor relation. Without successor, only the (trivial) implications \((i) \iff (ii) \iff (iii)\) survive whereas the reverse directions fail.

**Symmetric transitive closure.** Symmetric logspace (SL) is the class of languages accepted by symmetric Turing machines in logarithmic space (see [23]). Symmetric Turing machines are nondeterministic Turing machines with symmetric transition relation; if the machine can move from configuration \(C\) to configuration \(C'\), then also from \(C'\) to \(C\). A perhaps more natural definition of this class can be given using the symmetric transitive closure operator \(STC\): When \(\varphi(\bar{x}, \bar{y})\) is a formula defining a binary relation on \(k\)-tuples, then \([STC_{x, y} \varphi(\bar{x}, \bar{y})]\) defines its reflexive, symmetric and transitive closure, i.e. the transitive closure

\[
[TC_{x, y} (\varphi(\bar{x}, \bar{y}) \lor \varphi(\bar{y}, \bar{x}))].
\]

Immerman [12] showed that SL is precisely the class of problem which, in the presence of the successor relation, are definable in the logic \((FO + pos STC)\), i.e. the fragment of \((FO + STC)\) where the STC operator occurs only positively. It is an open problem whether \((FO + pos STC)\) = \((FO + STC)\), i.e. whether SL is closed under complementation. It is known that the undirected graph accessibility problem is complete in this class via first order translations and that every formula in \((FO + pos STC)\) has a normal form \([STC_{x, y} \varphi(\bar{x}, \bar{y})]\)(\(\bar{0}, \bar{e}\)) where \(\varphi\) is a quantifier-free first order formula.
Deterministic transitive closure. The deterministic version DTC of the transitive closure operator first omits all edges starting at a node with out-degree greater than one and then builds the transitive closure. Thus

\[
[DTC_{\bar{x}, \bar{y}} \varphi(\bar{x}, \bar{y})] = \\
[TC_{\bar{x}, \bar{y}} (\varphi(\bar{x}, \bar{y}) \land (\forall \bar{z})(\varphi(\bar{x}, \bar{z}) \rightarrow \bar{y} = \bar{z}))].
\]

Immerman [12] proved that (FO + DTC) captures L and that there is an analogous normal form theorem for (FO + DTC) as for (FO + TC) and (FO + STC).

3 Restrictions of Second Order Logic

Let \( \sigma \) be a vocabulary. We consider second order formulae of the form

\[
(Q_1 P_1) \cdots (Q_r P_r) (\forall \bar{z}) \bigwedge C_i
\]

whose second order quantifiers \( Q_i \) are over relations (not functions) and whose first order part is a universal formula over a conjunction of clauses \( C_i \) of vocabulary \( \sigma \cup \{P_1, \ldots, P_r\} \) that satisfy certain restrictions concerning occurrences of the relations \( P_1, \ldots, P_r \).

Definition 3 Second order Horn logic, denoted SO-HORN, is the set of formulae of type (1) where every clause is a disjunction of atoms and negated atoms with at most one positive occurrence of a predicate \( P_i \), occurrences of the predicates from \( \sigma \) and of equalities and inequalities are not restricted. It is sometimes convenient to write the clauses in "logic programming notation" (\( H \leftarrow B_1 \land \cdots \land B_m \)). The conjunction \( B_1 \land \cdots \land B_m \) is the body of the clause; \( H \) is is either an atom \( P_j(u) \) or the symbol \( \Box \) indicating a contradiction and is called the head of the clause. (In this notation the predicates \( P_1, \ldots, P_r \) always appear unnegated.)

Thus the quantifier-free part of the formulae in SO-HORN are Horn formulae with respect to the 'working predicates' \( P_1, \ldots, P_r \), but not with respect to the 'input predicates' from the underlying signature. (SO 3)-HORN denotes the existential fragment of SO-HORN, i.e. the formulae where all second order quantifiers are existential.

