Superposed Stochastic Automata: a class of Stochastic Petri nets amenable to parallel solution

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Abstract

In [14] Plateau has introduced an efficient way for solving stochastic processes that are derived from the composition of stochastic automata by making extensive use of the Kronecker (tensor) algebra for matrices. In this paper we apply that efficient solution to a class of Stochastic Petri nets (SPN) that has been called Superposed Stochastic Automata (SSA). The solution has been implemented both with sequential and parallel programs. SSA are a rather restricted subclass of SPN, but the extension to the general case doesn't appear to pose any theoretical problem.

1 Introduction

It is well known that Stochastic Petri nets (SPNs) offer a powerful model for the modeling and performance evaluation of systems involving concurrency and non determinism (choices), the major application areas being that of parallel and distributed systems and of communication networks. The major drawback of using SPNs is the large state space that can be generated even by rather simple models.

There have been many attempts to simplify SPN solution, in particular trying to identify subclasses of Petri nets that have a solution of the "product form" type [10,9,8]. All these approaches try to solve the stochastic process associated with an SPN without generating the complete Markov chain. Till now these attempts have provided rather interesting insights on SPNs, but the actual applicability of these results to real problems is rather limited.

In [14,13] Plateau presents a solution method for network of Markovian stochastic automata that has two main advantages. The first is that there is no need to store the infinitesimal generator of the complete Markov chain, as it is expressed as Kronecker sum and product (see [3]) of the infinitesimal generators of the single automata and of a certain number of "correcting matrices"; the solution process exploits the particular form of the Kronecker expression to avoid the computation of the infinitesimal generator of the global process. The second one is that the solution process is easily amenable to parallel implementation. The method has been implemented in a tool called PEPS [15] that takes the infinitesimal generator of the single automata and the correcting matrices given by the user, it performs a certain number of checks to control that the dependencies expressed by the network of automata are correct and computes (for the time being with a sequential algorithm) the steady state solution.

In this paper we bring the work described in [14] in the SPN environment. In particular we present a subclass of SPN that we call Superposed Stochastic Automata (SSA) from which we can automatically derive an expression of the type utilized by PEPS. By using SSA we lose some flexibility with respect to networks of stochastic automata, but we gain the advantages of being in a well known framework for which we have well developed tools and a clear semantics. Being in a clear, although rather restricted, framework as that defined by SSA we are able to automatically generate for an SSA a Kronecker expression of the type used by PEPS.

An SSA net is basically a set of stochastic state machines (automata) that interact through synchronization of the rendez-vous type (the classical synchronization in Petri nets). Given an SSA described in graphical form using the GreatSPN package [2], we automatically determine the Kronecker expression for the infinitesimal generator of the SSA and produce all matrices involved in the expression, so that the solution method is completely transparent to the user.

We have used SSA as a testing class of SPN to study the applicability of the efficient solution method based on Kronecker algebra in the field of stochastic Petri nets. The extension to the more general class of Superposed Stochastic Petri nets, where SPNs (in-
stead of state machines) interacts through transition synchronization, doesn’t appear to bring in any major theoretical problem.

We know of only another attempt to exploit Kroncker algebra to solve complex stochastic Petri nets. In [1], Buchholz applies it to the study of hierarchical markovian models described through queueing networks and Petri nets, but the type of interactions considered is not of the synchronization type as submodels interacts through the exchange of tokens (asynchronous interaction).

This paper is organized as follows. Section 2 introduces the basic definitions of SSA and in Section 3 we derive the Kroncker expression of SSA systems. Section 4 applies the techniques presented in [3,14] to show how to efficiently compute the steady state solution of the Markov process associated to an SSA net without having to explicitly compute (and store) its infinitesimal generator. We also discuss the complexity of a sequential implementation, while the major line of a parallel implementation are presented in Section 5 where we also present a few comparative results of sequential vs. parallel implementation. The extension to Superposed Stochastic Petri nets is discussed in Section 6 that concludes the paper.

2 Superposed Stochastic Automata

We introduce here the definition of “Superposed Stochastic Automata,” (SSA) a subclass of stochastic Petri nets (SPN) the class of timed nets defined by Mollusc in [12,11]. The name for the subclass comes from the fact that SSA are the timed, stochastic counterpart of a class of Petri nets introduced by DeMicheli et al. in [4] and that is called “Superposed Automata” nets (SA). The distinctive feature of SSA nets is that they can be considered as a set of stochastic automata combined through a synchronization operator of the rendez-vous type. In Petri net terminology a SSA net is a set of SPN nets of the state machine type composed by superposition of transitions.

We first give a formal definition of a stochastic machine

**Definition 1** A stochastic state machine SSM is defined as the 4-tuple SSM = (P, T, F, A) where

- P is the non empty set of places
- T is the non empty set of transitions
- F ⊆ P × T ∪ T × P with dom(F) ∪ codom(F) = P ∪ T is the flow relation. It has to satisfy the following restriction: ∀t ∈ T: |t| = |t'| ≥ 1
- A : T → R⁺, where A(t) is the rate of the exponential probability distribution associated to transition t

With respect to SPN nets, SSA have the additional restriction of not allowing any type of synchronization (all transitions are of the 1-in 1-out type), although they allow non determinism.

We now define Superposed stochastic automata net.

