ESTIMATING THE MEAN CYCLE TIME FOR STOCHASTIC SAFE MARKED GRAPHS

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ABSTRACT

Stochastic safe marked graphs with exponentially distributed firing time of timed transitions are investigated in the paper. An approximate method for estimating the mean cycle time is proposed. The method gives a value that tends to be greater than the exact one. It is of polynomial computational complexity. The estimate is much better than the best PERT network upper bound for networks with exponentially distributed durations of activities. The method is compared with the best upper bound for mean cycle time of stochastic marked graphs. The reasons of error of the above two methods are different. Therefore, one of the methods completes the other, and for given stochastic safe marked graphs we can choose the better one from the two estimates. The approximate method can be used for estimating the mean cycle time of stochastic safe marked graphs with NBUE distributed firing times.

1 INTRODUCTION

Stochastic safe marked graphs (SSMGs) are generalization of PERT networks because the PERT network is acyclic graph however, the cycles are allowed for SSMG. Hence, the methods for solving the problems associated with SSMGs are not simpler than the methods for PERT networks.

In order to understand the complexity of problems for SSMGs, now we consider PERT networks with durations of activities described by exponential random variables (RVs) that are assumed to be independent. Such networks are the Markov ones [10]. To compute the exact values of time characteristics for Markov PERT networks, one need to explore the state space of Markov chain [10]. This method suffers from exponential state explosion. Therefore, approximate method and bounding techniques with good accuracy and complexity properties are of special importance for SSMGs.

The emphasis of the paper is put on SSMGs with exponentially distributed firing times.

In many models of stochastic Petri nets [1],[7],[14] the firing time is expressed by the exponentially distributed RV.

Additionally, the SSMG with exponentially distributed firing times can be used to obtain the upper bounds of time characteristics (mean cycle time, distribution function of cycle time) for SSMG with NBUE (New Better than Used in Expectation)[2],[16] distributed firing times. For NBUE distributed firing time the following holds: a mean time until the firing is completed given that the firing has already lasted for α time units is less or equal to the original mean firing time. NBUE distributions class is a wide class of distributions. E.g. the class contains almost all distributions proposed for expressing the durations of activities for PERT networks, among the others: the Erlang, the hypoexponential. The exponential distribution is the maximal one in the class in the sense of convex ordering [2], [16].

In order to obtain the lower bound (LB) for the mean cycle time for stochastic marked graph a marked graph (MG) with deterministic model of firing times is created.

Polynomial algorithms for cycle time of bounded MGs with deterministic model of firing times are given in the papers [3], [11]. These algorithms are based on linear programming [9].

For safe MGs with deterministic model of firing times, there exists an algorithm [5] that is computationally more effective than the algorithm based on the linear programming.

An upper bound (UB) for bounded MGs is given in the paper [3]. The bound is polynomial one and it is based on the concept of degree of enabling of a transition.

The other UB for SSMGs with exponentially distributed firing times can be obtained using the best PERT network UB for networks with exponentially distributed durations of activities [8]. The bound is not the polynomial one.

The main problem is the fact that the above UBs are often too loose.

We formulate the estimate that is not the UB in the strict sense but it tends to be greater than the exact value. Now we sketch the idea of our method.

In order to obtain the LB for the mean cycle time for the SSMGs a MG with deterministic model of firing times is created [3], [5],[11]. Therefore, the coefficient

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of variation $C_X = \frac{\sqrt{Var[x]}}{E[X]}$ of RV X used for bounding is equal to 0.

For the UB based on paper [8], exponentially distributed RVs are used. The coefficient of variation of exponential RV is equal to 1.

Our approximate method is based on application of hypoexponentially distributed RVs. The coefficient of variation of these RVs satisfies the inequality $0<C_X\leq 1$.

In order to compute the performance of SSMGs the sum and maximum operations are executed.

The main point of our method is the approximation of maximum of two hypoexponentially distributed RVs by hypoexponential distribution.

The paper is organized as follows. The quality of the lower and the upper bounds for mean cycle time based on papers [3],[8] are presented in Section 2. The approximation of the maximum and sum of two hypoexponential RVs by the hypoexponential one is given in Section 3. The quality of our approximate method is examined in Section 4

2 QUALITY OF THE BOUNDS FOR MEAN CYCLE TIME OF THE STOCHASTIC SAFE MARKED GRAPHS

Marked graph (MG) is such a Petri net that for each place p there exists exactly one arc going into the p and exactly one arc going out of the p. We consider stochastic MGs with timed and immediate transitions [1]. The firing times of timed transitions are described by exponential RVs. These RVs are assumed to be independent.

The strong connectivity of MG is required.

