On Periodic Schedules for Deterministically Timed Petri Net Systems

K. Onaga* M. Silva♯ T. Watanabe*

* Dept of Circuits and Systems
Faculty of Engineering, Hiroshima University
Higashi-Hiroshima 724, Japan

♯ Dept of Electrical Engineering and Computer Science
University of Zaragoza
María de Luna, 3
(ACTUR) E-50015 Zaragoza, Spain

Abstract

This paper introduces a Petri Net based perspective to periodic schedules. These are classified, according to the time interpretation in multiple-*server semantics and/or transitions firing periodicity constraints, into* strict *and* general *periodic schedules. Using a net transformation rule, the computation of the general schedule class can be done through techniques for the strict subclass. A necessary and sufficient condition for the feasibility of a strict periodic schedule is given. Estimating the truncation error at initial firing time epoch τ_1 allows to get linear systems of inequalities as necessary conditions for a schedule to be feasible. An linear programming (LP) based heuristic approach is proposed to deal with the computation of "optimum" (i.e. minimal time) strict periodic schedules under the single-server semantics.*

1 Introduction

Scheduling is a fundamental problem in the control of any organization (e.g. computer and manufacturing systems). Vast amount of literature can be found in operations research journals, frequently relied upon mathematical tools of Gantt charts and task-precedence digraphs.

Petri nets have been widely recognized as the best mathematical tool for describing phenomena of conflicts, concurrency and synchronization among events, tasks and resources [1]. *Transition timed Petri nets* allow *tasks* (represented by transitions), *shared resources* (represented by shared places) and *constraints* (i.e. synchronization, precedence) to be modeled within *a single formalism.* The firing *delay* associated to a transition represents the duration of a single execution of the represented task. Moreover, net models allow in a natural way the study of *repetitive schedulings.*

A few results are known for scheduling Petri Net models in general (see, for example, [2, 3, 4]). Two classical scheduling strategies (or modes) are: (1) *immediate progress* or *earliest scheduling mode* (i.e. fire transitions as soon as enabled) and (2) *periodic scheduling mode.* In periodic schedulings the firing epoch of transitions is defined through $\{\pi, \tau_1\}$, where π is the *period*, and τ_1 is a vector representing the *firing epochs in the first period. A quasi-periodic* schedule is composed of a *warm-up* (or transient) schedule, followed by a *periodic* schedule.

Given a timed net system $\langle \mathcal{N}, d, M_0 \rangle$ and I a T-semiflow (or T-invariant) of $\mathcal{N}$ representing the firing count vector of the single period, a problem we address in this paper is to compute an optimal schedule $\{\pi, \tau_1\}$ (i.e. one minimizing π). For *Marked Graphs*, the earliest scheduling mode can reach the minimum period (cycle time)[2], which is also realized by the periodic scheduling mode in which each enabled transition is conveniently delayed for its first firing.

This paper presents a linear programming based formulation to compute optimal periodic schedules for structurally bounded Petri Nets. Section 2 introduces net's basic definitions and some results. Section 3 deals with lower bounds for the period and a cost function of the initial marking, independent of any particular scheduling strategy. These bounds can be useful for the derivation of heuristics to solve scheduling problems. Section 4, 5, 6, 7 discuss feasibility of periodic schedules. Section 8, 9 deal with a linear programming based heuristic to solve the optimal periodic scheduling problem. The last section concerns conclusions and enumerates some research items left to be done in future works.

2 Petri Nets Notations and Some Results

We assume the reader is familiar with the structure, firing rules, and basic properties of net models (see [1] for a recent survey). Let us recall some notations here: $\mathcal{N} = \langle P, T, F, W \rangle$ is a net with:

$P = \{p_1, p_2, \ldots, p_n\}$, the set of $n = |P|$ *places*,

$T = \{t_1, t_2, \ldots, t_m\}$, the set of $m = |T|$ *transitions*,

$F \subseteq (P \times T) \cup (T \times P)$, the set of *arcs* (or *edges*) representing *Flow relation.*

$W : F \rightarrow \mathbb{N}^+$, the *weight function* (the net is said to be *ordinary* if arcs weights are all equal to

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1), where $\mathbb{N}$ is the set of natural numbers and $\mathbb{N}^+ = \mathbb{N} - \{0\}$.