**Definition 2** We call Superposed Stochastic Automata net the tuple N = (P, T, F, P, A) where

- P is the non empty set of places
- T is the non empty set of transitions
- F ⊆ P × T ∪ T × P with dom(F) ∪ codom(F) = P ∪ T is the flow relation. It has to satisfy the following restriction: ∀t ∈ T: |t| = |t'| ≥ 1
- P = {P_i} is a partition of P such that ∀t ∈ T and ∀i ∈ 1,...,m : |Pi| ∩ |t| = |Pi| ∩ |t'| ≤ 1
- A : T → R⁺, where A(t) is the rate of the exponential probability distribution associated to transition t

The condition on the partition P identifies a set of stochastic state machines that compose the SSA net. SSA are a subclass of SPN nets in which only a certain type of synchronization is allowed, in particular each transition must have the same number of input and output arcs.

SSA transitions that have |t| = |t'| > 1 are called synchronization (or synchronized) transitions, as they represent synchronization between subnets.

Another way to interpret SSA nets is to say that they are obtained from a set of SSMs by synchronizing on transitions of equal rate.

We can now define SSM and SSA systems which can be obtained from SSA and SSA nets by assigning an appropriate initial marking.

**Definition 3** We call stochastic state machine system the tuple R = (P, T, F, P, A, p) where

- (P, T, F, P, A) is a SSM net.
- p ∈ P is the initial marking of the SSM

The dynamics of SSMs is the same as that of SPNs. It is interesting to note that, due to the particular choice of the initial marking and to the requirement of the transitions to be 1-in 1-out, exactly one place is marked in any reachable marking of an SSM, so that we can specify the state of an SSM simply by the name of the marked place.

**Definition 4** We call Superposed Stochastic Automata system the tuple R = (P, T, F, P, A, M_0) where

- N = (P, T, F, P, A) is a SSA net
- M_0 ⊆ P and ∀i ∈ 1,...,m : |Pi| ∩ M_0 = 1

The requirement on the initial marking is such that there is exactly one place marked in each element of the partition in any reachable marking. Therefore an SSA system can be considered as the superposition on transitions of equal rate of a set of SSA systems.
Figure 1 provides a pictorial representation of an SSA system where the subset of places is partitioned into four subsets. $T_1$ is the synchronization transition of SSA systems identified by $P_0, P_1, P_2$, and transition $T_6$ is the synchronization between $P_0$ and $P_1$.

Figure 1: An SSA system

3 Infinitesimal generator of SSA systems

In this section we explain how to write the infinitesimal generator of an SSA system as a Kronecker expression of the infinitesimal generators of the component automata.

It is well known in the literature [3] that there exists a strong relation between the matrix operator known as Kronecker sum and the composition of independent, continuous stochastic processes. In this section we exploit this property to write the infinitesimal generator of an SSA as the Kronecker expression of the infinitesimal generators of some independent processes plus some adequate matrices that play the role of "correction matrices.

3.1 Kronecker sum and product and the composition of stochastic processes

As far as this work is concerned we only consider matrix product and sum of matrices for square matrices (infinitesimal generator are square matrices by definition). Let $M(p)$ be the set of all $p \times p$ matrices of real values. We can then give the following definitions.

Definition 5 Let $A \in M(n)$ and $B \in M(p)$; $C$ is the Kronecker (tensor) product of $A$ and $B$ and we write $C = A \otimes B$ if $C \in M(np)$ is defined by:

$$C = \{ c_{ij} : c_{ij} = a_{1i} b_{1j} \text{ with } i = (i_1, i_2), j = (j_1, j_2) \}$$

As a simple example consider the tensor product of two $2 \times 2$ matrices. We have

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$C = A \otimes B = \begin{pmatrix} a_{11} b_{11} & a_{11} b_{12} & a_{12} b_{11} & a_{12} b_{12} \\ a_{11} b_{21} & a_{11} b_{22} & a_{12} b_{21} & a_{12} b_{22} \\ a_{21} b_{11} & a_{21} b_{12} & a_{22} b_{11} & a_{22} b_{12} \\ a_{21} b_{21} & a_{21} b_{22} & a_{22} b_{21} & a_{22} b_{22} \end{pmatrix}$$

If we interpret $A$ and $B$ as the state transition matrix of two discrete time Markov chains, it is immediate to recognize (see Davio in [3]) that $C$ is the transition probabilities matrix of the process obtained as independent composition of the two original processes.

Let us now define the Kronecker (or tensor) sum of two matrices

Definition 6 Let $A \in M(n)$ and $B \in M(p)$; $D$ is the Kronecker (tensor) sum of $A$ and $B$ and we write $D = A \oplus B$ if $D \in M(np)$ is defined by:

$$D = A \oplus B = A \oplus I_d + I_d \oplus B$$

where $I_d$ is the $k \times k$ identity matrix.

Let's consider again the two matrices $A$ and $B$ of the example above: we can express their Kronecker (tensor) sum as follows:

$$A \oplus I_d_2 = \begin{pmatrix} a_{11} & 0 & a_{12} & 0 \\ 0 & a_{11} & 0 & a_{12} \\ a_{21} & 0 & a_{22} & 0 \\ 0 & a_{21} & 0 & a_{22} \end{pmatrix}$$

$$I_d_3 \oplus B = \begin{pmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & b_{21} & b_{22} \end{pmatrix}$$

$$D = \begin{pmatrix} a_{11} + b_{11} & b_{12} & a_{12} & 0 \\ b_{21} & a_{11} + b_{22} & a_{12} & 0 \\ a_{21} & 0 & a_{22} + b_{11} & b_{12} \\ 0 & a_{21} & b_{21} & a_{22} + b_{22} \end{pmatrix}$$

Again if we consider $A$ and $B$ as the infinitesimal generator of two, time continuous markovian processes, then $D$ is the infinitesimal generator of the process obtained by independent composition of the two original one. It can be observed that in $D$ all transition
rates among states that differ by more than a single component are set equal to zero, as it is the case when we are in a continuous-time environment.