Example. The problem GEN is a well-known P-complete problem [17]. It can be defined as the set of structures \((n; S, f, a)\) in the language of one unary predicate \( S \), one binary function \( f \) and a constant \( a \), such that \( a \) is contained in the closure of \( S \) under \( f \). Clearly, the complement of GEN is also P-complete. It is defined by the following formula from (SO 3)-HORN:

\[
(\exists r)(\forall x)(\forall y)[(R x \leftarrow S x) \land \\
(R f x y \leftarrow R x \land R y) \land (\Box \leftarrow Ra)].
\]

Example. The circuit value problem CVP is also P-complete [21], even when restricted to circuits with fan-in two over NAND (Sheffer's stroke) gates. Such a circuit can be considered as a structure \((n; E, S^+, S^-, a)\) where \( E \) is a binary acyclic predicate, \( S^+ \) and \( S^- \) are monadic and \( a \) is a constant; \( E x y \) means that node \( x \) is one of the two input nodes for \( y \), \( S^+ \) and \( S^- \) contain the input nodes with value 1 and 0, respectively, and \( a \) stands for the output node.

We will take for granted that \( E \) is a connected, acyclic graph with fan-in two, sources \( S^+ \cup S^- \) and sink \( a \). Then the formula:

\[
(\exists T)(\exists F)(\forall x)(\forall y)(\forall z) \varphi
\]

where \( \varphi \) is the conjunction of the clauses

\[
Tx \leftarrow S^+ x \\
Fx \leftarrow S^- x \\
Ty \leftarrow Fx \land Ey \\
Fz \leftarrow Tx \land Ey \land Tz \land Fz \\
\Box \leftarrow 0 \land Fz
\]

states that the circuit \((n; E, S^+, S^-, a)\) evaluates to 1

Definition 4 Second order Krom logic, denoted SO-KROM, is the set of formulae of type (1) where every clause \( C_i \) is a disjunction of atoms and negated atoms that contains at most two occurrences of predicates \( P_1, \ldots, P_r \). Again, we can say that such formulae are Krom-formulae (i.e. formulae in 2-CNF) with respect to the 'working predicates' \( P_1, \ldots, P_r \).
As above, \((SO^3)\)-KROM is the existential fragment of SO-KROM, and the intersection of \((SO^3)\)-HORN and \((SO^3)\)-KROM is denoted by \((SO^3)\)-KROM-HORN.

**Example.** The graph accessibility problem ("Is there a path in the graph \((n, E)\) from \(a\) to \(b\)?") is complete for NL via first order translations, in fact even via projection translations [E]. Its complement is expressible by a formula from \((SO^3)\)-KROM-HORN:

\[
(\exists t)(\forall y)(\forall z)[Tzx \land \\
(Tzx \leftarrow Tzy \land Eyz) \land (\square \leftarrow Tab)].
\]

To justify the definition of our second order fragments we note that if we would allow quantification over functions, or first order prefixes of more general form, then the restriction to Horn clauses would be pointless: in this case already SO-KROM-HORN would have the full power of second order logic [9].

4 Second order Horn logic

We summarize the most important properties of second order Horn logic. The proofs can be found in [9].

**Theorem 2** For every formula \(\psi \in SO\)-HORN there exists a formula \(\psi' \in (SO^3)\)-HORN which is equivalent to \(\psi\) on all (finite and infinite) structures.

**Proof.** (Hint) It suffices to prove the theorem for formulae of the form

\[
\psi \equiv (\forall P)(\exists Q_1) \cdots (\exists Q_r)(\forall z)\varphi
\]

where \(\varphi\) is a conjunction of Horn clauses. It can be shown that such a formula is true (for a given structure) if and only if it is true for all relations \(P\) that are either universally true or that are false at precisely one point. We introduce new relations \(Q'_1, \ldots, Q'_r\) with \(\text{arity}(Q'_i) = \text{arity}(Q_i) + \text{arity}(P)\) and replace the quantifier-free part \(\varphi\) by the new formula \(\varphi_0 \land (\forall y)\varphi_1\) where \(\varphi_0\) is \(\varphi\) with atoms \(Pz\) replaced by an equality \(x = z\) and \(\varphi_1\) is obtained from \(\varphi\) by replacing every atom \(Pz\) by the inequality \(x \neq z\) and every atom \(Q_i(\bar{x})\) by \(Q'_i(\bar{x}, \bar{y})\). Then the formula

\[
\psi' \equiv (\exists Q_1) \cdots (\exists Q_r)(\forall y)(\forall z)(\varphi_0 \land \varphi_1)
\]

is equivalent to \(\psi\).