A marked net (or net system) is denoted as $\langle \mathcal{N}, M_0 \rangle$, where M_0 is the initial marking, $M_0 : P \rightarrow \mathbb{N}$.

The pre-incidence (post-incidence) matrix $C^- = [C^-_{ij}]$ ($C^+ = [C^+_{ij}]$) is an $n \times m$ matrix where $C^-_{ij} = W(p_i, t_j)(C^+_{ij} = W(t_j, p_i))$. The incidence matrix of net $\mathcal{N}$ is defined by $C = C^+ - C^-$. $C_t(C_p)$ represents the column (row) associated with transition t(place$p$) of C.

The preset of $x \in (P \cup T)$ is ${}^\bullet x = \{y \in (P \cup T) \mid (y, x) \in F\}$, and the postset of $x \in (P \cup T)$ is $x^\bullet = \{y \in (P \cup T) \mid (x, y) \in F\}$. The notation is extended to sets $X \subseteq (P \cup T)$ by ${}^\bullet X = \cup_{x \in X} {}^\bullet x$, and similarly for $X^\bullet$.

Let σ represent a firable sequence, while $\overline{\sigma}$ is the firing count vector associated to σ and $|\sigma|$ the length of sequence σ: $\sum_{i=1}^{m} \overline{\sigma}_i$. If M is reachable from M_0 (i.e. $\exists \sigma$ s.t. $M_0[\sigma\rangle M$), then $M = M_0 + C \cdot \overline{\sigma} \geq 0$ and $\overline{\sigma} \geq 0$ (this is called the net's *State Equation*). The set of the reachable markings from M_0 is denoted by $R(\mathcal{N}, M_0)$.

A net $\mathcal{N}$ is *structurally bounded* iff any net system $\langle \mathcal{N}, M_0 \rangle$ is bounded. An integer solution of $C \cdot X = 0, X > 0$ ($Y^T \cdot C = 0, Y > 0$) is a *T-semiflow* or *T-invariant* (*P-semiflow* or *P-invariant*) where $X = (X_1, X_2, \ldots, X_m) > 0$ means $X_i \geq 0$ for all i but $X_j > 0$ for at least one j. A T-semiflow X is realizable in $\langle \mathcal{N}, M_0 \rangle$ iff $\exists M \in R(\mathcal{N}, M_0)$ such that $M[\sigma\rangle M$ and $\overline{\sigma} = X$. The following are well-known properties.

In this work we are interested in periodic behaviours of net $\mathcal{N}$ in which $I \in \mathbb{N}^m$ represents the firing count vector during the single period.

Assumption. *Without loss of generality we will assume in the sequel that $I_t > 0$ $\forall t \in T$. This assumption is not a constraint at all because we are interested only in behaviours of $\mathcal{N}$ such that the T-semiflow I is periodically realized.*

A *transition timed net* is obtained by assigning (deterministic) time delays to the firing of transitions: $\langle \mathcal{N}, d \rangle$ is a transition timed net in which $d : T \rightarrow \mathbb{R}^+$ (positive real numbers) is a function assigning d_t time units to complete the firing of transition t. A timed marked net (or timed net system) is defined by $\langle \mathcal{N}, d, M_0 \rangle$. In this paper we adopt a very simple transition firing rule defined in *three phases*:

1. Transition t is *enabled* at marking $M \in R(\mathcal{N}, M_0)$ if $M \geq C^-_t$ holds.

2. If transition t starts to fire, then C^-_t tokens are inmediately removed from the input places of t, $p \in {}^\bullet t$. $M' = M - C^-_t$.

3. The firing of t finishes d_t time units later and C^+_t tokens are added to the output places of t, $p \in t^\bullet$,
$$M'' = M' + C^+_t = M + C^+_t - C^-_t.$$

4. No transition is allowed engaging in concurrent multiple firings even if there are abundant input tokens. This mode of firing is called "the single-server firing". However this constraint is recommended to be built into the net structure as a self-loop of a mono-marked place around each transition. This topics will be treated more fully in Section 4.