From an applicative point of view we cannot take a big advantage from what has just been said, because usually the stochastic processes involved in an application are far from being independent.

Another aspect that deserves attention is that even if the infinitesimal generator of \( D \) is rewritten using only the infinitesimal generators of the components SSMs, nevertheless \( D = A \oplus B \) has a number of elements equal to \(|A| \times |B|\) and the same is true also for \( C = A \otimes B \). The interesting point is that the solution of \( e^C = 0 \), the characteristic equation of the Markov process defined by \( C \), does not require to compute \( C \), but the solution can be done in terms of \( A \) and \( B \) (see [3]). This is not particularly surprising since the solution of a stochastic process which is obtained as independent composition of two other processes can always be computed from the solution in isolation of the two processes.

### 3.2 Infinitesimal generator of an SSA system

What happens if we want to express the infinitesimal generator of a set of non-independent stochastic processes in terms of the infinitesimal generator of the component processes? In [14] Plateau suggests (and applies) the idea of separating in each process the independent behaviour from the dependent one and to express the infinitesimal generator of the composed process as the sum of two parts: one is the composition of the independent behaviours of the component automata, and the other one takes into account all dependencies.

Let \( \Sigma \) be a SSA system whose associated partition \( \Pi \) individuates \( m \) SSM systems. Let \( Q_i \) be the infinitesimal generator of \( SSM_i \) and \( Q \) that of \( \Sigma \). Our goal is to study the relationships among the single \( Q_i \) and \( Q \). We shall therefore try to interpret the stochastic process associated to \( \Sigma \) as the composition, of course not independent, of the stochastic processes associated to the single \( SSM_i \).

Before moving on to a more formal treatment of the problem, let us introduce a very simple example. Figure 2 depicts two simple SSMs, \( SSM_1 \) and \( SSM_2 \) that have a transition of equal rate \( \mu \). The SSA system \( \Sigma \) in Figure 3 is obtained from \( SSM_1 \) and \( SSM_2 \) by superposition on transition \( t_2 \). The infinitesimal generators associated to \( SSM_1 \) and \( SSM_2 \) can be expressed as:

\[
Q_1 = \begin{bmatrix} -\mu & \mu \\ \nu & -\nu \end{bmatrix} \quad Q_2 = \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix}
\]

While the matrix \( Q \) of the SSA system is given by

\[
Q = \begin{bmatrix} 1,3 & 1,4 & 2,3 & 2,4 \\ 1,3 & -\mu & 0 & 0 & 0 \\ 1,4 & \lambda & -\lambda & 0 & 0 \\ 2,3 & \nu & 0 & -\nu & 0 \\ 2,4 & 0 & \nu & \lambda & -\lambda -\nu \end{bmatrix}
\]

If we delete from \( Q_1 \) and \( Q_2 \) the elements that concern the non-independent behaviour of the two SSMs, that is to say, if we delete from \( Q_1 \) and \( Q_2 \) the elements that represent the firing of the synchronized transition, we obtain the two following matrices \( Q'_1 \) and \( Q'_2 \):

\[
Q'_1 = \begin{bmatrix} 0 & 0 \\ \nu & -\nu \end{bmatrix} \quad Q'_2 = \begin{bmatrix} 0 & 0 \\ \lambda & \lambda \end{bmatrix}
\]

The composition of the independent behaviour of \( SSM_1 \) and \( SSM_2 \) leads to

\[
Q' = Q'_1 \oplus Q'_2 = \begin{bmatrix} 1,3 & 1,4 & 2,3 & 2,4 \\ 1,3 & 0 & 0 & 0 & 0 \\ 1,4 & \lambda & -\lambda & 0 & 0 \\ 2,3 & \nu & 0 & -\nu & 0 \\ 2,4 & 0 & \nu & \lambda & -\lambda -\nu \end{bmatrix}
\]

The resulting matrix \( Q' \) differs from \( Q \) only in the first row, because the first row corresponds to the state from which we can fire the synchronized transition.
We can rewrite $Q$ as

$$Q = Q' + K$$

where $K$ is an appropriate “correcting factor” (correcting matrices). If we want to express also the correcting factor as Kronecker expression of matrices which are related to the single components, then we can remark that

$$K = \mu \cdot \bar{K} = \begin{bmatrix} 1,3 & 1,4 & 2,3 & 2,4 \\ 1,3 & -\mu & 0 & 0 & \mu \\ 1,4 & 0 & 0 & 0 & 0 \\ 2,3 & 0 & 0 & 0 & 0 \\ 2,4 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\bar{K}$ can be rewritten as $-K_1 \otimes K_2 + K_3 \otimes K_4$.

Please note the use of the Kronecker product, an operator which is usually associated to discrete processes. This is because the firing of a synchronized transition cause a change of state in all the SSMs involved in the synchronization. This type of situation is what was called "simultaneous jump" in [15]. Finally, we can express $K_1, K_2, K_3, K_4$ as

$$K_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad K_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The product $K_1 \otimes K_2$ is the correcting factor for the diagonal elements, while $K_3 \otimes K_4$ inserts the effects of the synchronized transition.