We suppose that MG is live. For live and safe MG there exists a firing sequence after which the same marking as before is reached. Hence, we can consider a cyclic behaviour.

The idea of mean cycle time for stochastic MGs is defined in the paper [3].

The concept of discrete event dynamic systems (DEDSs) with stochastic processing times [15] is similar to the stochastic MGs.The number of dimensions for DEDSs is equal to the number of transitions for stochastic MGs. An example of DEDS with two-dimensional state space and exponential distributions of processing times is given in [15]. The example illustrates how the problem of calculating the exact value of mean cycle time is complicated. Therefore, approximate methods and bounding techniques with good accuracy and complexity properties are of special importance.

First we compare the quality of LB with quality of UB based on PERT network UB.

The LB is obtained for deterministic model of firing times [3], [5], [11]. In the deterministic model the transition firing times are assumed to be equal to the means of RVs of firing times for original SSMG.

Now we present the PERT network UB [8]. In order to obtain the UB the exponential RV is used for bounding the maximum (or sum) of exponential RVs. Coefficient of variation of RV S that is equal to maximum (or sum) of exponential RVs satisfies the relation $C_S \leq 1$. To get the UB for RV S the exponential RV T with the mean E[T]=E[S] and coefficient of variation C_T=1 is used. Hence, the RV T has greater variance then the RV S.

The quality of the UB is analyzed in the paper [6]. The UB is much better than a bound based on the mean and variance of activity durations [4].

In order to use the PERT network UB [8], we study the PERT network with exponentially distributed durations of activities. PERT network is represented by the acyclic graph. To apply the PERT network UB, we execute the following operations. We start from some repetitive marking M, and we fire all transitions exactly once. Finally we obtain the marking M again. Hence, we get an UB on the mean cycle time.

Let us consider a SSMG represented by Fig. 1a.

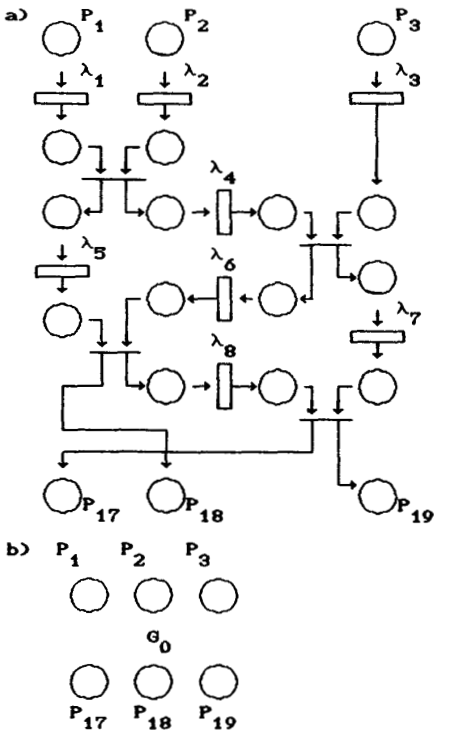


Fig. 1.

The immediate transitions are illustrated by bars and the timed ones by rectang-

les [1].

We suppose that the firing times for timed transitions are expressed by the exponential RVs with parameters $\lambda_1=\lambda_4=\lambda_6=\lambda_8=3$, $\lambda_2=\lambda_3=\lambda_5=1$, $\lambda_7=0.6$.

A brief representation of SSMG from Fig. 1a is given at Fig. 1b.

We study three SSMGs expressed by Fig.2.