3 Lower Bounds for the Period and the Resource Cost

This section presents two lower bounds. The first concerns the cycle time of a timed net system $\langle \mathcal{N}, d, M_0 \rangle$, while the second concerns a marking-based linear function describing the cost of the system.

3.1 Sifakis Bound for the Period

Let us assume that firing sequence σ works in $\langle \mathcal{N}, d, M_0 \rangle$ long run with I as *a mean relative firing count vector*:

$$\lim_{|\sigma| \rightarrow \infty} \frac{\overline{\sigma}}{|\sigma|} = \frac{I}{|I|} \text{ where } |I| = \sum_{i=1}^{m} I_i \,.$$

If $\mathcal{N}$ is structurally bounded, I is a T-semiflow, and the *mean firing rate vector* F is given by: $F = I/\pi$, where π is the mean cycle time or the period.

An explicit formula for a lower bound of the period of a deterministic all timed net system was obtained by Sifakis in [5] (where he considered timed places instead of timed transitions).

$$\overline{M} \geq C^+ \cdot (diag\ d) \cdot \frac{I}{\pi} = C^+ \cdot \frac{D}{\pi} \qquad (1)$$

where:

* $\overline{M}$ is the limit average marking for any scheduling such that in long run the transitions are fired in the relative ratio given by I.

* $diag\ d$ is the diagonal matrix of d_i, $i = 1, 2, \ldots, m$.

* $D \triangleq (diag\ d) \cdot I$

Vector $\overline{M}$ is unknown, thus π cannot be directly computed. Premultiplying (1) by a P-semiflow J, the following can be written:

$$J^T \cdot \overline{M} \cdot \pi = J^T \cdot M_0 \cdot \pi \geq J^T \cdot C^+ \cdot D \qquad (2)$$

and hence

$$\pi \geq \pi^* \triangleq \max_J \frac{J^T \cdot C^+ \cdot D}{J^T \cdot M_0} \text{ where } J^T \cdot C = 0, J > 0. \qquad (3)$$

Eq. (3) gives the explicit formula of Sifakis lower bound π^*. This bound has been shown to be reachable for Marked Graphs (in Reiter [4] for the periodic schedule and in Chretienne [2] for the earliest firing schedule), but for more general net systems it is not reachable [6].

Property 1 *[5] Sifakis bound π^* for periods of periodic schedules realizing a relative firing count vector I in $\langle \mathcal{N}, d, M_0 \rangle$ can be computed by the following linear programming problem:*

$$\begin{aligned} \pi^* = \quad & \max \quad J^T \cdot C^+ \cdot D \\ & \text{s.t.} \quad J^T \cdot C = 0\,, J > 0 \ \textit{(i.e. J is a P-semiflow)} \\ & \qquad J^T \cdot M_0 = 1 \quad \textit{(a normalization constraint)} \end{aligned} \qquad \text{(LPP1)}$$

Therefore Sifakis bound π^* can be computed in *polynomial time* by solving (LPP1).

3.2 Minimum Cost Initial Marking

Let us assume the cost of a system modeled by a net to be *aditive* (i.e. a linear function of the number of resources). The number of resources in $\langle \mathcal{N}, M_0 \rangle$ is given by M_0. Thus a natural cost function is:

$$v(M_0) = V^T \cdot M_0 \text{ where } V \in (I\!R^+)^n. \qquad (4)$$

Therefore, the following property can be stated:

Property 2 *Let $v(M_0) = V^T \cdot M_0, V \in (I\!R^+)^n$ be a cost function. The linear programming problem (LPP2) gives a lower bound for $v(M_0)$, while π^* is the lower bound for the period (reachable for live and bounded Marked Graphs) computed by means of (LPP1):*

$$\begin{aligned} v^* = \quad & \min \quad v(M) = V^T \cdot M \\ & \text{s.t.} \quad C \cdot z + \pi^* \cdot M \geq C^+ \cdot D \\ & \qquad M \leq M_0 \\ & \qquad M \geq 0 \end{aligned} \qquad \text{(LPP2)}$$