**Corollary 3** Let \(\psi \in SO\)-HORN. Then the set of finite models of \(\psi\) is in \(P\).

**Proof.** \(\psi\) is equivalent to a formula \(\psi' \in (SO^3)\)-HORN. A structure \(\mathcal{B}\) of appropriate vocabulary yields a propositional Horn formula of polynomial length (with respect to the cardinality of \(\mathcal{B}\)) which is satisfiable iff \(\mathcal{B} \models \psi'\). It is well known that the satisfiability problem for propositional Horn formulae can be solved in linear time.

In the presence of a successor relation, SO-HORN has the same expressive power as fixpoint logic and provides therefore an alternative logical characterization for polynomial time.

**Theorem 4** [9] Every collection \(L \subseteq \mathcal{O}\) which is definable by a formula in \(FO + LFP\) is also definable in SO-HORN.

**Corollary 5** Let \(L \subseteq \mathcal{O}\). Then the following are equivalent

(i) \(L \in \mathcal{P}\);

(ii) \(L\) is expressible by a formula in \((SO^3)\)-HORN;

(iii) \(L\) is expressible by a formula in SO-HORN.

5 Collapse of second order Krom logic

We first consider propositional Krom formulae \(\varphi\) (i.e. formulae in 2-CNF). Let \(X\) and \(Y\) be literals (variables or negated variables) of \(\varphi\); we write \(X \underbrace{\rightarrow}_r Y\) if and only if there exists a sequence \(Z_0, \ldots, Z_t\) of literals such that \(Z_0 = X, Z_t = Y\) and for all \(i < t\), the implication \((Z_i \rightarrow Z_{i+1})\) is equivalent to some clause of \(\varphi\). Note that \(X \underbrace{\rightarrow}_1 Y\) if and only if \(\neg Y \underbrace{\rightarrow}_1 \neg X\).

The following criterion for satisfiability of Krom formulae is well known [20].
Proposition 6 A propositional Krom formula \( \varphi \) is unsatisfiable if and only there exists a literal \( X \) such that both \( X \models \neg X \) and \( \neg X \models X \).

This criterion immediately implies that that the satisfiability problem for propositional Krom formulae is in NL (since NL is closed under complementation); actually it is complete for NL \([18]\).

We generalize this criterion to the case of quantified propositional formulae of the form

\[ \psi \equiv \forall X_1 \ldots \forall X_m \exists Y_1 \ldots \exists Y_n \varphi(X, Y) \]

where \( \varphi \) is a Krom formula. We call literals \( X_i, \neg X_i \) \( \forall \)-literals whereas \( Y_i, \neg Y_i \) are \( \exists \)-literals. As above we write \( U \models V \) if and only if there exist literals \( Z_0, \ldots, Z_t \) such that \( Z_0 = U \), \( Z_t = V \) and for all \( i < t \), the implication \( (Z_i \rightarrow Z_{i+1}) \) is equivalent to some clause of \( \varphi \), but in addition we require that the intermediate literals \( Z_1, \ldots, Z_{t-1} \) be \( \exists \)-literals.

Proposition 7 A propositional \( \forall \exists \exists' \)-Krom formula \( \psi \) is false if and only if at least one of the following conditions holds

(i) there exist distinct \( \forall \)-literals \( X, X' \) such that \( X \models X' \);

(ii) there exists an \( \exists \)-literal \( Y \) such that both \( \neg Y \models Y \) and \( Y \models \neg Y \).