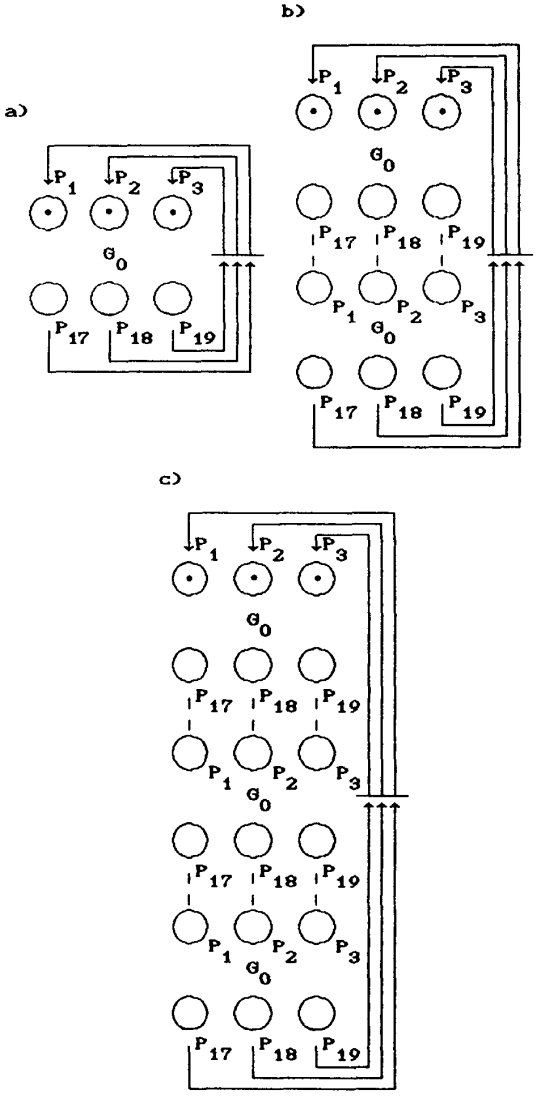


Fig. 2

The dashed lines mean that two connected places represent the same place.

The quality of the LB [3], [5], [11] and UB based on paper [8] is represented by Table 1. The error of LB is relatively small for different sizes of SSMG. The error for the UB increases very quickly if the size of SSMG increases.

Hence, the UB from the paper [8] is not acceptable for the greater SSMGs.

Table 1

SSMG	LB error [%]	LB	Exact Value	UB [8]	UB [8] error [%]
G_1	-21.68	3.00	3.83	5.26	37.37
G_2	-23.05	5.67	7.36	20.34	176.24
G_3	-23.51	8.33	10.90	66.71	512.30

Now we present the UB from the paper [3].

Let R(MG) be a reachability set of a MG and $^\bullet t$ be the set of input places for transition t.

Definition

Enabling bound K(t) for transition t is the greatest number k such that
$(\exists\, M \in R(MG))\ (\forall\, p \in {}^\bullet t)\ (M(p) \geq k)$.

Enabling bound expresses the greatest degree of enabling of the transition.

The UB for the mean cycle time is given by [3]:

$$\Gamma_{max} = \sum_{t_j \in T} \frac{E[X_j]}{K(t_j)} \qquad (1)$$

where $E[X_j]$ is the mean firing time for transition t_j, $K(t_j)$ is the enabling bound for t_j, T is the set of transitions.

We consider live and safe MGs so $K(t_j)=1$ for each $t_j \in T$.

Now we examine the quality of the above UB for an extreme case.

Let us analyze the MG from Fig. 3.

We suppose the deterministic model of firing time. The probability distribution that is concentrated in one point is the minimal one in NBUE distributions class in the sense of convex ordering. The RV X concentrated in one point m_0 is a limit $X=\lim_{k\to\infty} kP$ where P is the exponential RV with the parameter $\lambda=\frac{k}{m_0}$.

Let the firing time for transition t_0 be equal $X_0=0$ and for each t_i, $i \in \overline{1,n}$, $X_i=\tau$. Hence, the UB from [3] is equal $\Gamma_{max}=n\tau$. The exact value of cycle time is equal to τ. Therefore, when adding each new transition we obtain the error that is greater by 100% .

3 APPROXIMATION OF THE MAXIMUM AND SUM OF TWO HYPOEXPONENTIAL RANDOM VARIABLES BY THE HYPOEXPONENTIAL ONE

First, we recall the foundations of the

approximation technique from [12]. In the paper [12] the approximation of distribution functions was not considered. In second part of this Section we present the results of statistical tests for distribution functions.

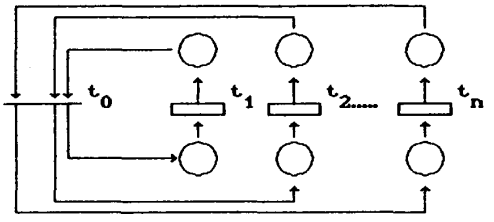


Fig. 3

In order to obtain the approxdmate value of mean cycle time, we execute the sum and maximum operations on hypoexponentially distributed RVs.

Definition [17]

If $X_1,...,X_n$ are mutually independent exponentially distributed RVs with different parameters $\lambda_1,...,\lambda_n$ ($i \neq j \Rightarrow \lambda_i \neq \lambda_j$), respectively, then

$$X=\sum_{i=1}^{n} X_i$$

is n-stage hypoexponentially distributed RV with parameters $\lambda_1,...,\lambda_n$.

The probability density function (PDF) of hypoexponentially distributed RV X is given by:

$$f_X(x)=\sum_{i=1}^{n} a_i\lambda_i e^{-\lambda_i x}, x>0, \quad (2)$$

where

$$a_i=\prod_{\substack{j=1\\ j\neq i}}^{n} \frac{\lambda_j}{\lambda_j-\lambda_i}, \quad 1\le i\le n. \quad (3)$$

The distribution function (DF) of X is expressed by:

$$F_X(x)=1-\sum_{i=1}^{n} a_i e^{-\lambda_i x}.$$

Let EXP and HYPOEXP denote the class of exponentially and hypoexponentially distributed random variables, respectively.