It is natural to assume that the cost of resources of a system is the same for all reachable markings (i.e. $v(M_0) = v(M)$, where $M_0[\sigma\rangle M$). Therefore:

$$V^T \cdot M_0 = V^T \cdot M = V^T(M_0 + C \cdot \overline{\sigma}) \Rightarrow V^T \cdot C \cdot \overline{\sigma} = 0$$

must hold for any $\overline{\sigma} > 0$. This is possible only when $V^t \cdot C = 0$ holds.

The lower bound for the resource cost, $v_{min}(M_0)$, can be used to guide the selection of initial markings. The *Minimal Initial Marking* (MIM) problem (i.e. given $\mathcal{N}$ and a firing count vector X, find an M_0 such that $M_0[\sigma >$ and $\overline{\sigma} = X$) has been considered in [7]. It has been shown to be NP-complete even for Marked Graphs under very weak conditions. Thus practical solutions are necessarily heuristic.

4 Firing Semantics and Definitions of Periodic Schedules

There are two extreme possibilities for the interpretation of transition firing:

- *infinite-server semantics,*
- *single-server semantics,*

In the first case transitions can be fired concurrently with themselves, while in the second case no transition can fire concurrently with itself (i.e. there exists *firing self-inhibition*). The single-server semantics can be made explicit by introducing a place loop with a single token around each transition.

In principle, the transition must be interpreted by the infinite-server semantics: Any restriction must be explicitly expressed by net structures and attached restrictions should be allowed in texts only when no net mechanism is available. A given T-invariant I can be converted into the unity T-invariant $\tilde{I} = 1\!\!1$ by adopting the sequential I_t-server semantics for each t.

The lower bound of the period computed in Section 3.1 assumes the ∞-server semantics. If the single-server semantics is considered (then additionnal constraints on firing are introduced), the linear programming problem (LPP1) is valid but a self-loop monomarked place must be added to each transition of $\langle \mathcal{N}, M_0 \rangle$.

Three periodicity interpretations can be distinguished:

1. *Strict-periodic schedule,* in which firing of each transition strictly follows a fixed regular time-interval from its first firing. The single server with self-inhibition time $h_t \geq 0$ produces this strict periodicity,

$$\sigma_{SP} = \sigma_{cyc}\sigma_{cyc} \cdots = \sigma^*_{cyc}, \ \overline{\sigma}_{cyc} = I,$$

   where * is the concatenation operator.

2. *Periodic schedule,* in which the time regularity exists among successive periods, but within the single period, the firing may not be in regular intervals for individual transitions. The sequential I_t-server semantics with setup times produces this perodicity when the converted net $\tilde{\mathcal{N}}$ is used. Now the server plays the role of the transition.

$$\tilde{\sigma}_P = \tilde{\sigma}_{cyc}\tilde{\sigma}_{cyc} \cdots = \tilde{\sigma}^*_{cyc}, \ \overline{\tilde{\sigma}}_{cyc} = 1\!\!1.$$

3. *Quasi-periodic schedule,* in which a transient (warm-up) firing is allowed before entering into the periodic firing. The sequential I_t-server semantics produces this quasi-perodicity if applied to $\langle \tilde{\mathcal{N}}, M \rangle$ for some reachable marking M, $M_0[\sigma_{warm-up}\rangle M$.

$$\tilde{\sigma}_{QP} = \tilde{\sigma}_{warm-up} \cdot \tilde{\sigma}^*_{cyc}, \ \overline{\tilde{\sigma}}_{cyc} = 1\!\!1.$$

It is obvious that a strict periodic schedule is more restricted than a periodic schedule, which in turn is more restrictive than a quasi-periodic schedule.