Proof. It is clear that any of the two conditions implies the falseness of \( \psi \). Conversely, assume that \( \psi \) is false. Then there exists an assignment \( \varepsilon : \{X_1, \ldots, X_m\} \rightarrow \{0, 1\} \) such that \( \varphi' \equiv \varphi(\varepsilon, Y) \) is unsatisfiable. Either this formula is false because it contains a clause already interpreted false by \( \varepsilon \), then this clause is equivalent to \( X \models X' \) for distinct \( \forall \)-literals \( X, X' \) and therefore (i) holds. Otherwise, by Proposition 6, there exists an \( \exists \)-literal \( Y \) with \( \neg Y \models Y \) and \( Y \models \neg Y \), i.e. there exists a sequence \( Z_0, Z_1, \ldots, Z_t \) of \( \exists \)-literals such that \( Z_0 = Z_t = Y \) and \( Z_k = \neg Y \) for some \( k \) with \( 0 < k < t \), and such that all implications \( (Z_k \rightarrow Z_{k+1}) \) are equivalent to some clause of \( \varphi' \). If also \( Y \models \neg Y \) and \( -Y \models Y \), then condition (ii) is satisfied. Otherwise take the last implication \( (Z_t \rightarrow Z_{t+1}) \) which does not occur in \( \psi \). But this can only happen if \( Z_t = \neg Z_{t+1} \) and \( \varphi \) contains a clause \( (X \rightarrow \neg Z_{t+1}) \) where \( X \) is an \( \forall \)-literal with \( \varepsilon(X) = 1 \). It follows that \( X \models Y \). Analogously we infer that there is an \( \forall \)-literal \( X' \) such that \( \neg X' \models Y \). It follows that \( X \models X' \), moreover \( X \) and \( X' \) are distinct since \( \varepsilon(X) = \varepsilon(\neg X') = 1 \). Thus, condition (i) is satisfied.

The following results can be inferred from Proposition 7:

Theorem 8 For every formula \( \psi \in \text{SO-KROM} \) there is a formula \( \psi' \in (\text{SO } \exists)-\text{KROM} \) which is equivalent to \( \psi \) on all (finite and infinite) structures.

Proof. It suffices to prove the theorem for formulae \( \psi \equiv (\forall X)\varphi \) where \( \varphi \in (\text{SO } \exists)-\text{KROM} \). Indeed, an arbitrary formula in SO-KROM may then be brought to existential form by successively removing the innermost universal second order quantifier. The basic idea of the proof is the following: If \( \psi \) is false in a given structure \( \mathcal{B} \), then, by Proposition 7, this fact is witnessed already by an interpretation of \( X \) at two points \( \bar{u} \) and \( \bar{v} \). The universal second order quantifier \( (\forall X) \) may therefore be replaced by a string of universal first order quantifiers. Inside the clauses, occurrences of \( X \) are replaced by appropriate equalities and inequalities. Finally the universal quantifiers can pulled inside the scope of the existential second order quantifiers using elementary techniques.

When the language contains at least one constant one can even prove the collapse of SO-KROM to (SO \( \exists \))-KROM-HORN.

Corollary 9 Let \( \psi \in \text{SO-KROM} \). Then the set of finite models of \( \psi \) is in NL.

The proof is analogous to the proof of Corollary 5.

Remark. Proposition 7 can be generalized to a similar criterion for arbitrary quantified propositional Krom formulae which implies that QBF-KROM is in NL.
6 Second order Krom logics and logarithmic space classes

We first show that nondeterministic logarithmic space is captured by SO-KROM, in fact even by (SO 3)-KROM-HORN. To prove this we present a formula which expresses the negation of a transitive closure.

**Proposition 10** Let $\varphi(x, y)$ be a quantifier-free first order formula. Then

$$\neg[\text{TC}_{x,y} \varphi(x, y)](a, b)$$

is equivalent to a formula in the logic (SO 3)-KROM-HORN.