The mean of X ∈ HYPOEXP and variance of X are given by equations:

$$E[X]=\sum_{i=1}^{n} E[X_i]=\sum_{i=1}^{n}\frac{1}{\lambda_i} \quad (4)$$

$$Var[X]=\sum_{i=1}^{n} Var[X]=\sum_{i=1}^{n}\frac{1}{\lambda_i^2}. \quad (5)$$

The coefficient of variation of random variable X

$$C_X=\frac{\sqrt{Var[X]}}{E[X]}$$

for X ∈ HYPOEXP satisfies the relation $0 < C_X \le 1$.

Let X and Y be such hypoexponentially distributed RVs with parameters $\lambda_1,...,\lambda_n$ and $\mu_1,...,\mu_m$ that are assumed to be independent. RV W=X+Y is (n+m)-stage hypoexponentially distributed with parameters $\lambda_1,...,\lambda_n,\mu_1,...,\mu_m$ and with the mean

E[W] = E[X] + E[Y],

and with the variance

Var[W] = Var[X] + Var[Y].

We now consider the RV Z = max (X,Y), where X,Y ∈ HYPOEXP. Let DFs of RVs X,Y ∈ HYPOEXP be described by

$$F_X(x) = 1 - \sum_{i=1}^{n} a_i e^{-\lambda_i x}$$

$$F_Y(y) = 1 - \sum_{i=1}^{m} b_i e^{-\mu_i y}$$

where a_i, b_i are given by expression (3).

The DF of RV Z satisfies the equation:

$F_Z(z) = F_X(z)\,F_Y(z)$.

Hence,

$$F_Z(z) = 1 - \sum_{i=1}^{n} a_i e^{-\lambda_i z} - \sum_{j=1}^{m} b_j e^{-\mu_j z} + \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j e^{-(\lambda_i+\mu_j)z}.$$

Expressions for the first and the second moments are derived using Laplace transform [12], and are the following:

$$E[Z]=\sum_{i=1}^{n} a_i \frac{1}{\lambda_i}+\sum_{j=1}^{m} b_j \frac{1}{\mu_j}-\sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \frac{1}{\lambda_i+\mu_j} \quad (6)$$

$$E[Z^2] = 2\left\{ \sum_{i=1}^{n} a_i \frac{1}{\lambda_i^2}+\sum_{j=1}^{m} b_j \frac{1}{\mu_j^2} - \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \frac{1}{(\lambda_i+\mu_j)^2} \right\}.$$

The variance of RV Z is given by:

$$Var[Z] = E[Z^2] - (E[Z])^2. \quad (7)$$

In order to approxdmate the RV Z=max(X,Y) (X,Y∈HYPOEXP) by means of RV U∈HYPOEXP, we choose such U that

E[U]=E[Z],

Var[U]=Var[Z],

hence $C_U=C_Z$.

Coefficient of variation for RV V ∈ HYPOEXP satisfies the inequality $C_V \le 1$. Therefore, the relation $C_Z \le 1$ should be satisfied.

Theorem [13]

If $Z=\max(X,Y)$, where $X,Y \in HYPOEXP$, then $C_Z \leq 1$.

In order to approximate the random variable $Z = \max(X,Y)$ $(X,Y \in HYPOEXP)$ by means of random variable $U \in HYPOEXP$, we choose such U that

$C_U = C_Z$,

$E[U] = E[Z]$,

$Var[U] = Var[Z]$.

Let n be the number of stages of U. The number n is selected according to the equation:

$$n = \left\lceil \frac{1}{C_Z^2} \right\rceil + 1,$$

where $\lceil a \rceil$ is the least integer number that is not less than a.

The parameters $\lambda_1,\ldots,\lambda_n$ of stages of RV U satisfy the equation

$$\lambda_i = k^{i-1}\lambda, \quad 1 \leq i \leq n, \quad 0 < k < 1.$$

Squared coefficient of variation C_U^2 is given by [12]:

$$C_U^2 = \frac{\sum_{i=1}^{n} \frac{1}{\lambda_i^2}}{\left(\sum_{i=1}^{n} \frac{1}{\lambda_i}\right)^2} = \frac{1+k^{-n}}{1+k^{-1}} \frac{1-k^{-1}}{1-k^{-n}}.$$

Hence, the parameter k of RV U is computed by solving the equation:

$$\frac{1+k^{-n}}{1+k^{-1}} \frac{1-k^{-1}}{1-k^{-n}} = C_Z^2, \qquad (8)$$

where C_Z is the coefficient of variation of $Z = \max(X,Y)$ $(X,Y \in HYPOEXP)$.