5 Feasibility Conditions for a Strict Periodic Schedule under Single Server Semantics

Let τ_k^t denote the k-th firing epoch of transition t. Then τ_k^t can be explicitly written as:

$$\tau_k^t = \tau_1^t + (k-1)\frac{\pi}{I_t} \quad (5)$$

where τ_1^t is the initial firing epoch to be adjusted and $\pi_t = \pi/I_t$ is the period of transition t. The strict periodic schedule is defined by π (the period) and τ_1 (the vector of first firing epochs): $\{\pi, \tau_1\}$.

Let us introduce the firing number function $f_t(\tau)$ as the number of times transition t has fired up to and including time τ. Then we have:

$$f_t(\tau) = \lfloor \frac{(\tau - \tau_1^t)I_t}{\pi} \rfloor + 1. \quad (6)$$

where floor function $\lfloor a \rfloor$ is the greatest integer not exceeding a.

Let us consider the token number in place p at time epoch τ, $M^p(\tau)$:

$$M^p(\tau) = M_0^p + \sum_t C_{tp}^+ f_t(\tau - d_t) - \sum_t C_{pt}^- f_t(\tau) \geq 0 \quad (7)$$

Time displacement d_t appears in the second term because at place p the token produced by the firing of input transition t appears d_t time units later.

Property 3 *Suppose the T-semiflow I be realizable as a firing count vector of a some firing sequence σ from the initial marking M_0. A strict periodic firing is feasible iff there exists a non-negative solution $\{\pi, \tau_1\}$ that satisfies $M^p(\tau) \geq 0$ for any $p \in P$ and at any time instant τ. Moveover $M^p(\tau) \geq 0$ holds at any τ iff $M^p(\tau_1^t) \geq 0$ holds for any transition t, where τ_1^t is the t-component of verctor τ_1.*

6 Transforming a Periodic Schedule into a Strict-Periodic Schedule

The difference between strict-periodic and periodic schedules resides in the fact that the latter does not require a regular interval for the firing of each transition inside a single period but among periods. Periodic and strict-periodic schedules coincide obviously if $I = \mathbb{1}$, because any transition will fire only once in each period.

The following net transformation for each tansition t allows to relax strict-priodic to periodic schedules. Thus only the strict case will be considered later.

Transformation rule. Let $\langle \mathcal{N}, M_0 \rangle$ be the original net system and I the T-semiflow to be executed cyclically. For any t, for which a periodic schedule is required, do[See Fig.1]:

a) I_t instantiations of t: $\{t_1, t_2, \ldots, t_{I_t}\}$. At this point all the instances possess *identical* flow relationship to outside (i.e. same pre and same post-incidences).

b) I_t instantiation of setups of t: $\{\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_{I_t}\}$

c) Add $2I_t$ ordinary places $\{p_1, p_2, \ldots, p_{I_t}\}$, $\{\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{I_t}\}$ to form a ring of $2I_t$ transitions and $2I_t$ places in such that $(t_1, \tilde{t}_2, t_2, \tilde{t}_3, \ldots, t_{I_t}, \tilde{t}_{t_1})$ is the order of the transitions in the ring.

d) d_{t_i} is the firing delay associated with t_i and h_{t_i} is the setup time associated with $\tilde{t}_i$. In this paper we deal with the simplest case of $d_{t_i} \equiv d_t$, $h_{t_i} \equiv 0, i = 1, 2, \ldots, I_t$. There is no need to provide the setup transitions $\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_{I_t}$ and the associated place $\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{I_t}$.

e) The input place $\tilde{p}_1$ of t_1 contains a single token but no other place contains tokens.

7 Linearization of the Floor Function and Matrix Notations

Let us linearize the firing number function $f_t(\tau)$ by introducing the error term $\varepsilon_t(\tau)$:

$$f_t(\tau) = 1 + \frac{(\tau - \tau_1^t)I_t}{\pi} - \varepsilon_t(\tau) \quad (8)$$

where $0 \leq \epsilon_t(\tau) < 1$ holds. Substituting Eq.(8) into Eq.(7), we obtain

$$\begin{aligned} M^p(\tau) = M_0^p &+ \{\sum_t C_{tp}^+(1 - \varepsilon_t(\tau - d_t)) \\ &- \sum_t C_{pt}^-(1 - \varepsilon_t(\tau))\} \\ &+ \frac{1}{\pi}\{\sum_t C_{tp}^+ I_t - \sum_t C_{pt}^- I_t\}\tau \\ &- \frac{1}{\pi}\{\sum_t C_{tp}^+ I_t(\tau_1^t + d_t) \\ &- \sum_t C_{pt}^- I_t \tau_1^t\} \geq 0 \end{aligned} \quad (9)$$

where we note that the third term vanishes due to I being a T-semiflow.