**Proof.** Let $\bigvee_i \varphi_i$ be the DNF of $\varphi$, i.e. every $\varphi_i$ is a conjunction of atoms and negated atoms. Let $\psi$ be the formula

$$(\exists R)(\forall z)(\forall y)(\forall x)(R \exists \bar{z} \land 
\bigwedge_i (R \exists z \leftarrow R x y \land \varphi_i(y, z)) \land \neg \exists a b).$$

This formula obviously belongs to (SO 3)-KROM-HORN. It expresses that there is a relation $R$, containing the transitive closure of $\varphi$, but not the pair $(a, b)$.

**Theorem 11** Let $L \subseteq \mathcal{O}$. Then the following are equivalent

(i) $L \in \text{NL}$;

(ii) $L$ is expressible by a formula in (SO 3)-KROM-HORN;

(iii) $L$ is expressible by a formula in SO-KROM.

**Proof.** (i) $\implies$ (ii): Let $L \in \text{NL}$; then the complement $\overline{L}$ is also in NL and hence expressible in (FO + TC). Immelman has proved that (in the presence of the successor relation), every formula in (FO + TC) has a normal form $[\text{TC}_{x,y} \varphi(x, y)](0, \bar{c})$, where $\varphi$ is a quantifier-free first order formula (Theorem 1). If $L$ is expressible by such a formula, $L$ is expressible by its negation and therefore by a formula in (SO 3)-KROM-HORN.

(ii) $\implies$ (iii) is trivial and (iii) $\implies$ (i) is Corollary 9.

**Symmetric Krom formulae.** We now define a symmetric variant of SO-KROM.

**Definition 5** Let SO-SymKROM be the language of second order formulae

$$(Q_1 P_1) \cdots (Q_r P_r)(\forall y) \land C_i$$

whose first order part is a universal formula over a conjunction of clauses of the form

$$\varphi \rightarrow A \text{ or } \varphi \rightarrow (A \oplus B)$$

where $\varphi$ is a conjunction of atoms and negated atoms without occurrences of $P_1, \ldots, P_r$ and $A$ and $B$ are arbitrary atoms or negated atoms.

**Example:** The set of all bipartite graphs is expressed by the formula

$$(\exists R)(\forall x)(\forall y)(E x y \rightarrow (R x \oplus R y)).$$

The set of bipartite graphs is known to be complete for Co-SL.

**Remark.** We could equivalently define SO-SymKROM as the set of formulae in SO-KROM that contain with every clause of the form $\varphi \lor A \lor B$ also the clause $\varphi \lor \neg A \lor \neg B$ (where $\varphi$ does not contain the 'working predicates' $P_1, \ldots, P_r$). In particular, we can use the bi-conditional $\leftrightarrow$ instead of the exclusive or $\oplus$.

The symmetric version of second order Krom logic captures Co-SL:

**Theorem 12** Let $L \subseteq \mathcal{O}$. Then the following are equivalent

(i) $L \in \text{Co-SL}$;

(ii) $L$ is expressible by a formula in (SO 3)-SymKROM;

(iii) $L$ is expressible by a formula in SO-SymKROM.

**Proof.** (i) $\implies$ (ii): If $L$ is in Co-SL then it is expressible by a formula of the form $\neg [\text{STC}_{x,y} \varphi(x, y)](0, \bar{c})$ where $\varphi$ is quantifier-free. Let $\bigvee_i \varphi_i$ be the disjunctive normal form of $\varphi$ and build the formula

$$(\exists R)(\forall x)(\forall y)(\forall \bar{z})(R \exists \bar{z} \land 
\bigwedge_i (\varphi_i(y, z) \rightarrow R x y \leftrightarrow R \exists z) \land \neg R x \bar{c}).$$
This is a symmetric Krom-Horn formula which expresses \( L \).

(ii) \( \Rightarrow \) (iii) is trivial. To prove (iii) \( \Rightarrow \) (i) we observe that the collapse of SO-KROM to its existential fragment preserves symmetric formulae. If we have a fixed formula \( \psi \) from (SO \( \exists \))-SymKROM then every finite structure \( B \) yields a propositional symmetric Krom formula, (i.e. a conjunction of clauses each of which is a single literal or an exclusive or of two literals) which is satisfiable if and only if \( B \models \psi \). The satisfiability problem for such formulae is known to be in Co-SL [18].