Synchronization of two automata on a single transition. Let $SSM_1$ and $SSM_2$ be two SSMs with respectively $n$ and $m$ states and infinitesimal generators $Q_1$ (an $n$ x $n$ matrix) and $Q_2$ (a $m$ x $m$ matrix). Let $Q$ (a $n$ x $m$ matrix) be the infinitesimal generator of the SSA obtained from $SSM_1$ and $SSM_2$ by superposition on the transition $t$ of rate $\mu$.

Let us call $i \in j_1$ ($i_2 \in j_2$) the two states of $SSM_1$ ($SSM_2$) such that $i[j > j_1$ ($i_2[j > j_2$), where we use the usual notation $i[j > j$ to say that $t$ fires in $i$ and its firing produces the new state $j$. We can build from $Q_1$ the matrix $Q_1'$ where all state transitions out of state $i_1$ due to $t$ are inhibited (please note that this would require a modification also in the diagonal elements). We can write

$$Q_1' = Q_1 - \mu E_{i_1,j_1}$$

where $E_{i_1,j_1}$ is the $n$ x $n$ matrix defined by

$$E_{i_1,j_1}(i,k) = \begin{cases} 1 & \text{if } i = i_1 \text{ and } k = j_1 \\ -1 & \text{if } i = k = i_1 \\ 0 & \text{otherwise} \end{cases}$$

We can build also $Q_2$ in a similar way. $Q_1' \otimes Q_2'$ is the independent part of the expression. We now have to determine the correcting factor that allows the firing of $t$ when $SSM_1$ is in state $i_1$ and $SSM_2$ is in state $i_2$.

We have therefore to add to $Q_1' \otimes Q_2'$ a $(n \times m) \times (n \times m)$ matrix where the only non null values are

$$-\mu((i_1, i_2), (i_1, i_2)) = \mu((i_1, i_2), (j_1, j_2)) = \mu$$

Hence $Q = Q_1 \otimes Q_2 - \mu(K_1 \otimes K_2) + \mu(K_3 \otimes K_4)$, where $K_1$ is the $n$ x $n$ matrix where the only non null element is that of index $(i_1, i_2)$, $K_3$ is the same but for the states $(i_1, j_1)$, while $K_2$ and $K_4$ are $m$ x $m$ matrices where the only non null elements are $(i_2, i_2)$ and $(i_2, j_2)$.

Synchronization of $N$ automata on a single transition. In this case there are $N$ SSMs, $SSM_i, i = 0, \ldots, N-1$ of associated matrices $Q_i$. Let us indicate with $(h_k, j_k)$ the two states of $SSM_i$ that have $h_k[t > j_k]$, when $SSM_i$ is considered in isolation. If $m_i$ is the set of places (and therefore of states) of $SSM_i$, then $Q_i$ is a $m_i \times m_i$ matrix...

From $Q_i$ we can build $Q_i' = Q_i - \mu E_{h_i,j_i}$, and $Q = \Pi_{i=0}^{N-1} Q_i'$ which differ from $Q$ only in the firing of transition $t$, which is enabled in state $(h_0, h_2, \ldots, h_{N-1})$ with $(h_0, h_2, \ldots, h_{N-1})[t > (j_0, j_2, \ldots, j_{N-1}) - 1]$. This addition requires an adjustment in the diagonal element $((h_0, h_2, \ldots, h_{N-1}), (h_0, h_2, \ldots, h_{N-1}))$. Finally

$$Q = \Pi_{i=0}^{N-1} Q_i' - \mu \Pi_{i=0}^{N-1} K_i + \mu \Pi_{i=0}^{N-1} K_i'$$

where $K_i$ is the matrix identically equal to zero, but for the element $K_i(h_i, j_i) = 1$, while $K_i'$ is the matrix whose only non zero element is $K_i'(h_i, j_i) = 1$. Note that $E_{h_i,j_i} = K_i' - K_i$.

It is now clear that, for a given synchronization transition, it is necessary to add two correcting factors, and each correcting factor is the Kronecker product of $N$ matrices.

Synchronization of $N$ automata on two transitions. We now consider the case of $N$ automata $SSM_i$ that are synchronized on two transitions $t_1$ and $t_2$ of associated rates $\mu_1$ and $\mu_2$.

We indicate with $h_k[t_1 > h'_k[t_2 > j'_k]$ the state transition caused by $t_1$ in $SSM_i$, if any, and $j_k[t_2 > j'_k$ is the state transition caused by $t_2$ in $SSM_i$, if any.

We can group the $N$ automata as follows:

$G_1$: automata that synchronize only on $t_1$

$G_2$: automata that synchronize only on $t_2$

$G_3$: automata that synchronize on both $t_1$ or $t_2$
The three groups have empty intersections.

