On the Power of Probabilistic Polynomial Time: 
\( \text{pNP}[^{\log}] \subseteq \text{PP} \) 
(Extended Abstract)

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Abstract

We show that probabilistic polynomial time is closed under polynomial-time parity reductions. Therefore every set polynomial-time truth-table reducible to SAT (every set in the \( \Theta^P_2 \) level of the polynomial hierarchy) is accepted by a probabilistic polynomial-time Turing machine. Equivalently, \( \text{pNP}[^{\log}] \subseteq \text{PP} \).

1 Main Results

Comparing the power of various computation paradigms is a core concern of computational complexity theory. In this paper, we study which classes in the polynomial hierarchy are contained in probabilistic polynomial time, PP.

Near the bottom of the polynomial hierarchy sits \( \text{pNP}[^{\log}] \), the class of languages accepted by polynomial-time Turing machines allowed \( O(\log n) \) calls to an NP oracle, which was first studied by Papadimitriou and Zachos in [PZ83]. Recently, the class has taken on new importance. The class \( \text{pNP}[^{\log}] \) defines the \( \Theta^P_2 \) level of Wagner's refined polynomial hierarchy, has natural complete sets [Kre88, KSW86, Kad87, Wag87a], is equal to the class of sets polynomial-time truth-table reducible to SAT [Hem87, Wag87b, BH88], and is the level to which the polynomial hierarchy collapses under the assumption that NP has sparse Turing-complete sets [Kad87].

In [Gill77], Gill showed that NP is contained in PP. In [Rus85], Russo showed that the class PP is closed under symmetric difference. Using this observation, Papadimitriou and Yannakakis [PY84] showed that \( \text{D}^P \subseteq \text{PP} \), and Balcázar, Díaz, and Gabarro [BDG88] showed that the entire Boolean hierarchy is contained in PP.\(^1\)

In this paper we extend Russo's approach, by showing that PP is closed under polynomial-time truth-table reductions in which the truth-table implements the parity operation. This yields the corollary that \( \text{pNP}[^{\log}] \subseteq \text{PP} \).

Definition 1 A set \( A \) is polynomial-time parity reducible to \( B \) (denoted \( A \leq_{\text{parity}}^P B \)) if \( A \leq_{\text{parity}}^P B \) via a truth-table that tests whether an odd number of its inputs belong to \( B \).

Theorem 2 PP is closed under \( \leq_{\text{parity}}^P \) reductions.

Proof: Suppose that \( B \in \text{PP} \) via Turing machine \( N \), and that \( A \leq_{\text{parity}}^P B \). We define a machine \( N' \) accepting \( A \) that behaves as follows on input \( x \).

i. Simulate the \( \leq_{\text{parity}}^P \) reduction from \( A \) to \( B \), until it produces a list of strings \( x_1, \ldots, x_k \) such that \( x \in A \) if and only if an odd number of those strings belong to \( B \).

ii. Guess paths \( p_1, \ldots, p_k \) of machine \( N \). (This is possible because \( k \) is bounded by a polynomial in \( |x| \).)

iii. Compute the number of strings \( x_i \) such that \( p_i \) is an accepting path of machine \( N \) on input \( x_i \). Accept if that number is odd.

Though \( N' \) is defined in a most naive fashion, it nonetheless accepts the language \( A \). We proceed to prove this claim. By standard techniques [Gill77], we may assume that \( N \) accepts if more than one-half of

\(^1\)This improved on a result by Papadimitriou and Zachos, who showed that the Boolean hierarchy is contained in \( \text{PH}^\#P \) [PZ83].
its paths accept, and that \( N \) rejects if less than one-half of its paths accept. Let, 
\[ r_i = \frac{1}{2} (1 - p_i) \]

denote the probability that a path of \( N \) accepts input \( x_i \). We define an operation on real numbers as follows
\[ r_1 \circ r_2 = r_1 (1 - r_2) + r_2 (1 - r_1). \]

Note that \( r_1 \circ r_2 \) is the probability that exactly one of two randomly chosen paths \( p_1, p_2 \) of machine \( N \) accepts its input string \( x \). It is easily verified that
\[ \frac{1}{2} (1 - p_1) \circ \cdots \circ \frac{1}{2} (1 - p_k) = \frac{1}{2} (1 - p_1 \cdots p_k). \]

Moreover, that is the probability that a path of \( N' \) accepts \( x \). Thus \( N' \) accepts \( x \) if and only if an odd number of the real numbers \( p_i \) are negative. Equivalently, \( N' \) accepts \( x \) if and only if an odd number of the probabilities \( r_i \) are less than one-half. In other words \( N' \) accepts \( x \) if and only if an odd number of the strings \( x_1, \ldots, x_k \) are accepted by \( N \). Thus \( N' \) accepts the language \( A \).

Klaus Wagner has reported a clever proof of Corollary 3 (personal communication). That result can also be obtained as corollary to work by Toda [Tod88].

The results in this paper, [Tod88], and [Gil77] relativistic, as do the results of Köbler, Schöning, Toda, and Torán [KSTT89]. Therefore, Hoene has noted that \( PP \) contains as a subset
\[ \{ A : \exists B \in \text{NP}^F_{\text{classical}} [\text{log}] \ | A \leq_{\text{P-time}} B \}, \]

and in particular \( PP \) contains \( \text{NP}^F_{\text{classical}} \).

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References


[KSTT89] Johannes Köbler, Uwe Schöning, Seinosuke Toda, and Jacobo Torán. Turing machines with few accepting computations and low sets for \( PP \).