### Deterministic Krom formulae.

To capture deterministic logarithmic space we introduce also a deterministic variant of SO-KROM. To ensure that a formulae \( \psi \) in this logic can be evaluated deterministically with logarithmic space, the chain criterion of Proposition 6 must be 'deterministic', i.e. it must apply to a graph with outdegree at most one. This means that for every structure \( B \) and every instance \( \bar{P}a \) of an atom (where \( P \) is a working predicate and \( a \) a tuple of elements from \( B \)), there can exist at most one other instance \( \bar{R}b \) such that \( \psi \) contains a clause

\[
\varphi(\bar{z}, \bar{y}, \bar{z}) \rightarrow (P\bar{z} \lor R\bar{y})
\]

with \( B \models (\exists \bar{z})\varphi(\bar{a}, \bar{b}, \bar{z}) \) (where \( \varphi \) does not contain working predicates).

This is of course a semantic condition. There are several possibilities to formulate a syntactic condition which implies this, but leaves the logic strong enough to describe log space computations. We present one that is rather restrictive:

**Definition 6** SO-DetKROM is the set of formulæ in \( \psi \in \text{SO-KROM} \) that satisfy the following conditions:

- the clauses of \( \psi \) either have at most one occurrence of a working predicate, or they have the form
  \[
  \varphi \lor (\neg)P\bar{z} \lor (\neg)R\bar{y}
  \]
  where \( \varphi \) does not contain working predicates and \( \bar{z} \) and \( \bar{y} \) are all distinct variables (no constants).

- if \( \psi \) contains a clause
  \[
  \varphi \lor (\neg)P\bar{z} \lor (\neg)R\bar{y}
  \]
  then it must also contain a clause
  \[
  \varphi \lor \varphi(\bar{z}/\bar{y}) \lor \bar{z} = \bar{y}
  \]
  where \( \varphi(\bar{z}/\bar{y}) \) is obtained by substitution of \( \bar{y} \) by new variables \( \bar{z} \);

- if two clauses \( C \) and \( C' \) of \( \psi \) contain the same working predicate \( P \) either both positively or both negatively, say,

\[
C \equiv \varphi \lor P\bar{z} \lor R\bar{y} \\
C' \equiv \varphi' \lor P\bar{u} \lor Q\bar{u}
\]

then \( \psi \) must also contain a clause

\[
\varphi(\bar{z}/\bar{y}) \lor \varphi(\bar{z}/\bar{u})
\]

where \( \bar{z} \) is a tuple of new variables.

**Remark.** The first condition is not really a restriction, because identification of variables with each other or with constants can be incorporated into \( \varphi \); e.g. the clause \( (Px0 \lor Puu) \) can be written \( y \neq 0 \lor y \neq u \lor Pxy \lor Puy \). However, without this convention, the formulation of (ii) and (iii) would be more complicated.

**Example.** A (directed) forest is an acyclic directed graph with outdegree at most one. The following formula expresses that a given graph \((n, E)\) is a forest:

\[
(\exists T)(\forall x)(\exists y)(\forall z)\varphi
\]

with

\[
\varphi \equiv (Exy \rightarrow Txz) \land (Exy \land Tyz \rightarrow Tzx) \land \\
(Exy \land Exz \rightarrow y = z) \land \neg Txz
\]

Clearly this formula belongs to SO-DetKROM.

The proof of Theorem 8 also shows that SO-DetKROM collapses to its existential fragment. The definition of SO-DetKROM thus implies
Proposition 13 Let $\psi \in \text{SO-DetKROM}$. There is a deterministic algorithm that determines with logarithmic space whether a given finite structure is a model of $\psi$.

A deterministic variant of the graph accessibility problem is $1\text{GAP}$, the set of directed graphs $(n, E)$ with two distinguished points $a, b$ such that

$$(n, E) \models [\text{DTC}_{x,y} \text{ Exy}](a, b).$$

$1\text{GAP}$ is complete for deterministic logarithmic space.