Following the usual approach, we build \( Q' \) from \( Q \) as follows, depending on what is the group of \( SSM_i \):

\[
\begin{align*}
SSM_i \in G_1 & \Rightarrow Q'_i = Q_i - \mu_1 E_{i},K_i' \\
SSM_i \in G_2 & \Rightarrow Q'_i = Q_i - \mu_2 E_{i},K_i' \\
SSM_i \in G_3 & \Rightarrow Q'_i = Q_i - \mu_1 E_{i},K_i' - \mu_2 E_{i},K_i' \\
\end{align*}
\]

We can rewrite the above expression as \( Q'_i = Q_i - \mu_1 E((\mathcal{t}_1 \cap \mathcal{t}_2), \mathcal{t}_1 \cap \mathcal{SM}_i) - \mu_2 E((\mathcal{t}_1 \cap \mathcal{t}_2), \mathcal{t}_2 \cap \mathcal{SM}_i) \), where \((\mathcal{t}_1, \mathcal{t}_2) = (\mathcal{t}_1 \cap \mathcal{G}_1, \mathcal{t}_2 \cap \mathcal{G}_2)\). The correcting factors are of the type:

\[
Q = \prod_{i=1}^{n} Q_i'
\]

where

\[
K_i(t_1) \text{ is a matrix of all zeros, but for element } (\mathcal{t}_1 \cap \mathcal{SM}_i, \mathcal{t}_1 \cap \mathcal{SM}_i) \text{ which is equal to one. If } \mathcal{t}_1 \cap \mathcal{SM}_i = \emptyset \text{ then } K_i \text{ is the identity matrix.}
\]

\[
K_i(t_2) \text{ is the same as above, but with } (\mathcal{t}_2 \cap \mathcal{SM}_i, \mathcal{t}_2 \cap \mathcal{SM}_i).
\]

\[
K'_i(t_1) \text{ is a matrix of all zeros, but for element } (\mathcal{t}_1 \cap \mathcal{SM}_i, \mathcal{t}_1 \cap \mathcal{SM}_i) \text{ which is equal to one. If } \mathcal{t}_1 \cap \mathcal{SM}_i = \emptyset \text{ then } K'_i \text{ is the identity matrix.}
\]

\[
K'_i(t_2) \text{ is the same as above, but with } (\mathcal{t}_2 \cap \mathcal{SM}_i, \mathcal{t}_2 \cap \mathcal{SM}_i).
\]

Identities matrices have been introduced to take into account those automata that do not have the synchronization.

Synchronization of \( N \) automata on an arbitrary number of transitions. We are now ready to deal with the most general case of \( N \) SSMS systems that are synchronized on an arbitrary set of transitions. Let \( TS \subset T \) be the set of synchronization transitions. A generic transition \( t \) is a synchronization transition iff \(|t| > 1\), i.e. \( TS = \{ t \in T : |t| > 1 \} \) (we remind the reader that, by definition of SSA, \(|t| = |t^*|\)). We say that an automata participates to the \( t \) synchronization iff \(|t \cap \mathcal{SM}_i| = 1\).

We have therefore the following formula that, given an SSA systems, computes its infinitesimal generator \( Q \) as Kronecker expressions of matrices of the \( m \times m \) type, where \( m_i \) is the number of states of \( SSM_i \) (that also coincides with the number of places of the SSM).

\[
Q = \prod_{i=1}^{n} Q'_i - \sum_{t \in TS} \mu_i \prod_{t^*} K_i(t) + \sum_{t \in TS} \mu_i \prod_{t^*} K'_i(t)
\]

where \( Q'_i = Q_i - \sum_{t \in TS} \mu_i E_i(t, t^*). \) The correcting factors \( K \) and \( K' \) are defined by

\[
K_i(t) = \begin{cases} 
\text{identity} & \text{if } |t \cap \mathcal{SM}_i| = 0 \\
\text{all zeros but } (t \cap \mathcal{SM}_i, t \cap \mathcal{SM}_i) & \text{if } |t \cap \mathcal{SM}_i| = 1 
\end{cases}
\]

\[
K'_i(t) = \begin{cases} 
\text{identity} & \text{if } |t \cap \mathcal{SM}_i| = 0 \\
\text{all zeros but } (t \cap \mathcal{SM}_i, t \cap \mathcal{SM}_i) & \text{if } |t \cap \mathcal{SM}_i| = 1 
\end{cases}
\]

Where we have used \( S_i \) for \( SSM_i \) for typesetting reasons. We have finally obtained an expression for the infinitesimal generator of an SSA system in terms of Kronecker sum and product of simple matrices. In particular if the SSA system under study is composed of \( N \) automata and \( TS \) synchronization transitions, then the number of matrices involved in the expression is \( N + 2N|TS| \).

It is important to observe that the expression of \( Q \) is only a more compact way of writing the infinitesimal generator of an SSA system, but actually the number of elements of \( Q \) is still \( Nm \times Nm \).

We have mentioned in the previous section that the solution of an equation of the type \( \pi Q = 0 \) where \( C \) is the Kronecker product of a certain number of smaller matrices, does not require to compute \( C \). This doesn't hold true any longer, at least it is not so immediate. In the next section we shall deal with the problem of finding a solution of \( \pi Q = 0 \) that does not require the computation and the storage of \( Q \).

4 Solution of equation \( \pi Q = 0 \)

In the previous section we have derived an expression for the infinitesimal generator of an SSA system. If the SSA system is composed of \( N \) SSMS, and if each SSM has \( m \) states, then the expression derived involves matrices of the \( (m \times m) \) type, and many of them are very sparse. If we compute the expression of \( Q \), then the resulting matrix is of size \( (Nm \times Nm) \).

In order to take advantage of the particular structure of the expression it is necessary to find an appropriate solution method that does not require to develop the Kronecker products and sums.