The parameter λ of RV U is given by

$$\lambda = \frac{1}{E[Z]} \frac{1-k^{-n}}{1-k^{-1}}. \qquad (9)$$

Therefore, two numbers λ and k only are sufficient to express the RV U.

The sum $W=X+Y$ $(X,Y \in HYPOEXP)$ is the hypoexponential RV. When X,Y, respectively, are n,m - stage ,respectively, hypoexponentially distributed RVs then W has n+m parameters $\lambda_1,\ldots,\lambda_n,\mu_1,\ldots,\mu_m$.

The RV $W \in HYPOEXP$ can be approximated in the same way as it is done for $Z=\max(X,Y)$ $(X,Y \in HYPOEXP)$. The RV W is approximated by RV $V \in HYPOEXP$ with parameters:

$E[V]=E[W]$,

$Var[V]=Var[W]$,

hence $C_V=C_W$,

according to the schema for $Z=\max(X,Y)$ $(X,Y \in HYPOEXP)$.

Now we present results of statistical tests for the above approximations.

We use Kolmogorov one sample test. The values of the exact DF for maximum (or sum) are treated as random values from an undetermined distribution.

We test a hypothesis that the maximum (or sum) of two hypoexponential RVs is the hypoexponential RV that could be characterized by two parameters k and λ according to the above approximation schema.

The error involved in the hypothesis is measured by

$$D_n = \max_i \left| F(z_i) - E(z_i) \right| \qquad i=1,\ldots,n$$

where

E is the approximating hypoexponential DF,

F is the exact DF of maximum (or sum) of $X,Y \in HYPOEXP$.

Table 2

E(X)/E(Y)	C_X^2	C_Y^2	D_{100}max	D_{100}sum
0.2	0.10	0.10	0.00005	0.00691
0.2	0.10	0.25	0.00200	0.01507
0.2	0.10	0.50	0.01447	0.02972
0.2	0.25	0.10	0.00028	0.00274
0.2	0.25	0.25	0.00276	0.01097
0.2	0.25	0.50	0.01266	0.02398
0.2	0.50	0.10	0.00083	0.00116
0.2	0.50	0.25	0.00350	0.00607
0.2	0.50	0.50	0.01176	0.01704
0.4	0.10	0.10	0.00413	0.00625
0.4	0.10	0.25	0.02034	0.01676
0.4	0.10	0.50	0.05336	0.03654
0.4	0.25	0.10	0.00474	0.00190
0.4	0.25	0.25	0.01480	0.00989
0.4	0.25	0.50	0.03628	0.02285
0.4	0.50	0.10	0 00392	0.00847
0.4	0.50	0.25	0.00917	0.00548
0.4	0.50	0.50	0.02209	0.01011
0.6	0.10	0.10	0.01555	0.00419
0.6	0.10	0.25	0.03612	0.01780
0.6	0.10	0.50	0.07754	0.03554
0.6	0.25	0.10	0.00697	0.00477
0.6	0.25	0.25	0.01729	0.00637
0.6	0.25	0.50	0.04322	0.01692
0.6	0.50	0.10	0.01647	0.01532
0.6	0.50	0.25	0.01053	0.00569
0.6	0.50	0.50	0.01998	0.00750
0.8	0.10	0.10	0.01436	0.00285
0.8	0.10	0.25	0.04179	0.01378
0.8	0.10	0.50	0.07699	0.03187
0.8	0.25	0.10	0.01584	0.00813
0.8	0.25	0.25	0.01361	0.00587
0.8	0.25	0.50	0.03258	0.01473
0.8	0.50	0.10	0.04214	0.02321
0.8	0.50	0.25	0.01323	0.00688
0.8	0.50	0.50	0.01039	0.00632

The above test provides a useful indication of the closeness of the approximation. We have executed calculations for n=100 and significance level α=0.05. The critical value $D_{100}(0.05)$ is equal to 0.13403. The hypothesis should be rejected if D_{100} obtained from calculations is greater than $D_{100}(0.05)$.

We have computed the error involved in

the hypothesis for different relations between RVs X and Y upon that the maximum (or sum) operation is performed.

Table 2 contains the D_{100} for maximum and sum operations for different relations between the means of X and Y (E[X]/E[Y]), and for different values of squared coefficients of variation (C_X^2,C_Y^2).

The table does not contain such a case when $D_{100}>D_{100}(0.05)$. Hence, there is no reason to reject the hypothesis.

The approximation is better for sum than for the maximum.

For series-parallel (S-P) PERT networks with exponentially distributed durations of activities the method based on the above approximations gives almost exact value of mean completion time [12], [13]. Results of the above statistical tests indicate the reason.

