Dealing with arbitrary time distributions with the Stochastic Timed Petri Net model – Application to queueing systems

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Abstract

The ability of the Stochastic Timed Petri Net model for dealing with a great variety of firing time distributions is presented.

The distributions can be: continuous (exponential or uniform); discrete (including the particular case of a deterministic distribution with a zero firing time (immediate transition) or a non zero firing time); mixed.

This ability is based on a method of tractable computation whatever the distribution (in particular the difficult cases of the discrete and mixed distributions), for obtaining a randomized state graph (which represents the dynamic behaviour of the system being modelled).

Applications to queueing systems are considered: the queue M/G/1; the queue M/G/1/K. A general method for analysing queueing systems, which is based on an interpretation of the randomized state graph, is presented.

1 Introduction

A correct analysis of the time dependency of the system behaviour requires an accurate representation of the time parameter of the different mechanisms.

The time parameter can be a purely continuous (everywhere differentiable), a discrete or a mixed random variable.

Furthermore, about the past duration or age, either we must consider that it influences the future duration (non-forgetfulness property) or we can consider that it does not influence the future duration (forgetfulness or memoryless property).

The Timed Petri Net models [1] are models for dealing with the time dependency. However, many models dedicated to a particular time distribution or a particular combination of distributions exist, but they do not consider the problem in a global way.

The first works [2] [3] have considered firing times with exponential distribution (definition of the Stochastic Petri Net (SPN) model).

Then other models have still been introduced. The GSPN model [4] considered two types of transitions (transitions with exponential distributions and transitions firing in zero time (immediate transitions)) and the DSPN model [1] equally considered two types of transitions (always transitions with exponential distributions and transitions with deterministic firing times, these last ones being still used with some restrictions). Furthermore, in these two models, the analysis is much more complex than by only considering exponential distributions (necessity to make a discrimination between states and/or transitions).

The ESPN model [5] makes still the discrimination between the immediate transitions and the other transitions.

The GTPN model [6] only considers discrete distributions and a specific way of analysis.

In this context, we consider that it is important to try to define a general method which allows:
• to deal with firing times with arbitrary distributions (continuous, discrete, mixed; non-forgetfulness or forgetfulness property),
• and to make an analysis transparent to the distribution types.

It is the goal of this paper to contribute to this definition. This is made by considering the Stochastic Timed Petri Net model which is a tool that we are implementing in " C langage " on a SUN 4 station. The Stochastic Timed Petri Net model (STPN) is based on the Extended Time Petri Net model presented in the previous workshops [7] [8] (the main ideas are examined thoroughly and general formulas are given).

This paper includes three sections:
• Section 2 recalls the necessary background on the STPN model and gives the time distributions which are considered,
• Section 3 presents the main idea to have tractable computations whatever the distribution characteristics.
• Section 4 finally presents a model application to the queues M/G/1 and M/G/1/K analysis.

2 Stochastic Timed Petri Net (STPN) model

2.1 Background [7] [8]

1. An STPN is a triplet < PN, I0, FO > where
• PN is the underlying Petri Net model,
• I0 defines the initial firing interval functions for the transitions. A firing interval $[\theta_{mi}, \theta_{Mi}]$ is associated with every transition t_i.

We have: $0 \leq \theta_{mi} \leq \theta_{Mi} \leq \infty$; θ_{mi} is the Earliest Firing Time (EFT) and θ_{Mi} is the Latest Firing Time (LFT). θ_{mi} and θ_{Mi} are relative to the instant where the transition is enabled (origin time for the transition). The firing is instantaneous.

• F0 defines the initial firing probability density functions for the transitions. A probability density function $f_i(x)$ is associated with every transition t_i on its firing interval (we still say that $f_i(x)$ is the intrinsic probability density function of t_i). We have: $\int_{\theta_{mi}}^{\theta_{Mi}} f_i(x)dx = 1$.

2. The STPN behaviour is described by a randomized state graph : states and transitions between states.
• A state is a triplet $< M, I, F >$ where M is the marking, I is the set of the time intervals of the transitions enabled by the marking M and F is the set of the intrinsic probability density functions associated to these intervals.
• A transition between two states is a triplet $< t_i, p_i, \theta_i >$ where t_i is the name of the transition inducing the state change, p_i is the branching probability (i.e the probability of firing the transition t_i, at the latest, at the smallest last firing time (sLFT)) and θ_i is the average firing date (or still the conditional sojourn time).

In the case of an exclusive transition t_i, p_i and θ_i only depend on $f_i(x)$. On the other hand, if the transition t_i is simultaneously enabled with other transitions, as we consider the race policy, p_i and θ_i too depend on the probability density function of these other transitions (concept of the extrinsic density probability function $f_ie(x)$).

After the firing of an exclusive or a competitive transition, the I and F values are the initial specification values (I0 and F0). After the firing of a concurrent transition, the values of I and F, for the transitions still enabled, must be computed by considering the memory of these transitions.

3. The randomized state graph provides the following informations:
• With the branching probabilities, we get the transition matrix [P] (which represents the embedded chain); from [P], we get the equilibrium state probality vector.
• With the combination of the conditional sojourn times and the branching probabilities, we get the unconditional sojourn times.
• From the equilibrum state probabilities and the unconditional sojourn time, we get the steady state probabilities.

Note that the concept of randomized state graph at (1-x)% has also been introduced [7].