The solution of the characteristic equation of the Markov process \( \pi Q = 0 \) is typically done numerically, and there are two basic techniques: direct method and iterative method. If we want to work on the expression without computing \( Q \), then we cannot apply direct methods, that typically requires to work with the complete infinitesimal generator. We have hence to use an iterative method, like, for example, the power methods, that is the one that is actually used in our implementation. At each iteration we compute the \( n \)-th approximation of the probability vector \( \pi \) as:

\[
\pi^n = \pi^{n-1} + \frac{1}{\omega(1+\epsilon)} \pi^{n-1} Q
\]
where $\omega$ is the maximum of the diagonal elements and $\varepsilon$ is an arbitrarily small constant.

In order for the power method to converge we need to have a single unitary eigenvalue and a sufficient condition for a stochastic matrix to have a unique unitary eigenvalue is that the matrix is irreducible. This condition corresponds to the requirement that the Markov process be ergodic. But this is not the case for the matrix expressed by Equation 1, even if the process associated to the SSA is actually ergodic.

Observe in fact that we consider as possible states of an SSA system all possible combinations of the states of the single SSMs, but not all states are possible. For example consider the SSA system of Figure 1. All states that are obtained as combinations of the states of the single automata are actually reachable from the initial marking of the system. If we now modify the SSA in Figure 1 so as to superpose transition $T7$ with transition $T8$, then all states where place $P6$ is marked and $P16$ is not are not reachable any longer from the initial marking.

The key observation that allows to use the power method even in this case is that, starting from an initial value of $\pi_0$ in which we assign probability 1 to the state that corresponds to the initial marking, and probability 0 to all other states, then by computing at each iteration the formula $\pi^n = \pi^{n-1} + \frac{1}{\omega(1+\varepsilon)} \pi^{n-1} Q$, we never put to non zero the probability of states which are not reachable from the initial marking. This is because the matrix $Q$ contains only the rate transitions among reachable states.

We also have to note that the choice of the initial value of $\pi$ influence the speed of convergence of the method in a negative sense (the mass probability is all accumulated in a single point and it may require a great number of iterations to distribute it among all reachable states). Better methods are actually under study to provide a more convenient initialization for reachable states. Better methods are actually under study to provide a more convenient initialization for reachable states. Better methods are actually under study to provide a more convenient initialization for reachable states. Better methods are actually under study to provide a more convenient initialization for reachable states. Better methods are actually under study to provide a more convenient initialization for reachable states.

We now show how the expression for $Q$ can be rewritten in order to allow the computation of the iterative equation also without explicitly computing $Q$. This part is mainly a rewriting in our context of the results obtained by Davio in [3] which have been applied to the composition of non independent stochastic processes by Plateau in [14] and [15].

The iterative equation with $Q$ substituted by its expression has the form:

$$\pi^k = \pi^{k-1} + \frac{1}{\omega(1+\varepsilon)} \pi^{k-1}(\Pi_{i=0}^{N-1} Q_i(t) + \sum_{i \in \mathcal{T}} \mu_i \Pi_{x_{i0}}^{N-1} K_i(t))$$

(2)

The Kronecker sum of Equation (2) can be rewritten as ordinary matrix sum of Kronecker products. As we shall see next, Kronecker products are easier to deal with. We obtain:

$$\Pi_{i=0}^{N-1} Q_i(t) = Q_0 \otimes I_d + Q_1 \otimes I_d + \ldots \otimes I_d_{N-1} + I_d \otimes Q_1 \otimes Q_2 \otimes I_d \otimes \ldots \otimes I_d_{N-1} + I_d \otimes \ldots \otimes I_d_{N-2} \otimes Q_{N-1}$$

That is to say a sum with $N$ elements. Each element is the Kronecker product of $N$ matrices and $N-1$ of these matrices are identity matrices.

Equation (2) can therefore be rewritten as

$$\pi^k = \pi^{k-1} + \frac{1}{\omega(1+\varepsilon)} \pi^{k-1}(\sum_{i=0}^{N-1} I_d_{p_i} \otimes \ldots \otimes I_d_{p_{i-1}}) + \sum_{i \in \mathcal{T}} \mu_i \Pi_{x_{i0}}^{N-1} K_i(t) + \sum_{i \in \mathcal{T}} \mu_i \Pi_{x_{i0}}^{N-1} K_i(t))$$

(3)

In [3] it is shown that it is possible to rewrite the Kronecker product of $N$ matrices $M_i$ each of the type $(p_i, q_i)$, $0 \leq i \leq N$ as follows:

$$\otimes_{i=0}^{N-1} M_i = \Pi_{i=0}^{N-1} I_d_{p_{i-1}} \otimes \ldots \otimes I_d_{p_{i-1}} \otimes M_i \otimes I_d_{p_{i+1}} \otimes \ldots \otimes I_d_{p_{N-1}}$$

$$= \Pi_{i=0}^{N-1} S_{p_i, q_i}(I_d_{q_i} \otimes M_i)$$

where $(q_i = \Pi_{j=0}^{N-1} I_d_{p_j})$) and $S_{p_i, q_i}$ is the matrix that realizes a perfect shuffle permutation of the $(p_i, q_i)$ type. Equation (3) can hence be transformed into

$$\pi^k = \pi^{k-1} + \frac{1}{\omega(1+\varepsilon)} \pi^{k-1}(\sum_{i=0}^{N-1} I_d_{p_i} \otimes \ldots \otimes I_d_{p_{i-1}}) + \sum_{i \in \mathcal{T}} \mu_i \Pi_{x_{i0}}^{N-1} S_{p_i, q_i}(I_d_{q_i} \otimes K_i(t))$$

(4)

where

$$A_{ij} = \begin{cases} Q_i & \text{if } i = j \\ I_{d_{p_j}} & \text{otherwise} \end{cases}$$