It is straight forward to prove the following matrix notations:

$$\begin{aligned} &\{M_0 + C \cdot \mathbb{1} - C^+ \cdot \varepsilon(\tau\mathbb{1} - d) + C^- \cdot \varepsilon(\tau\mathbb{1})\}\pi \\ &-C \cdot (diag\ I) \cdot \tau_1 - C^+ \cdot D \geq 0. \end{aligned} \quad (10)$$

where we use the following notations:

$\mathbb{1} \triangleq (1, 1, \ldots, 1)^T$

$\varepsilon(\tau\mathbb{1}) \triangleq (\varepsilon_{t_1}(\tau), \varepsilon_{t_2}(\tau), \ldots, \varepsilon_{t_m}(\tau))^T$

$\varepsilon(\tau\mathbb{1} - d) \triangleq (\varepsilon_{t_1}(\tau - d_{t_1}), \varepsilon_{t_2}(\tau - d_{t_2}), \ldots, \varepsilon_{t_m}(\tau - d_{t_m}))^T$

$diag\ I \triangleq$ square diagonal matrix of $I_{t_1}, I_{t_2}, \ldots, I_{t_m}$.

$diag\ d \triangleq$ square diagonal matrix of $d_{t_1}, d_{t_2}, \ldots, d_{t_m}$.

$D \triangleq$ column vector of $d_{t_1}I_{t_1}, d_{t_2}I_{t_2}, \ldots, d_{t_m}I_{t_m}$.

Remark 1 *Let us assume $I = \mathbb{1}$ (this is the case when the net is transformed in accord with the sequential I_t-server semantics and for which periodic schedules are obtainable). Then: Eq (10) reduces to:*

$$\{M_0 - C^+\varepsilon(\tau\mathbb{1} - d) + C^-\varepsilon(\tau\mathbb{1})\}\pi - C\cdot\tau_1 - C^+d \geq 0. \tag{11}$$

8 Evaluation of Trancation Errors

One way of relaxing the computational difficulty in Eq (11) is to compute its average value in some intervals of time. Its non negativity will be a necessary (but not sufficient) condition for $\{\pi, \tau_1\}$ to be a strict-periodic schedule. Averaging in time, τ, over one period, the following is obtained:

$$\begin{aligned}\{M_0 - C^+\cdot\hat{\varepsilon}_d^\pi + C^-\hat{\varepsilon}^\pi\}\pi \\ -C\cdot\tau_1 - C^+\cdot d \geq 0\end{aligned} \tag{12}$$

where $\hat{\varepsilon}_d^\pi$ and $\hat{\varepsilon}^\pi$ are the average values of $\varepsilon(\tau\mathbb{1} - d)$ and $\varepsilon(\tau\mathbb{1})$ over one period.

As the start of the computation, it appears that $\hat{\varepsilon}_d^\pi = \frac{1}{2}\mathbb{1} - d$, $\hat{\varepsilon}^\pi = \frac{1}{2}\mathbb{1}$ is reasonable guess. Therefore Eq (12) reduces to:

$$(M_0 + C^+\cdot d)\pi - C\cdot\tau_1 - C^+\cdot d \geq 0. \tag{13}$$

Property 4 *Without loss of qenerality we can assume $I = \mathbb{1} \triangleq (1, 1, \ldots, 1)$ because the net syatem is expanded already in accordance with sequential I_t-server semantics for each transition t. Suppose $\{\pi, \tau_1\}$ be a periodic schedule and the transitions be renumbered so as to satisfy the following montone ordering,*