We want to express the complement of $1\text{GAP}$ by a deterministic Krom formula assuming the presence of the successor relation $\text{Szy}$ and the constants 0 and $e$. Let $F$ and $G$ have arities 3 and 4, respectively and let $\varphi$ be the conjunction of the following clauses

$$
\begin{align*}
F(a, a, 0) & \iff G(x, y, z, v) \land Suv \land \neg Eyv \\
F(x, y, u) & \iff F(x, y, z) \land \neg Eyz \land Szu \\
F(x, z, 0) & \iff G(x, y, z, u) \land u = e \\
G(x, y, z, z) & \iff F(x, y, z) \land Eyz \\
& \quad \land \neg F(a, b, x).
\end{align*}
$$

It is easy to see that the formula

$$
\psi \equiv (\exists F)(\exists G)(\forall x)(\forall y)(\forall z)(\forall u)(\forall v)\varphi
$$

is equivalent to a formula in $\text{SO-DetKROM}$. We claim that $\psi$ expresses the complement of $1\text{GAP}$.

Indeed if there is no 'deterministic' path in $(n, E)$ from $a$ to $b$ then $\psi$ is satisfied by the following predicates $F$ and $G$

$$
\begin{align*}
F(x, y, z) & \iff [\text{DTC}_{x,y} \text{ Exy}](x, y) \land \neg Eyv \\
& \land (\forall u < z) \neg Eyu \\
G(x, y, z, u) & \iff F(x, y, z) \land Eyz \land (\forall u, z < u \lor u < v) \neg Eyv.
\end{align*}
$$

On the other hand, suppose that there is a 'deterministic' path from $a$ to $b$. We prove, by induction on the length of the path that we can derive the the atom $F(a, b, 0)$ (and hence a contradiction) using the clauses in $\varphi$. Assume that $u$ is a node on the path from $a$ to $b$, that $F(a, u, 0)$ has already been derived and that there is an edge from $u$ to $v$. We then derive $F(a, u, 1), F(a, u, 2), \ldots, F(a, u, v)$, then $G(a, u, v, e), \ldots, G(a, u, v, e)$ (where $e = n - 1$) and finally $F(v, 0, 0)$. Thus, $\psi$ is false in $(n, E, a, b)$.

In fact, $1\text{GAP}$ is complete for logspace even via projection translations [12, 13], i.e. via translations by quantifier-free formulae

$$
\varphi(x, y) \equiv \bigvee_i \alpha_i \land \beta_i
$$

where

- each $\alpha_i$ is a conjunction of equalities, successor relations and their negations,
- distinct $\alpha_i$ and $\alpha_j$ are mutually exclusive,
- each $\beta_i$ as an atom or a negated atom.

From the deterministic Krom formula for (the complement of) $1\text{GAP}$ we can therefore construct a formula for any problem that is solvable in deterministic logarithmic space and prove

**Theorem 14** $\text{SO-DetKROM}$ captures $L$.

7 Unordered structures

Finite structures can be considered as relational databases. In the context of database queries, the presence of a linear ordering is often undesirable because the queries should be independent of the representation of the input, i.e. the ordering of the universe, that is chosen. However, it is well-known that fixed point logic without a linear ordering is too weak to express the polynomial time computable queries; there is, e.g. no formula in this logic that says that the cardinality of the universe is even [5]. On the other hand, every problem in $P$ (without order) is expressible in fixed point logic (with order) by a formula that is order independent; but the set of order independent formulae should not be considered as a logic because order independence is an undecidable property [10]. It is an important open problem whether there exists an order independent logic for polynomial time; it is discussed in the papers [1, 4, 10, 16].
It is therefore natural to ask, whether SO-HORN captures P even in the absence of the successor relation. This is not the case. In fact, without successor, the second order fragments of this paper are strictly weaker than fixpoint logic and the respective versions of transitive closure logic.