By distributing $\pi^{k-1}$ over the terms of the sum we obtain

$$\pi^k = \pi^{k-1} + \frac{1}{\omega(1+\varepsilon)}(\sum_{i=0}^{N-1} \pi^{k-1} I_d_{q_i} \otimes A_{ij}) - \sum_{i \in \mathcal{T}} \mu_i (\pi^{k-1} \Pi_{x_{i0}}^{N-1} S_{p_i, q_i}(I_d_{q_i} \otimes K_i(t))) + \sum_{i \in \mathcal{T}} \mu_i (\pi^{k-1} \Pi_{x_{i0}}^{N-1} S_{p_i, q_i}(I_d_{q_i} \otimes K_i(t)))$$

(5)

By taking advantages of the following equalities:

$$\Pi_{i=0}^{N-1} S_{p_i, q_i}(I_d_{q_i} \otimes A_{i}) = \Pi_{i=0}^{N-1} S_{p_{i+1}, q_{i+1} \ldots p_{N-1}, q_{N-1}}(I_d_{q_i} \otimes A_i) S_{p_{i+1}, \ldots p_{N-1}, q_{N-1}}(I_d_{q_i} \otimes A_i)$$

60
and

$$S_{P_{i+1} \cdots P_{N-1} P_{0} \cdots P_{i}} = S_{P_{0} \cdots P_{i} P_{i+1} \cdots P_{N-1}}$$

where $S^T$ indicates the inverse of the permutation realized by $S$. We finally get to

$$\pi^k = \pi^{k-1} + \frac{1}{\omega(1+\epsilon)} \sum_{j=0}^{N-1} (\pi^{k-1})$$

\[\Pi_{t=0}^{N-1} S_{P_{0} \cdots P_{i} P_{i+1} \cdots P_{N-1}} (Id_{t} \otimes A_{ij}) S_{P_{0} \cdots P_{i} P_{i+1} \cdots P_{N-1}} + \sum_{t \in T_5} \mu((\pi^{k-1} \Pi_{t=0}^{N-1} S_{P_{0} \cdots P_{i} P_{i+1} \cdots P_{N-1}} (Id_{t} \otimes K_{t})) S_{P_{0} \cdots P_{i} P_{i+1} \cdots P_{N-1}} + \sum_{t \in T_6} \mu((\pi^{k-1} \Pi_{t=0}^{N-1} S_{P_{0} \cdots P_{i} P_{i+1} \cdots P_{N-1}} (Id_{t} \otimes K_{t})) S_{P_{0} \cdots P_{i} P_{i+1} \cdots P_{N-1}})$$

(7)

Where the matrix multiplication by $S^T$ is at no cost if indices are appropriately stored. The most costly operation is the computation of an expression like that in the first term of the sum:

$$\pi^{k-1} \Pi_{t=0}^{N-1} S_{P_{0} \cdots P_{i} P_{i+1} \cdots P_{N-1}} (Id_{t} \otimes A_{ij}) S_{P_{0} \cdots P_{i} P_{i+1} \cdots P_{N-1}}$$

To efficiently compute all these products we can take advantage of the left associative law of the ordinary matrix multiplication, and we can therefore compute the following sequence of expressions:

$$\pi^0 = (((\pi^{k-1} S_{P_{0} P_{1} \cdots P_{N-1}} (Id_{t} \otimes A_{ij})) S_{P_{0} \cdots P_{i} P_{i+1} \cdots P_{N-1}})$$

$$\pi^1 = (((\pi^0 S_{P_{0} P_{1} \cdots P_{N-1}} (Id_{t} \otimes A_{ij})) S_{P_{0} \cdots P_{i} P_{i+1} \cdots P_{N-1}})$$

$$\pi^N_{k-1} = (((\pi^{k-2} S_{P_{0} \cdots P_{i} P_{i+1} \cdots P_{N-1}} (Id_{t} \otimes A_{ij})) S_{P_{0} \cdots P_{i} P_{i+1} \cdots P_{N-1}})$$

and the computation of $\pi^N_{k-1}$ gives the final value for (8).

By taking advantage of the special properties of the Kronecker product we can therefore solve the characteristic equation of an SSA system by simply using a vector of $\Pi_{t=0}^{N-1}$ (number of (potential) states of the SSA), and matrices of size $(p_{i} \times p_{i})$ (0 ≤ i ≤ N) without being forced to explicitly compute and store the infinitesimal generator of the SSA, which has a (potential) size of $(\Pi_{t=0}^{N-1} p_{i} \times N_{t=0}^{N-1} p_{i}).$

We have therefore found a way of computing each iterative step in such a way as to save storage. But what is the computational complexity of the proposed procedure? The complexity of the computation of Equation 7 is of the order

$$N^2 \alpha + 2|TS| N \alpha = O(N^2 \alpha)$$

and the term $\alpha$ expresses the complexity of the inner computation of Equation 7. In fact the product $PS_{t}(Id_{t} \otimes A_{ij}) S^T_{t}$, the cost due to permutations is of the order $\Pi_{t=0}^{N-1} p_{i}$ and to execute the product $P_{s_{t}}(Id_{t} \otimes A_{ij})$, where $P_{s_{t}} = PS_{t}$, it is necessary to execute $q_{i}$ ($q_{i} = \Pi_{t=0}^{N-1} p_{i}$) products of order $p_{i}$.  