$$\tau_1^{t_1} \leq \tau_1^{t_2} \leq \cdots \leq \tau_1^{t_m}.$$

Then the truncation error vector ε_d and ε evaluated at time instant $\tau_1^{t_i}$ can be expressed by a very simlpe form

$$\varepsilon(\tau_1^{t_i}) = \frac{1}{\pi}\begin{bmatrix} (\tau_1^{t_i} - \tau_1^{t_1}) \\ \vdots \\ (\tau_1^{t_i} - \tau_1^{t_{i-1}}) \\ 0 \\ \pi + (\tau_1^{t_i} - \tau_1^{t_{i+1}}) \\ \vdots \\ \pi + (\tau_1^{t_i} - \tau_1^{t_m}) \end{bmatrix},$$

$$\varepsilon_d(\tau_1^{t_i}) = \frac{1}{\pi}\begin{bmatrix} (\tau_1^{t_i} - \tau_1^{t_1} - d_1) \\ \vdots \\ (\tau_1^{t_i} - \tau_1^{t_{j-1}} - d_{j-1}) \\ \pi + (\tau_1^{t_i} - \tau_1^{t_j} - d_j) \\ \vdots \\ \pi + (\tau_1^{t_i} - \tau_1^{t_m} - d_m) \end{bmatrix} \tag{14}$$

where j is the first index $\tau_1^{t_i} - \tau_1^{t_j} - d_j < 0$, $j \leq i$.

Property 5 *If a periodic schedule $\{\pi, \tau_1\}$ exixsts, then it satisfies the following condition with $y := \pi$, $\mathbf{x} := \tau_1$*

$$\{M_0 - C^+\cdot\varepsilon_d(x_i) + C^-\cdot\varepsilon(x_i)\}y - C\cdot\mathbf{x} - C^+\cdot d \geq \mathbf{0} \tag{15}$$

$$\mathbf{x} = (x_1, x_2, \ldots, x_m) \geq \mathbf{0}, x_1 \leq x_2 \leq \cdots\cdots \leq x_m$$

$$\textit{for all } i = 1, 2, \ldots, m$$

where $\varepsilon_d(x_i)$, $\varepsilon(x_i)$ are given by Eq (14) of Property 4.

Conversely if, starting from an initial state $\{y^0, \mathbf{x}^0\}$, Eq (15) is solved by some iteration scheme to yield $\{y^k, \mathbf{x}^k\}$ that satisfies the condition Eq (15), then $\pi := y^k, \tau_1 := \mathbf{x}^k$ constitutes a periodic schedule of the net system.

9 An Heuristic LP Approach to the Design of Optimal Periodic Schedules

The problem to be solved is to compute the minimal period π and associated initial firing-time vector τ_1 that allows the execution of a repetitive sequence σ such that $\overline{\sigma} = I$ (a T-semiflow) under the firing regularity condition.

⟨⟨LP algorithm for optimal periodic scheduling⟩⟩

Input :

- $\langle\mathcal{N}, d, M_0\rangle$: the incidence matrix, $C = C^+ - C^-$, the transition firing delay vector d, the initial marking M_0.
- T-semiflow to be executed periodically, I.
- Control variables:
  - *max_nb_of_iterations* ≥ 1
  - relative period updating threshold, $\delta << 1$.

Output : Period $\pi := y$, and initial firing-epoch vector $\tau_1 := \mathbf{x}$

Step 0°. Replace each transition t by the sequential I_t-server ring. Hence after we assume T-semiflow, $I = \mathbb{1}$.

Step 1°. Let $\varepsilon(\tau\mathbb{1}) \equiv \frac{1}{2}\mathbb{1}$, $\varepsilon(\tau\mathbb{1} - d) \equiv \frac{1}{2}\mathbb{1} - d$ and solve the following LPP :

$$\begin{aligned} Y = & \min y \\ & \text{subject to:} \\ & (M_0 + C^+ \cdot d)y - C \cdot \mathbf{x} \quad (LPP) \\ & -C^+ \cdot d \geq \mathbf{0} \\ & y \geq 0, \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{16}$$

Step 2°. If there exists no feasible solution to LPP, then stop (i.e. M_0 is unable to produce a periodic schedule).