4. <u>Remark</u> the computation of $f_ie(x)$ and of the new I and F values (after the firing of a concurrent transition) are critical points which must be carefully analysed (see section 3).

2.2 The initial intrinsic probability density functions

These functions are presented by considering that we have two types of time intervals associated to a transition t_i ($[\theta_{mi}, \theta_{Mi}]$)
• type 1: $\theta_{mi} \geq 0$; θ_{Mi} has a superior limit,
• type 2: $\theta_{mi} = 0$; $\theta_{Mi} = \infty$

In the type 1 case, we consider that $f_i(x)$ can have two components: a continuous component ($f_ic(x)$) and a discrete component ($f_id(x)$).

The continuous component $f_ic(x)$ is uniform on the time interval ($f_ic(x) = C_i$ for $x \in [\theta_{mi}, \theta_{Mi}]$).

The discrete component $f_id(x)$ has the general form $\sum_n K_{in}\delta(x - \theta_n)$ where K_{in} represents the weight of the Dirac impulse occuring at the instant θ_n ($\theta_n \in [\theta_{mi}, \theta_{Mi}]$).

With these components, we define the following functions $f_i(x)$:

1. only uniform ($f_i(x) = f_ic(x)$); we represent a distribution with a mean value and a variance,
2. only discrete ($f_i(x) = f_id(x)$). We can cover several practical cases
   - We only have a Dirac impulse (deterministic distribution): if the occurrence time is zero, we represent what it is called an immediate transition; if the occurrence time is different from zero, we have a constant delay,
   - We have several impulses: we represent a distribution with a mean value and a variance
3. mixed (uniform and discrete); we can represent very sophisticated distributions.

In the type 2 case, we consider the (continuous) exponential distribution:
$f_i(x) = f_ic(x) = \lambda_i e^{-\lambda_i x}$. By this way, we want to be able to meet again classical results.

3 Method for computing the extrinsic probability density functions and the memory of the concurrent transitions

3.1 Basic idea

• The decomposition of the firing interval of the transitions which are simultaneously enabled into a set of sub-intervals is **the key idea of our method.** This decomposition is really necessary for dealing with the discrete component of the distributions (with the continuous component it is not necessary). These sub-intervals define the integration intervals.

Two types of integration intervals are defined:
- type " A " i.e intervals of non–zero duration,
- type " B " i.e intervals of zero duration.

This decomposition is illustrated through the example shown on the figure 1, where we consider four transitions t_i, t_j, t_k and t_l which are simultaneously enabled.

The three transitions t_i, t_j, t_k have mixed distributions:
- the continuous components $f_ic(x), f_jc(x)$ and $f_kc(x)$ are uniform components,
- the discrete components are:

$$(\sum_n K_{in}\delta(x - \theta_n), \quad \sum_n K_{jn}\delta(x - \theta_n),$$

$$\sum_n K_{kn}\delta(x-\theta_n)).$$

The coefficients K_{in}, K_{jn} and K_{kn} are the weights associated to the Dirac impulses, occuring at the instant θ_n.

The transition t_l has an exponential distribution $\lambda_l e^{-\lambda_l x}$.

• This decomposition of the firing interval has six stages:
1. Definition of the smallest last firing time ($sLFT$). Here $sLFT = \theta_{Mi}$. $sLFT$ defines the upper integration bound.
2. Definition of the Earlest Firing Time (EFT) of each transition. Here we have $\theta_{mi}, \theta_{mj}, \theta_{mk}$ and θ_{ml}. They define for each transition the lower integration bound.
3. Definition of the Dirac impulse occurrence instants. Here we have θ_1, θ_2 and θ_3. Note that, at θ_3, the transitions t_i and t_k have a Dirac impulse.
4. Sequential view of all these instants: here we have $\theta_{ml}, \theta_{mi}, \theta_{mj}, \theta_1, \theta_2, \theta_{mk}, \theta_3, \theta_{Mi}$.
5. Renaming each Dirac impulse occurrence instant with the pair $\{instant^-, instant^+\}$. For example, the instant θ_1 is renamed $\{\theta_1^-, \theta_1^+\}$.
6. Definition of the sequence of the integration intervals (closed intervals). This definition results from the stages 4 and 5. Here we have:
$[\theta_{ml}, \theta_{mi}]$, $[\theta_{mi}, \theta_{mj}]$, $[\theta_{mj}, \theta_1^-]$, $[\theta_1^-, \theta_1^+]$, $[\theta_1^+, \theta_2^-]$, $[\theta_2^-, \theta_2^+]$, $[\theta_2^+, \theta_{mk}]$, $[\theta_{mk}, \theta_3^-]$, $[\theta_3^-, \theta_3^+]$, $[\theta_3^+, \theta_{Mi}]$.

In this set of intervals, two sub–sets must be distinguished: the type " A " sub-set (intervals with a non zero duration or still the index of the lower bound and the upper bound are different) and the type "B " sub-set (intervals with a zero duration or still the indices of the lower bound and the upper bound are the same).

On the type "A " intervals, we only have continuous distributions; on the type "B " intervals, we only have discrete distributions.

<u>**Remark**</u>: Note that a firing interval of the form $[\theta_i, \theta_i]$ (for example if the transitions have all a deterministic distributions with the value θ_i) becomes an integration interval of the form $[\theta_i^-, \theta_i^+]$.