5 Parallel solution

The expression 7 lends itself to parallel evaluation. We have developed a parallel solution algorithm that has been implemented on a transputers machine (a MEIKO Computing Surface) under the CS Tool programming environment. 

We can observe that the three summations in Equation 7 are independent: their evaluation is not influenced by the order, moreover they work on different data (matrices). The independence of data is an important parameter that we have to consider, because in a transputers based machine there is no shared memory: if we partition data among processes (and consequently among transputers) we can avoid data duplications.

A second observation is that each addendum of the three summations are independent (for the same above mentioned reasons) each one from the others. It's important to observe that the parallel architecture and the tool which have been used have influenced the choices that we have taken during the algorithm development. In particular every transputer has four links to connect it to the others processing elements and to avoid rerouting (made automatically by the CS Tool Run Time Environment) a good rule is to give to the set of transputers the structure of a ternary tree.

Using CS Tool, only a transputer of the transputers set allocated to the parallel application can directly access the file system. This observation suggest to limit file system accesses to one of the sequential processes that compose the parallel program.

In the parallel program we have three types of processes. Processes of type A compute (at each step of the iterative algorithm) a matrix product of the following form

$$\pi^{k-1} \Pi_{t=0}^{N-1} S_{P_{0} \cdots P_{i} P_{i+1} \cdots P_{N-1}} (Id_{t} \otimes M) S_{P_{0} \cdots P_{i} P_{i+1} \cdots P_{N-1}}$$

where $M$ is a $A_{i,j}$ matrix or a correction matrix. The number of these processes is $2|TS| + N$. Processes of type B act instead as results collectors: they receive partial values of $\pi^{k}$ from first type processes and add them to obtain the value of the relative summation in the k-th step. Finally, there is a single process of a
third type that coordinates the iterative algorithm. At every step, it computes the steady state vector from the partial steady state vectors that it receives from the process of type B; it checks the algorithm convergence to end the parallel program. This process is the only one that access the file system.

At each step, the computation starts when processes receive the steady state vector obtained in the previous step by the "father" process. The steady state vector is computed from the process of the third type.

The parallel algorithm computes each summation term in parallel so that its complexity is of the order of \( O(Na) \) and therefore the expected speed up is \( \approx N \), we therefore expect a speed-up which is proportional to the number of SSMs that compose the SSA system.

Table 1 shows a comparison among the execution time of sequential implementation on a single transputer and its parallel counterpart on \( Tr \) transputers. Each SSA system analyzed is described in terms of the number of places \( |P| \), of transitions \( |T| \), of synchronized transitions \( |TS| \), and of number of states \( |RS| \).

We provide here only results of parallel versus sequential implementation of our solution, while it should also be necessary to compare these results from that coming from the standard SPN packages. Unfortunately, the SPN package that is available here, which is GreatSPN [2] runs on a SUN workstation and therefore the comparison is unfeasible, both because it is difficult to estimate the relative speed of the two machines and because it is difficult to interpret the execution time information available under the SUN operating system. We plan to re-implement the solution part of the GreatSPN package under our transputer machine, so as to be able to test also the speed of the sequential solution proposed with respect to more classical techniques.

6 Conclusions

In this paper we have presented the application of a new solution method introduced by Plateau in [14] that exploits Kronecker algebra as nicely described in [3] to compute the steady state solution of SSA, a subclass of SPN. The use of this rather restricted class was justified by the need of keeping all formulas as simple as possible in order to arrive in a reasonable amount of time to have also a parallel implementation of the solution method.

Nevertheless, although SSA appear to be as rather restrictive on the kind of systems that can be modelled, they have a rather wide range of applicability. In particular the untimed counterpart of SSA, the class of net known as Superposed Automata (see [4]) from which SSA are derived, has been used to model distributed systems, especially in the office organization environment [5].

The extension to Superposed Stochastic Petri Nets (SSPN) doesn't appear to bring any major theoretical problem. SSPN can be defined as the class of nets obtained by a set of SPN systems that interact through transition(s) synchronization. It is therefore a very powerful model, perfectly suitable to describe systems that interact by message exchanges implemented in a rendez-vous style and other.

The automatic generation of the Kronecker expression of an SSPN doesn't appear to pose major problems. The main difference is that while in SSA a transition can be enabled in a single state, so that we have one correcting factor per synchronized transition, in SSPN a synchronized transition can be enabled in more than one state (and typically this is the case) so that we have to provide more correcting factors. It is clear that if there are a large number of correcting factors then part of, or all, the advantages introduced by the solution method proposed can be lost, although the advantages of the parallel implementation are still retained. It should also be noted that this is true also for SSA: if SSMs are very dependent one from the others, i.e., if they perform a lot of synchronizations, then the advantages of the method can be lost. The point here is that with SSPN it can be easier to describe systems that, even if they interact "graphically" on a single transition, the level of interactions among stochastic processes can be very high, if there are many different markings in which that transition is enabled.

We plan in the near future to extend our parallel implementation also to SSPN and to apply SSPN to the modelling of parallel CSP-like programs, as described in [7]. A more long term goal would be to extend the class of nets to the kind of hierarchical interactions described in [1].

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This work is part of the author's PhD dissertation [6] which highly profited from the supervision of Prof. Gianfranco Balbo. The parallel implementation was done in collaboration with Valerio Malechino, whose contribution is highly appreciated. Finally, the author is grateful to the anonymous referees for their detailed comments on the draft of this paper.

References


Table 1: Parallel versus sequential implementation

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