For non S-P networks, the approximation error of the mean completion time is greater [13]. The reason is the fact that some RVs upon that the maximum operation is performed (e.g. for network of Fig. 4, the RVs $X=X_1+X_4$ and $Y=\max(X_1+X_3,X_2)+X_5$ are associated ones [2]. There exist correlation between RVs X and Y because they both depend on RV X_1. Hence, for non S-P PERT networks our method tends to give a result that is greater than the exact one.

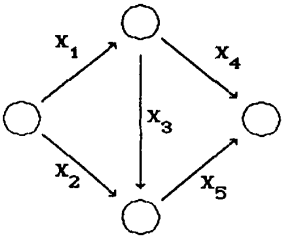


Fig. 4

For PERT networks, the method based on the above approximations does not calculate the UB for the mean completion time. The method cannot give the UB what is caused by the following fact. The RV $Z=\max(X,Y)$, where $X,Y \in$ HYPOEXP, is approximated by the RV U such that 1. E[U]=E[Z], 2. Var[U]=Var[Z]. The requirement 2. exclude the relation $Z \lhd U$ where $\lhd$ is the convex ordering [2] [16]. Our estimate is better than the UB from paper [8] (Theorem 4 in [13]).

4. APPROXIMATION OF MEAN CYCLE TIME FOR STOCHASTIC SAFE MARKED GRAPHS

First we compare our approximate method with the PERT network UB [8].

Table 1 (from Section 2) presents the quality of the UB [8]. For greater SSMGs the UB is not useful because of bound error. Now we analyse the reason. In order to obtain the UB for RVs being maximum (or sum) of exponential RVs, the exponential RV T with the mean E[T]=E[S] and coefficient of variation $C_T=1$ is used. The coefficient of variation $C_T=1$ is the reason of weakness of the above UB for greater SSMGs because T has greater variance. E.g. let us consider the RV S=nX where X is the exponential RV. In this case $C_S^2 = \frac{1}{n}$.

In our approximate method the hypoexponential RV T with $0 < C_T \leq 1$ is used.

In order to compare our method with the method based on UB [8] we choose the SSMG from Fig.2c. In this case the UB gives the error equal to 512.3% .

We perform the computations for different interval of squared coefficient of variation (SCV) of the approximating hypoexponential RV. The table 3 presents quality of approximation by hypoexponential RV with interval of SCV equal to $[C_{min}^2,1]$ for different C_{min}^2.

For given interval $[C_{min}^2,1]$ the approximate method works as follows.

Let S be the maximum (or sum) of two hypoexponential RVs.

If $C_{min}^2 \leq C_S^2 \leq 1$ the RV S is approximated by $T \in$ HYPOEXP with

$E[T] = E[S]$,

$C_T = C_S$,

hence, Var[T] = Var[S].

If $C_S^2 < C_{min}^2$ then RV S is approximated by $T \in$ HYPOEXP with

$E[T] = E[S], C_T^2 = C_{min}^2$,

hence, $\mathrm{Var}[T] = C_{min}^2 \left(E[S]\right)^2$.

For $C_S^2 < C_{min}^2$ the RV S is approximated by RV T with greater variance. It is the reason why our method tends to give value that is greater than the exact one of mean cycle time.

The table 3 illustrates how the greater interval of SCV gives the better approximation. The exact value of mean cycle time for SSMG is equal 10.90

The problem is that we cannot use the entire interval of values (0,1] for SCV of hypoexponential RV. The reason is the limited accuracy of computer computations. Let us analyse the form of expression for hypoexponential RV. Some coefficients a_i (expression 3) can have great positive or negative value with small difference of absolute values. This fact causes that even great precision is not sufficient for greater interval of SCV. The dependency of interval of SCV in the function of real number representation length is the problem for further investigation.

Table 3

Interval of c^2	Approximate Value	Approximation Error [%]
1	66.71	512.30
[1/2,1]	36.83	238.04
[1/3,1]	28.48	161.40
[1/4,1]	24.47	124.60
[1/5,1]	21.99	101.84
[1/6,1]	20.40	87.24
[1/7,1]	19.29	77.05
[1/8,1]	18.46	69.44
[1/9,1]	17.79	63.29
[1/10,1]	17.27	58.51
[1/11,1]	16.87	54.84
[1/12,1]	16.54	51.81
[1/13,1]	16.26	49.24
[1/14,1]	16.02	47.04

Now we consider a computational complexity of the approximate method. Algorithms for computing the mean and variance of maximum (or sum) of two RVs according to expressions (6) and (7) (or (4) and (5)) are polynomial. A solution for k according to expression (8) can be find by a secant method that is of polynomial complexity. The parameter λ can be computed (expression (9)) by polynomial calculations. The approximation is executed no more than p times where p is the number of timed transitions. Therefore, our method has polynomial computational complexity.