3.2 The formulas of the extrinsic probability density functions

The extrinsic probability density function $f_ie(x)$ of a transition t_i is the product of the intrinsic probability density function $f_i(x)$ and a coefficient which expresses "The probability that the other enabled transitions would be fired after x (if competitive transitions) or will be fired after x (if concurrent transitions)".

We now express $f_ie(x)$ on the two types of integration intervals.

3.2.1 Type " A " integration intervals:

These integration intervals have the form $[\theta_v, \theta_w]$ or $[\theta_v, \theta_w^-]$ or $[\theta_v^+, \theta_w]$ or $[\theta_v^+, \theta_w^-]$: $f_ie(x)$ is only continuous and is noted $f_iec(x)$.

Let us call L the set of the transition simultaneously enabled with the transition t_i.

• The general expression of $f_iec(x)$ is :

$$f_iec(x) = f_ic(x) \prod^{l\in L} \{[\sum K_l(\succ \theta_w \ or \ \theta_w^-)] + [\int_x^{\theta_{Ml}} f_lc(y)dy]\}$$

where $\sum K_l(\succ \theta_w$ or $\theta_w^-) + \int_x^{\theta_{Ml}} f_lc(y)dy$ represent the probability of firing transition t_l after x.

• We put now into practice the general expression with the different continuous probability density functions that we consider in the set L:

- only uniform (i.e $f_lc(x) = C_l$ for $\theta_{ml} \leq x \leq \theta_{Ml}$). We have:

$$f_iec(x) = f_ic(x) \prod^{l\in L} \{[\sum K_l(\succ \theta_w \ or \ \theta_w^-)] + [C_l(\theta_{Ml} - x)]\}$$

- only exponential (i.e $f_l(x) = f_lc(x) = \lambda_l e^{-\lambda_l x}$); note that in this case we have not Dirac impulses by definition. We have:

$$f_ie(x) = f_iec(x) = f_ic(x) \prod^{l\in L} \{e^{-\lambda_l x}\}$$

- uniform (call L' the set of these transitions) and exponential (call L'' the set of these transitions). We have:

$$f_iec(x) = f_ic(x) \prod^{l\in L'} \{[\sum K_l(\succ \theta_w \ or \ \theta_w^-)] + [C_l(\theta_{Ml} - x)]\} \prod^{l\in L''} \{e^{-\lambda_l x}\}$$

3.2.2 Type " B " integration intervals

These integration intervals have the form $[\theta_w^-, \theta_w^+]$. The extrinsic probability density function $f_ie(x)$ is now discrete and is noted $f_ied(x)$ with:
$f_ied(x) = K_ie\delta(x-\theta_w)$, K_ie being the extrinsic weight which must be computed.

• Two different cases must be considered: case 1, i.e the transition t_i has not a Dirac impulse at the instant θ_w (then $K_i = 0$), and case 2, i.e the transition t_i has a Dirac impulse at the instant θ_w (then $K_i \neq 0$).

In the case 1, we obviously have $K_ie = 0$.

• The case 2 is a case which must be attentively analysed (A first thought was given in [8] but the general case was far to be solved).

Let us suppose, in order to make a full analysis, that the cardinal of the set L of the other transitions simultaneously enabled is "m", that all these transitions have a Dirac impulse at the instant θ_w, and that we have a next integration interval (necessarily of type " A " with a lower bound θ_w^+) where all the transitions of the set L have still a probability density function.

For each transition $t_l \in L$, we must now consider two exclusive events: $event_n$ (n for now) i.e the transition t_l is firable at θ_w; $event_a$ (a for after) i.e the transition t_l is firable after θ_w (i.e in the type " A " integration interval with the lower bound θ_w^+). Then the probability, that 2, 3, ..., $(m+1)$ transitions are firable at the instant θ_w, exists but as, by definition of the STPN model, we only fire one transition (here t_i) at an instant, we must make to appear this choice (we consider an equiprobable choice). Then the different combinations of the transitions in the set L must be expressed.

The general expression of $K_i e$ is:

$$
\begin{aligned}
K_i e \; &= \; K_i B & (1)\\
&+ \; K_i\{\frac{1}{2}\sum_{j\in C_m^1} K_j\} B_{j-} & (2)\\
&+ \; K_i\{\frac{1}{3}\sum_{j,k\in C_m^2} K_j K_k\} B_{j-k-} & (3)\\
&+ \; ...\\
&+ \; ...\\
&+ \; K_i\{\frac{1}{m}\sum_{j,k,..,q\in C_m^{m-1}} K_j K_k..K_q\} B_{j-k-..q} & (4)\\
&+ \; K_i\{\frac{1}{m+1} K_j K_k..K_q K_r\} & (5)
\end{aligned}
$$

with:

$$B = \prod^{l\in L} \{[\sum K_l(\succ \theta_w \;)] + [\int_{\theta_w}^{\theta_{Ml}} f_l c(y)dy]\}$$

$$B_{j-} = \prod^{l\in L;\; l\neq j} \{[\sum K_l(\succ \theta_w \;)] + [\int_{\theta_w}^{\theta_{Ml}} f_l c(y)dy]\}$$

$$B_{j-k-} = \prod^{l\in L;\; l\neq j, l\neq k} \{[\sum K_l(\succ \theta_w \;)] + [\int_{\theta_w}^{\theta_{Ml}} f_l c(y)dy]\}$$

The different terms (1), (2), (3), ..., (5) give the probability of firing the transition t_i at the instant θ_w in the following exclusive conditions:

- (1) $event_a$ for all the transitions in the set L,
- (2) $event_n$ for all the combinations of one transition in the set L (the factor $\frac{1}{2}$ represents the equiprobable choice) and $event_a$ for all the other transitions of the set L,
- (3) $event_n$ for all the combinations of two transitions in the set L (the factor $\frac{1}{3}$ represents the equiprobable choice) and $event_a$ for all the other transitions of the set L,
- ...,
- (5) $event_n$ for all the transitions of the set L (the factor $\frac{1}{m+1}$ represents the equiprobable choice).