Step 3°. Let *iterations := 0* { iterations counter}.

Step 4°. Let $\Omega = \{x_t : \forall t \in T\}$ be the set of firing epochs provided by $\mathbf{x}$. Sort the elements of Ω in the increasing order and then remanber the transitions such that $x_{t_1} \leq x_{t_2} \leq \cdots \leq x_{t_m}$ holds.

Step 5°. **for** $i = 1, 2, \ldots, m$ compute the error vector:

$$\pi := y, \tau_1 := \mathbf{x}, \varepsilon(\tau_1^{t_i}), \varepsilon_d(\tau_1^{t_i})$$

Step 6°. **for** $i = 1, 2, \ldots, m$:

6.1° $Y^i = \min y^i$ $\quad (LPP^i)$
subject to:
$$\{M_0 - C^+ \cdot \varepsilon_d(\tau_1^i) + C^- \cdot \varepsilon(\tau_1^i)\}y^i - C \cdot \mathbf{x}^i - C^+ \cdot d \geq \mathbf{0}$$
$$\mathbf{x}^i \geq \mathbf{0}$$

6.2° If there exists no feasible solution to LPP^i,then stop.

Step 7°. Let $Y^* := max\{Y^i\}$; *iterations := iterations + 1*

Step 8°. **if** $(Y^* - Y^i)/Y^* > \delta$ for some i **and** *iterations* $\leq$ *max_nb_of_iterations*

then $y := Y^*; \mathbf{x} := \sum_i \mathbf{x}^i/m$; **go to** step 4°.

Step 9°. **if** $(Y^* - Y^i)/Y^* \leq \delta$ for all i , **then** Check the firing feasibility Eq (11) of $\{\pi, \tau_1\}$ with $\pi := Y^*$, $\tau_1 := \sum \mathbf{x}^i/m$.

10 Conclusions

Periodic schedules have been considered for Timed Petri Nets systems, $\langle \mathcal{N}, d, M_0 \rangle$. Conceptual classifications have been presented. The time specification leads to *single* or *infinite*-server semantics. The existence of *warm-up* sequences leads to *quasi-periodic* schedules. Finally, depending on how strong the firing periodicity is, *strict-periodic* and *periodic* schedules have been identified.

We have been concerned with single-server semantics periodic schedules. A *transformation rule* of linear complexity allows to transform periodic schedule problems into strict-periodic schedule problems with $I = \mathbb{1}$.

We hope that, even if we leave open many technical questions, this paper will provide the reader stimulus and insights for research in the vast and mostly unexplored area of Petri net scheduling.

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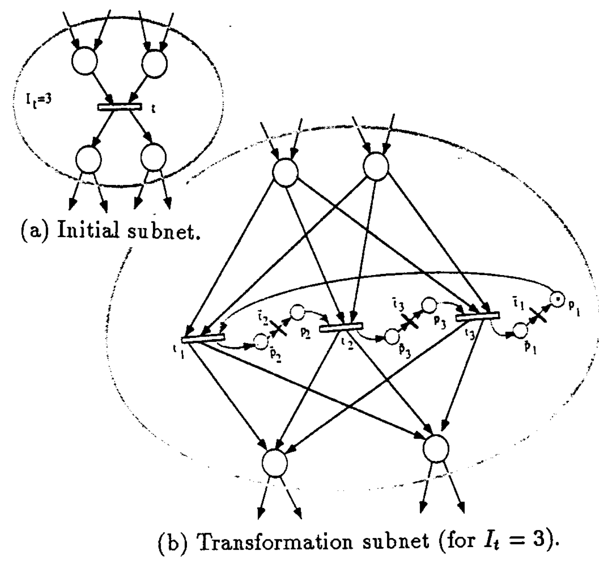

Figure 1: Net system transformation that allows to compute a periodic schedule by using strict-periodic scheduling techniques for $I_t = 3$.