From the above analysis we conclude that there are two reasons of errors in our method. First is the impossibility of using the entire interval of SCV for hypoexponential RV. The second reason is the association [2] of RVs upon that the maximum operation is performed.

The UB [8] is charged with association of RVs too.

Now we compare our method with UB from the paper [3].

In order to execute the comparison we examine five SSMGs: G_1, G_2, G_3 (from Fig. 2) and G_4, G_5, respectively, that are created from four G_0s (G_0 is given by Fig. 1a), five G_0s, respectively, in the same way as the SSMGs G_1, G_2, G_3, are built.

We choose the SCV interval equal to [1/14,1].

The table 4 contains the exact values of mean cycle time for the above SSMGs and the results obtained using our approximate method, and the UBs obtained according to the expression (1) [3].

We see that our method is better than the method based on [3] for G_1, G_2, G_3, and worse for G_4, G_5.

There are two factors that influence the quality of our estimate when comparing with the UB [3]. First one is the maximal length of circuits in MGs that is measured by the number of timed transitions. The second one is the degree of parallelism between the transitions.

Table 4

SSMG	Exact Value	Approximate Value	Upper Bound [3]
G_1	3.83	4.18	6
G_2	7.36	9.24	12
G_3	10.90	16.02	18
G_4	14.43	26.01	24
G_5	17.96	40.67	30

There are two reasons of approximation error of our method for SSMGs with circuits of greater length:

1. The RVs for some paths in SSMG are associated,
2. The limited interval of SCV of the approximating RVs.

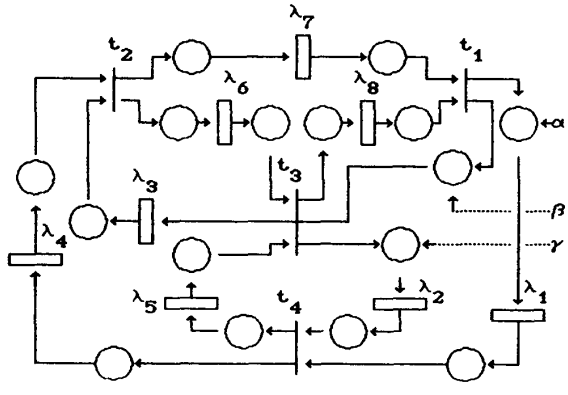
The table 4 illustrates the comparison of the two methods for different maximal lengths of circuits in MG.

The UB [3] is created by reducing the degree of parallelism between transitions. Hence, e.g. for the MG from Fig. 3 the UB gives too loose estimate. The above MG contains the parallel structure only. Hence, in this case our method gives almost exact result. For SSMG with greater degree of parallelism the approximate method is better than UB.

Therefore, one of the two estimates completes the other, and for given SSMG we can choose the better one.

Now we analyse an interesting aspect.

Let us consider a SSMG from Fig. 5.



$\alpha \leftarrow P_1(P_{17})$, $\beta \leftarrow P_3(P_{19})$, $\gamma \leftarrow P_2(P_{18})$

Fig. 5

The SSMGs from Fig. 2a, 2b, 2c, respectively, are the single, the twofold, threefold, respectively, extension of the origi-

nal SSMG from Fig. 5.

Now we study a convergency of estimates obtained from these extensions.

For strongly connected live and bounded MG, the marking is regenerated after such firing sequence that its firing vector has equal components.

Let $E[\Gamma_n]$ denote the exact value of mean cycle time of SSMG that is the n-fold extension of the original SSMG. For ergodic original SSMG the mean cycle time $E[\Gamma]$ is equal to

$$E[\Gamma] = \lim_{n \to \infty} \frac{E[\Gamma_n]}{n}.$$

The exact value of mean cycle time for original SSMG is equal to 3.53.

The table 5 presents the convergency of the quotient $\frac{E[\Gamma_n]}{n}$.

Let $E[A\Gamma_n]$ denote our estimate of mean cycle time for n-fold extension of original

Table 5

Stochastic Safe Marked Graph	Exact Value	Aproximate Value
G_1	3.83	4.18
G_2	3.68	4.62
G_3	3.63	5.34

SSMG. The quotient $\frac{E[A\Gamma_n]}{n}$ is not convergent (table 5). The reasons are the association of RVs and limited interval $[C^2_{min},1]$ for SCV of approximating RVs.

Our method gives the best estimate for single extension of original SSMG. Therefore, it is not useful to use more than one extension of original SSMG.