**Proposition 15** Every formula in SO-HORN is equivalent to a formula in fixed point logic.

**Proof.** By Theorem 2 we can assume that

\[ \psi \equiv (\exists X_1) \ldots (\exists X_r) \varphi \in (SO \exists)-HORN. \]

Furthermore, we may assume that the head of every clause in \( \varphi \) is either empty or an atom \( X_i(y) \); indeed, a clause of the form \( \alpha \leftarrow \beta \) whose leading predicate is negatable may be replaced by \( \square \leftarrow (\beta \land \neg \alpha) \). Thus we may write \( \varphi \) as a conjunction \( \varphi' \wedge \varphi'' \) where \( \varphi' \) contains the clauses with head \( X_i(z) \) and \( \varphi'' \) the clauses with empty head. The formula \( \varphi' \) can be considered as a Datalog program (with inequalities). It associates with every \( \sigma \)-structure \( B \) a tuple \( X^* \) of relations on \( \mathcal{B} \) which are computed by the usual fixed point semantic for Horn clauses. Therefore \( X^* \) is the simultaneous fixed point of a tuple of existential first order formulae.

Now suppose that \( B \models \psi \). Then, it follows by induction on the stages of the fixed point, that \( (B, X^*) \models \varphi \) and that all \( X \) with \( (B, X) \models \varphi \) are extensions of \( X^* \).

Thus \( \psi \) is equivalent to \( \varphi''(X^*) \). Substituting the fixed point formula that define \( X^* \) we obtain a formula from \((FO + LFP)\) equivalent to \( \psi \). \( \square \)

**Remark.** Proposition 15 is a variant of the well-known result in logic programming that fixed point semantics and minimal model semantics of a datalog program coincide.

Dahlhaus [7] and Kolaitis [19] have defined an existential hierarchy within \((FO + LFP)\): Let \( EFP_0 \) be empty, and \( EFP_{i+1} \) be the closure under disjunctions, conjunctions and existential quantification of the class of all formulae \( LFP (\exists g)\psi(x) \) where \( \psi \) is a quantifier-free formula which may contain (not necessarily positive) occurrences of \( EFP_i \)-definable predicates. Finally, the existential fragment \( EFP \) of \((FO + LFP)\) is \( \bigcup_{i \in \mathbb{N}} EFP_i \). Dahlhaus and Kolaitis proved that all levels of \( EFP \) are distinct and that \( EFP \) is strictly weaker than fixpoint logic. Moreover Kolaitis showed that every stratified logic program belongs to a fixed level \( EFP_k \) and concluded that stratified logic programs have less power than fixpoint logic, disproving a claim of Chandra and Harel [6].

The argument in the proof of Proposition 15 shows that every formula in SO-HORN is expressible in the closure of \( EFP_1 \) under disjunction and negation, hence in \( EFP_2 \).

For SO-KROM the collapse to existential form and the chain criterion from proposition 6 imply a converse to Proposition 10:

**Proposition 16** A collection of finite structures is definable by a formula in SO-KROM if and only if it is definable by the universal closure of a formula \( \neg \left[ TC_{\vec{x},\vec{y}} \varphi(\vec{u}, \vec{v}) \right] \) where \( \varphi \) is a quantifier-free first order formula. For SO-SymKROM an analogous statement holds with \( STC \) instead of \( TC \).

On the other side there are severe limits for the expressive power of second order Horn and Krom logic.

**Proposition 17** There exist first order formulae which are not expressible in SO-HORN or SO-KROM.

**Proof.** Since the first order part of all formulae in SO-HORN and SO-KROM is purely universal, they are closed under substructures, i.e. every substructure of a model is also a model. On the other hand, there are very simple first order formulae that are not closed under substructures, such as, e.g., \( (\exists x)(\exists y)(x \neq y) \).

This remains true even for structures that are ordered by a total ordering \( \leq \) rather than by a successor relation. (For successor structures the proof breaks down because no proper substructure of a successor structure is a successor structure.) However, it is not clear whether SO-HORN and SO-KROM capture the class of problems in \( P \) and \( NL \) which are closed under substructures.

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References