Now let us consider the simplification of this general formula in the cases where a transition t_l has not a Dirac impulse ($K_l = 0$) at the instant θ_w and/or has not a probability density function defined after θ_w (that means that θ_w is the $sLTF$ which is fixed by the transition t_l).

1. the transition t_l has not a Dirac impulse ($K_l = 0$) at the instant θ_w: the terms including combinations with K_l disappear (in particular we have no more the term (5)).
2. the transition t_l has not a probability density function defined after θ_w: the terms including the factor $\{[\sum K_l(\succ \theta_w \;)] + [\int_{\theta_w}^{\theta_{Ml}} f_l c(y)dy]\}$ relative to this transition, disappear (in particular the term (1) does not more exist).
3. if we have the cases 1 and 2, all the terms of the general formula disappear and then we still have $K_{ie} = 0$.

This formula has been implemented in an automatic way [11].

3.3 The memory of the concurrent transitions

The formulas : the memory appears at two levels : firing interval; probability density function.

Let us suppose that a transition t_i, concurrent with a transition t_j, has fired at the instant θ_i. The new firing interval (I'_j) and the new intrinsic probability density function ($f'_j(x)$) for the transition t_j are:

$$I'_j \;=\; [max(0,\; EFT_j - \theta_i),\; LFT_j - \theta_i]$$

$$f'_j(x) \;=\; \frac{f_j(x+\theta_i)}{1 - F_j(\theta_i^-)}$$

where $F_j(\theta_i^-)$ is the cumulated probability of $f_j(x)$ on the interval $[0, \theta_i^-]$ or still $[0, \theta_i[$. **This view of the open interval $[0, \theta_i[$ is essential for considering discrete distributions** (with only continuous distributions, the fact that the interval is $[0, \theta_i[$ or $[0, \theta_i]$ has no importance).

Examples : the memory aspect is illustrated on figure 2 by considering a transition t_j with a mixed distribution (uniform C_j and a Dirac impulse K_j at θ_j). We represent two cases: the case 1 where θ_i is after θ_j and the case 2 where θ_i is identical to θ_j. It is important to see in the case 2 that we have still the Dirac impulse in the new distribution $f'_j(x)$.

A particular but very interesting example by its consequences is the following: Let us consider two concurrent transitions t_i and t_j, each one with a deterministic firing time at the same moment θ_w:

$$I_i \;=\; [\theta_w,\, \theta_w] \quad f_i(x) \;=\; K_i\, \delta(x - \theta_w) \text{ with } K_i \;=\; 1$$

$I_j = [\theta_w, \theta_w] \quad f_j(x) = K_j \, \delta(x - \theta_w)$ with $K_j = 1$

Suppose that (after the equiprobable choice) the transition t_i is fired. The new firing interval and the new intrinsic probability density function for the transition t_j are:

$$I'_j = [max(0, \theta_w - \theta_w), \theta_w - \theta_w] = [0, 0]$$

$$f'_j(x) = \frac{K_j \, \delta(x + \theta_w - \theta_w)}{1 - F_j(\theta_w^-)} = K_j \, \delta(x)$$

Then the transition t_j is firable in zero time.

An extension of this example is the case where more than two transitions have a deterministic firing time with the same value θ_w. We can easily see that after the firing of one transition, the others are sequentially firable in zero time (we call this aspect " the procedure memory "). Note that the model automatically works out the problem of the immediate transitions ($\theta_w = 0$): the priority on the non immediate transitions and the interleaving (in zero time) of concurrent immediate transitions.

3.4 Power of the method: Ability to consider any combination of distributions

Let us consider the Petri Net of the figure 3 and the combinations of probability density functions of table 1.

	combination 1	*combination 2*
t1	*deterministic* $\delta(x - 1)$	
t2	*deterministic (immediate)* $\delta(x)$	
t3	*deterministic (immediate)* $\delta(x)$	*exponential* $\lambda 3 = 1$
t4	*mixed* • *deterministic* $\delta(x - 1)$ • *uniform 0.25 for* $0 \le x \le 2$	*exponential* $\lambda 4 = 1$
t5	*deterministic* $\delta(x - 2)$	
t6	*deterministic* $\delta(x - 2)$	*exponential* $\lambda 6 = 0.5$

table 1

The randomized state graphs relative to the combination 1 and the combination 2 are represented respectively on the figure 4 and the figure 5 (the transitions between the states are labelled with: the transition of the Petri Net; the branching probability; the firing time).

The graph of the figure 4 shows, in particular:

• the priority of the immediate transitions t_2 and t_3 (futhermore, the non immediate transition t_4 in competition with t_3 cannot be fired).

• the interleaving of the concurrent immediate transitions: t_2 ($p_2 = 0.5$ resulting from the equiprobable choice) and t_3 ($p_3 = 1$) or t_3 ($p_3 = 0.5$) and t_2 ($p_2 = 1$).