The repetitive marking that is used to obtain the approximate value has important impact on the quality of approximation. The repetitive marking obtained for the MG with the deterministic model of firing time seems to be the good solution.

5 CONCLUSIONS

In order to calculate the exact value of mean cycle time for SSMGs with exponentially distributed durations of activities, one need to explore the state space of Markov chain.

The approximate method for calculating the mean cycle time is proposed in the paper. The method does not require the generation of combinatorial state space. The quality of the method is compared with the bounds for the mean cycle time of SSMGs.

The LB based on deterministic model of firing time [3],[5], [11] is relatively good in spite of taking into consideration the first moment of firing time RVs only.

Our method gives a value that tends to be greater than the exact one.

The method gives much better estimate than the best PERT network UB for networks with exponentially distributed durations of activities [8].

The approximate method is of polynomial computational complexity.

A comparison of our estimate with the UB [3] is carried over.

The reason of bound error of the UB [3] is reducing the degree of parallelism.

The approximation error of our method is caused by:

1. RVs for different paths in MG are associated,
2. Limited interval of values for coefficient of variation of approximating hypo-exponential RVs.

The reasons of errors of the above two methods are different. Therefore, one of the methods completes the other, and for given SSMG we can choose the better one from the two estimates.

Our estimate can be used as the estimate not only for MG with exponentially distributed RVs of firing time. The method can be applied for estimating the mean cycle time for MG with NBUE distributed firing times. It is caused by the fact that exponential distribution is the maximal one in NBUE distributions class in the sense of convex ordering.

REFERENCES

[1] M. Ajmone Marsan, G.Balbo, G.Conte, A class of generalized stochastic Petri nets for the performance evaluation of multiprocessor systems, ACM Trans. on Computer Systems, Vol.2, No.2, May 1984, 93-122.

[2] R.E. Barlow, F.Proshan, Statistical Theory of Reliability and Life Testing Probability Models, Holt, New York 1975.

[3] J. Campos, G. Chiola, J.M. Colom, M. Silva, Tight polynomial bounds for steady-state performance of marked graphs, in: Proc. Third Int. Workshop on Petri Nets and Performance Models, Kyoto, Japan, Dec. 1989, IEEE Computer Society Press, 1989, 200-209.

[4] L.P. Devroye, Inequalities for the completion times of stochastic PERT networks, Math. of Operations Research, Vol.4, 1979, 441-447.

[5] D. Dubois, K.S. Stecke, Dynamic analysis of repetitive decision-free discrete event processes: The algebra of timed marked graphs and algorithmic issues, accepted for publication in Annals of Operations Research.

[6] S.E. Elmaghraby, The estimation of some network parameters in the PERT model of activity networks: review and critique, Chapter 1, Part III, in: R. Slowinski, J. Weglarz (eds.), Advances in Project Scheduling, Elsevier, 371-

432.

[7] G. Florin, S. Natkin, Les reseaux de Petri stochastiques, Technique et Science Informatiques. Vol.4, No.1, 1985, 143-160.

[8] J. Kamburowski, An upper bound on the expected completion time of PERT networks, European J. of Operational Res., Vol.21,1985, 206-212.

[9] N. Karmarkar. A new polynomial time algorithm for linear programming, Combinatorica, Vol.4, 1984, 373-395.

[10] V.G. Kulkarni, V.G. Adlakha, Markov and Markov regenerative PERT networks, Operations Res., Vol.34, No. 5, 1986, 769-781.

[11] J. Magott, Performance evaluation of concurrent systems using Petri nets, Information Processing Letters, Vol. 18, Jan. 1984, 7-13.

[12] J.Magott, K.Skudlarski, Combining generalized stochastic Petri nets and PERT networks for the performance evaluation of concurrent processes, in: Proc. Third Int. Workshop on Petri Nets and Performance Models, Kyoto, Japan, Dec. 1989, IEEE Computer Society Press, 1989, 249-256.

[13] J. Magott, K. Skudlarski, Estimating the mean completion time of PERT networks with exponentially distributed durations of activities, submitted for publication.

[14] M.K. Molloy, On the integration of delay and throughput measures in distributed processing models, Ph.D. dissertation, Univ. of California, Los Angeles, 1981.

[15] G.J. Olsder, J.A.C. Resing, R.E. De Vries, M.S. Keane, G. Hooghiemstra, Discrete event systems with stochastic processing times, IEEE Trans. Automatic Control, Vol.35, No.3, 1990, 299-302.

[16] D. Stoyan, Comparison Methods for Queues and Other Stochastic Models, Wiley, Chichester 1983.

[17] K.S. Trivedi, Probability and Statistics with Reliability, Queueing and Computer Science Applications, Edgewood Cliffs, Prentice-Hall, 1982