The graph of the figure 5 shows, in particular:

• the priority of the immediate transition t_2 over the concurrent non immediate transitions (t_3 or t_4),

• the natural consideration of the exponential distributions on the competitive transitions t_3 and t_4: $p_3 = \frac{\lambda_3}{\lambda_3 + \lambda_4} = 0.5$ and $p_4 = \frac{\lambda_4}{\lambda_3 + \lambda_4} = 0.5$.

4 Queueing systems analysis : M/G/1; M/G/1/K

The queue M/G/1 has a randomized state graph which is (obviously) unbounded (and then we cannot analyse it). However, by using the concept of graph at (1-x) % and by choosing a weak value for x, we get a graph, which informs us enough on the main characteristics of the behaviour, and from which interesting results can be obtained. We consider here a graph at 99.999 %. We show how this graph can be used to make an analysis comparable with the classical analysis [9].

The aim of the study of the queue M/G/1/K (here the randomized state graph is bounded) is to show a general method which is based on the interpretation of the information contained in the randomized state graph and which then can be used with any kind of queue.

4.1 The queue M/G/1

The Petri Net model is represented on the figure 6: the transition t_1 represents the arrival, from a place P_1, of a customer (exponential distribution: $\lambda = 0.5$); the place P_2 represents the server (it implements a FIFO policy) and the transition t_2 represents the service (general distribution).

4.1.1 Considering a deterministic service: $\delta(x - 1)$

The randomized state graph at 99.999 %

This graph, which is on the figure 7 (the big arrows represent the stopping transitions), visualizes the process of the arrivals (transition t_1) and the departures (transition t_2). The branching probabilities are only represented.

Obtaining informations on the embedded Markov chain (customer departure instants)

1. Background [9]

The matrix of the transition probabilities of the imbedded Markov chain $[\mathrm{M}] = [p_{ij}], (i, j = 0, 1, 2, ...)$ takes the form indicated on the next page (α_k is the probability of arrival of k customers during the service of the customer C_{n+1}).

$$[M]= \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \ldots \\ \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \ldots \\ 0 & \alpha_0 & \alpha_1 & \alpha_2 \ldots \\ 0 & 0 & \alpha_0 & \alpha_1 \ldots \\ 0 & 0 & 0 & \alpha_0 \ldots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

2. Extracting the coefficients α_k from the randomized state graph

• We look at the states resulting from a departure (input transitions labelled with t_2): S_0, S_1, S_4, S_7, S_{10}, S_{13}.

The largest number of coefficients α_k is got from states S_0 and S_1 (the two first lines of the matrix [M]. We obviously get less coefficients from the other states due to the graph stopping. But what it is important is to get enough coefficients from S_0 and S_1 .

• By considering the state S_0, we got the 6 coefficients below:

$$\begin{aligned}
\alpha_0 &= p[S_0 \rightarrow S_1 \rightarrow S_0] = 0.607 \\
\alpha_1 &= p[S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow S_1] = 0.300 \\
\alpha_2 &= p[S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow S_4] = 0.081 \\
\alpha_3 &= p[S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow S_{18} \\
&\rightarrow S_7] = 0.0114 \\
\alpha_4 &= p[S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow S_{18} \rightarrow S_{19} \\
&\rightarrow S_{10}] = 0.00082 \\
\alpha_5 &= p[S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow S_{18} \rightarrow S_{19} \\
&\rightarrow S_{20} \rightarrow S_{13}] = 0.000029
\end{aligned}$$

3. Evaluating the mean queue length $E[\widehat{q}]$

We remind the following equation [9] about the M/G/1 queue:

$$E\,[\widehat{v}] + E\,[\widehat{v}^2] - 2\,E\,[\widehat{q}] + 2\,E\,[\widehat{q}]\,E\,[\widehat{v}] - 2\,E\,[\widehat{v}]^2 \;=\; 0$$

where $\widehat{v}$ and $\widehat{q}$ are respectively the random variable representing the number of customers arriving during a service and the random variable representing the queue length. E is the expectation.

By considering the coefficients α_k (got from the graph), we define an approximate z - transform: $V(z) = \sum_{k=0}^{5} \alpha_k\, z^k$ for the variable $\widehat{v}$.

We use the properties of the derivatives of the z – transform [9]:

$$[\frac{dV(z)}{dz}]_{z=1} \;=\; E\,[\widehat{v}] \;\#\; 0.500$$

$$[\frac{d^2V(z)}{dz^2}]_{z=1} \;=\; E\,[\widehat{v}^2] \;-\; E\,[\widehat{v}] \;\#\; 0.240$$

• Finally we obtain: $E\,[\widehat{q}]\ \#\ 0.740$

4.1.2 Comparing with the classical analysis: the Pollaczek–Khinchin (P–K) mean value formula

The Pollaczek–Khinchin formula is:

$$E\,[\widehat{q}] \;=\; \rho \;+\; \rho^2 \frac{(1+C_b^2)}{2\,(1-\rho)}$$

ρ is the utilization factor, C_b is the coefficient of variation of the service ($C_b = \frac{\sigma}{\overline{x}}, \sigma$ being the square root of the variance and $\overline{x}$ being the mean value).

We have considered three service distributions with increasing values for the squared coefficient of variation C_b^2 . The results are presented on the table 2. They show the interest of our method (however if the variance increases we will need a randomized state graph with a better precision). But we have to note that the classical analysis is based on the computations of service time Laplace transform derivatives (evaluated at the argument $s = 0$) and then cannot be applied in all the cases (take the example of an uniform distribution). We have not these problems.

Service Time Distribution	Mean value : 1		
	$\delta(x-1)$	$0.25\,\delta(x) + 0.5\,\delta(x-1) + 0.25\,\delta(x-2)$	$0.5\,\delta(x) + 0.5\,\delta(x-2)$
	$C_b^2 = 0$	$C_b^2 = 0.5$	$C_b^2 = 1$
Pollaczeck - Khinchin formula	0.750	0.875	1
Randomized state graph at 99.999 %	0.740	0.870	0.970

table 2 Mean queue lenght $E\,[\hat{q}]$

4.2 The queue M/G/1/k

The Petri Net model is represented on the figure 8. The place P_3 represents the finite storage (the system can hold at most a total of K customers, here $K = 4$); the transition t_1 is exponentially distributed ($\lambda = 0.5$) and the transition t_2 represents a deterministic service ($\delta(x-1)$).

The randomized state graph (10 states) is represented on the figure 9. The transitions are labelled with: the transitions of the Petri Net; the branching probability; the firing time. The interpretation of these informations allows us to evaluate the main characteristics of the queue.

We indicate now how to get these characteristics. Let us call P_i the steady state probability of the state S_i.

1. Distribution of the customer number (N) in the system : $f(N)$

Let us call $P(N)$ the probability that the number of customers in the system is N. Here we have at the most: $N = 4$.

We see on the randomized state graph:

$$P(N \;=\; 0) = P_0; P(N = 1) = P_1;$$

$$P(N = 2) = P_2 + P_4;$$
$$P(N = 3) = P_3 + P_5 + P_7;$$
$$P(N = 4) = P_6 + P_8 + P_9;$$

Then :

$$\begin{aligned} f(N) &= P(N=0)\ \delta(N) + P(N=1)\ \delta(N-1) \\ &+ P(N=2)\ \delta(N-2) + P(N=3)\ \delta(N-3) \\ &+ P(N=4)\ \delta(N-4) \end{aligned}$$

Numerical application

$$\begin{aligned} f(N) &= 0.498\ \delta(N) + 0.323\ \delta(N-1) \\ &+ 0.136\ \delta(N-2) + 0.035\ \delta(N-3) \\ &+ 0.0047\ \delta(N-4) \end{aligned}$$

2. Distribution of the total time spent (t_s) by a customer in the system: $f(t_s)$

For this analysis we have, at first, to determine on the graph the states which can result from an arrival (input transition labelled with t_1) and, second, to follow the path (only composed of transitions labelled with t_2) which goes to the state S_0 (empty system) and to compute the path duration (adding the conditional sojourn times). These states are S_1 (one customer in the system), S_2 (two customers in the system), S_3 and S_5 (three customers in the system), S_6, S_8 and S_9 (four customers in the system).

From S_1, there is only one transition t_2 to the state S_0 (the conditional sojourn associated to this transition is the service time of duration 1).

For the path from the other states (S_2; S_3 and S_5; S_6, S_8 and S_9), the first transition t_2 represents the residual service time and each other transition t_2 represents a service time.

The path, from S_2 to S_0, is composed of the transition t_2 from S_2 to S_1 (residual service duration: 0.541) and of the transition t_2 from S_1 to S_0 (service time duration: 1)

The case of S_3 and S_5 is interesting to analyse: these states represent the arrival of a customer in a system which has already two customers but with different times to finish the service (the time memory is different and then we have two states). The path from S_3 to S_0 is composed of the transition t_2 from S_3 to S_4 (residual service time duration: 0.282), of the transition t_2 from S_4 to S_1 (service time duration: 1) and of the transition t_2 from S_1 to S_0 (service time duration: 1). We get, in the same way, the duration of the path from S_5 to S_0.

Note also the case of the states S_6, S_8 and S_9 which represents the arrival of a customer in a system which has already three customers but with different times to finish the service.

Let us call t_{si} the time spent in the system by a customer arriving in the state S_i.

Finally we have:

$$\begin{aligned} f(t_s) &= P_1\delta(t_s - t_{s1}) + P_2\delta(t_s - t_{s2}) \\ &+ P_3\delta(t_s - t_{s3}) + P_5\delta(t_s - t_{s5}) \\ &+ P_6\delta(t_s - t_{s6}) + P_8\delta(t_s - t_{s8}) \\ &+ P_9\delta(t_s - t_{s9}) \end{aligned}$$

Numerical application

$$\begin{aligned} f(t_s) &= 0.323\delta(t_s - 1) + 0.087\delta(t_s - 1.541) \\ &+ 0.010\delta(t_s - 2.282) \\ &+ 0.012\delta(t_s - 2.541) \\ &+ 0.001\delta(t_s - 3.282) \\ &+ 0.003\delta(t_s - 3.541) \\ &+ 0.0007\delta(t_s - 3.144) \end{aligned}$$

3. The mean duration of the busy period: D_{bp}

We use here a method of computation, often used in the reliability and safety analysis (MTFF and MTFMF computation [10]).

The duration of the busy period is the time which the system can spent after entering the state S_1 and before returning to the state S_0.

From the transition matrix [P] of the randomized state graph, if we consider the state S_0 as an absorbing state, we get the matrix [Q] (by removing the first line and the first columm) and also the fundamental matrix $[1 - Q]^{-1}$ [12]. Let us call $[\eta]$ the vector of the unconditional sojourns in the states S_i with $i \in \{0...9\}$

We have: $D_{bp} = [1 \quad \underbrace{0....00}_{\text{eight zeros}} \quad][1 - Q]^{-1}[\eta]$

Numerical application D_{bp} #2

5 Conclusion

The Stochastic Timed Petri Net (STPN) model has the following characteristics:

- it deals with a great variety of firing distributions: continuous (exponential, uniform); discrete (covering then the particular case of a deterministic distribution); mixed distributions,
- it provides a computation method for handling, in particular, the difficult problem of the discrete distributions,
- it allows the modeling and the analysis of systems with the forgetfulness property or the non–forgetfulness property,
- the qualitative and quantitative analysis of the behaviour of a system described by a STPN model is made with the object "randomized state graph", through an interpretation of the states, the transitions and the labels associated to the transitions.

Queuing systems (queues M/G/1 and M/G/1/K) have been modeled with the STPN model. In particular a general method has been presented which must allow to analyse any kind of queues.

We are applying now the model to other queuing systems: M/Er; M/H; G/G; queue networks.

Others applications have been equally made in the area of the real time systems where the ability, in particular, of dealing with deterministic distributions (zero time and time different of zero) is really a great asset.

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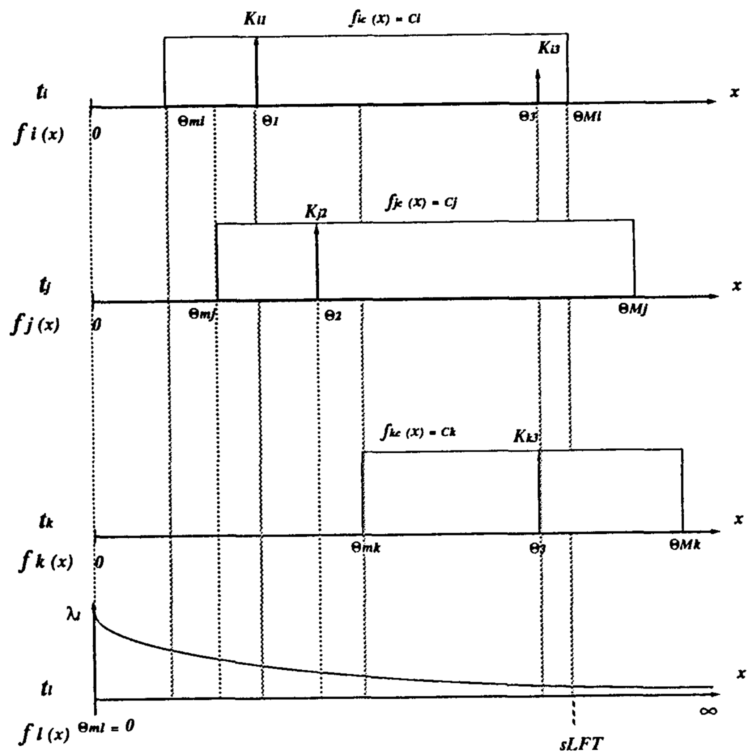
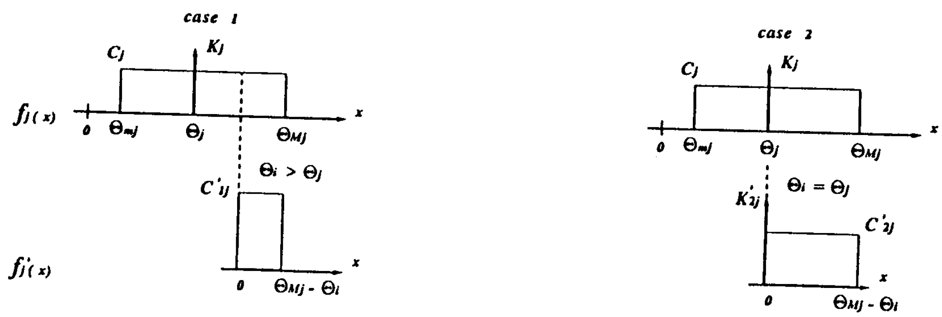


figure 1 Probability Density Functions



$$\bullet\ C'_{1j} = \frac{C_j}{1 - F_j(\Theta_i^-)}$$

with $F_j(\Theta_i^-) = K_j + C_j\,(\Theta_i - \Theta_{mj})$

$$\bullet\ C'_{2j} = \frac{C_j}{1 - F_j(\Theta_i^-)}$$

$$\bullet\ K'_{2j} = \frac{K_j}{1 - F_j(\Theta_i^-)}$$

with $F_j(\Theta_i^-) = C_j\,(\Theta_i - \Theta_{mj})$

figure 2 Dynamic evolution of the intrinsic time distribution

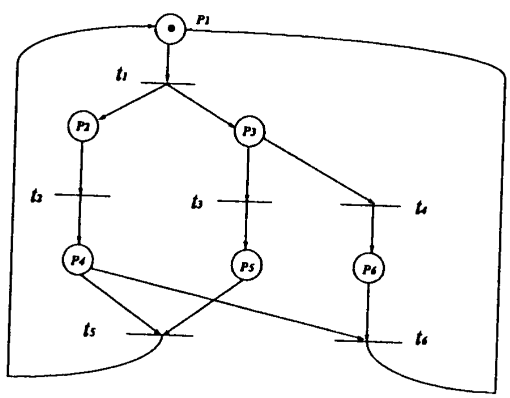


figure 3 Petri Net model

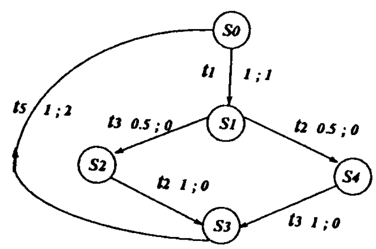


figure 4 Randomized state graph 1

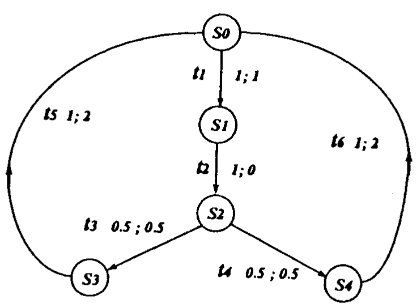


figure 5 Randomized state graph 2

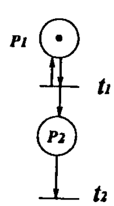


figure 6 Petri net model of the queue M/G/1

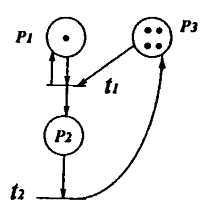


figure 8 Petri net model of the queue M/G/1/4

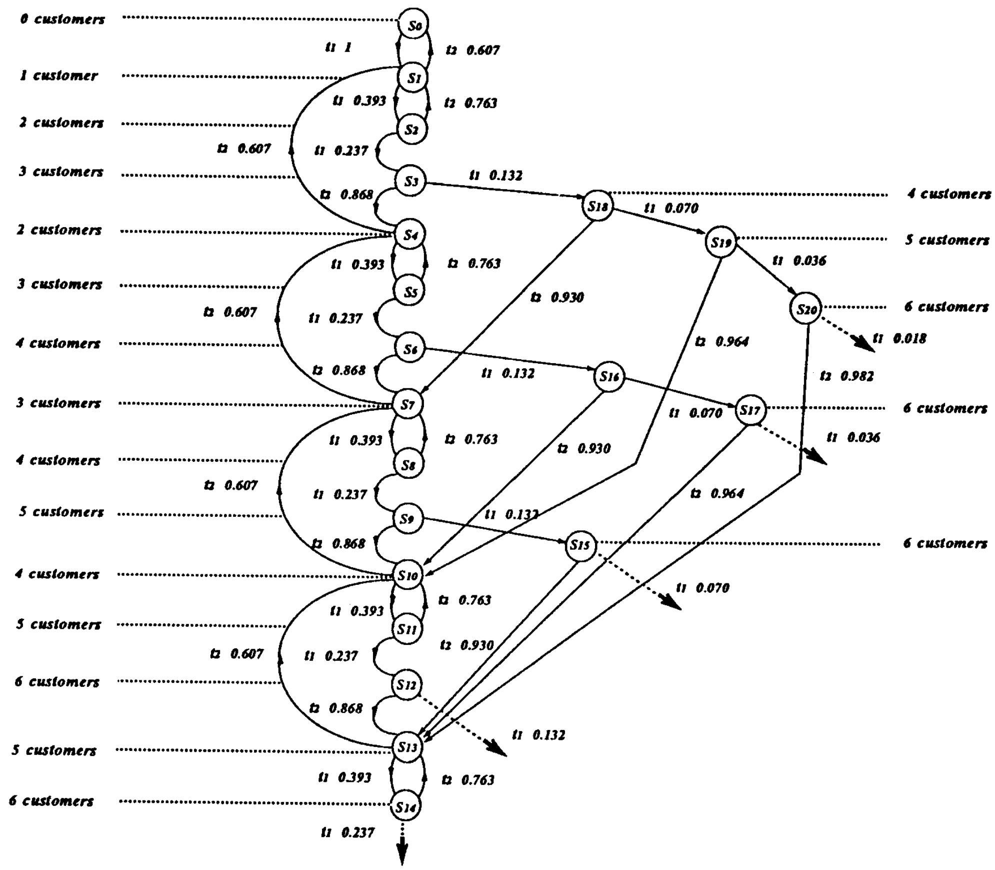


figure 7 Randomized state graph at 99.999 % (M/G/1)

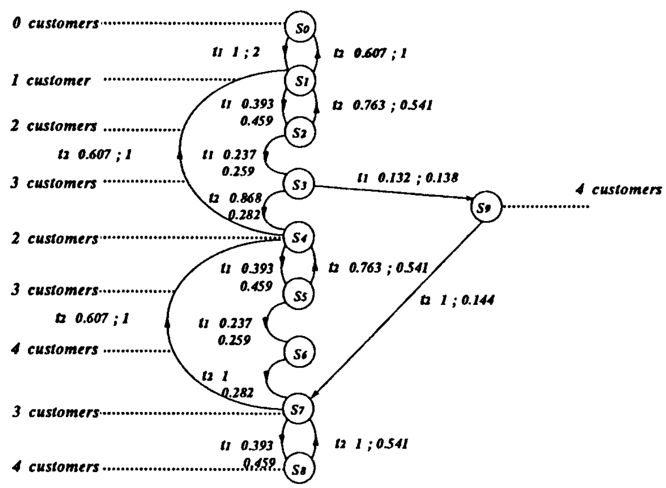


figure 9 Randomized state graph (M/G/1/4)