Simulation of Timed Petri Nets by Ordinary Petri Nets and Applications to Decidability of the Timed Reachability Problem and other related problems

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Abstract

In this paper we consider Timed Petri Nets, for which a timed step semantics generalizing the ordinary step semantics is presented. A codification of them, with regard to such semantics, by Ordinary Petri Nets is presented. This codification is presented in a gradual way, beginning with that corresponding to a more restricted and thus simpler step semantics. This semantics does not allow the overlapped firing of several instances of each transition. Next, we consider a step semantics which allows such overlapped executions, but not the simultaneous firing of several instances of the same transition. In both restricted cases the construction is based on an automaton which controls the system evolution. The construction for the most general semantics is rather more complicated, that is why it is presented gradually. We consider three versions of it, being the two first ones easier to understand, but having each one a different problem that we would probably like to avoid; this is made by the third construction, which refines the previous ones. Finally, we use this simulation to decide several problems on Timed Petri Nets.

1 Introduction

In this paper we introduce a class of Timed Petri Nets that are obtained from (Ordinary) Petri Nets by associating a duration to the execution of each transition of the net. We present a codification of them by means of Ordinary Petri Nets. This allows us to simulate their timed step sequence semantics by the ordinary occurrence sequence semantics of the corresponding translation. This simulation allows us to translate to Ordinary Petri Nets the timed versions of a collection of properties on Petri Nets, like Reachability, Liveness or Deadlock Freeness. Thus we can conclude their decidability, since they are decidable for Ordinary Petri Nets (see [4,5,6]). We have found in the literature two versions of timed extensions of Petri Nets. The first one, Time Petri Nets (see [7]), associates two integer numbers to each transition. These numbers represent, respectively, the moment at which the transition can be fired and the moment at which must fire at the latest (both with regard to the instant at which the transition is enabled). In Time Petri Nets the firing of transitions takes no time. The second version, Timed Petri Nets (see [9]) is the one that we study in this paper. It is based on the association of an integer number to each transition, which indicates its duration. The only scheme of codification of Timed Petri Nets by Non-Timed Petri Nets of which we have knowledge is that presented in [1]. It is done by modelling the passage of time by means of a clock controlling the net behaviour, represented by a transition generating ticks periodically. However, this scheme complicates the modelling of synchronizations. Furthermore, to model the coincidences of events at the same instant it is necessary to include some places with capacity 0, with the aim of maintaining the syntactic definition of Petri Nets. This means a violation of the net semantics rules, and thus we consider that such scheme of codification is not adequate, since it is not done in the strict ordinary net framework. On the other hand, that construction does not allow overlapped executions of the same transition, which would be natural to allow when generalizing the firing rule to Timed Petri Nets, in an straightforward way; since before an execution of a transition will finish we could be able to fire it again. It is even possible that taking the Step Semantics as a starting point, we would like to allow the simultaneous firing of several instances of the same transition at the same instant, as far as it would be allowed by the firing rule.

The construction that we present is based on the representation of the passage of time not by a 'special' transition, but by an 'absolutely standard' place. Thus we obtain a strict codification just using ordinary nets, which allows to use the full Petri Nets theory to study Timed Petri Nets. Besides we take a timed step semantics as semantics of the Timed Petri Nets. This one will be simulated by the standard semantics of the corresponding Petri Net. In this way, we let both the overlapping and the simultaneous firing of either the same or several transitions, as far as the firing rule allow. We also consider two more restrictive semantics, avoiding either the simultaneous firing of

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several instances of the same transition, or even overlapped executions of such instances. The main reason for considering these semantics is that the codification is much simpler than in the general case. These codifications, which are presented in sections 4.1 and 4.2, can be done by means of an automaton controlling the system evolution. These automata can be immediately codified in terms of Ordinary Petri Nets (see [8]).

The general construction covering the step semantics without restrictions, is presented in section 4.3. We consider three different constructions to realize the codification because the two first ones are simpler, and easier to understand. Nevertheless each construction has a problem that we would possibly like to avoid. The first solution has the problem that the simulation allows empty steps, and the second construction does not preserve liveness in general. Finally, the third construction has not any of these problems.

The paper is structured as follows: in section 2 we define Timed Petri Nets and their semantics; in section 3 we introduce several problems on Timed Petri Nets that can be solved using our simulation, by reducing them to related solved problems on Ordinary Nets. Finally, as described before, in section 4 we present the different constructions defining the codifications for each case.

2 Timed Petri Nets

Definition 1 (Timed Petri Net)
Let Σ be an infinite alphabet. We define a Timed Petri Net (TPN) as a tuple (P, T, F, W, δ), where:

P : Finite set of places
T : Finite set of transitions (disjoint with P)
$F \subseteq P \times T \cup T \times P$ (Set of arcs)
$W : F \longrightarrow \mathbb{N}^+$ (Weight of the arcs)
$\delta : T \longrightarrow \mathbb{N}^+$ (Durations)
□

The effect of the firing of timed transitions is formalized as follows: when a transition is fired, we will remove the corresponding tokens from its preconditions, and only when the transition ends (after the corresponding duration time has passed) we add the corresponding tokens to the postconditions. This implies the necessity of adding to the ordinary markings a second component which indicates what transitions are in execution, and for each one of them how much time is left till its termination.

Definition 2 (Markings of Timed Petri Nets)
Let $N = (P, T, F, W, \delta)$ be a TPN. A marking M for N is a pair (M_1, M_2), where $M_1 \in \mathbb{N}^P$ and M_2 is a finite multiset of pairs in $T \times \mathbb{N}^+$ such that $\forall t \in T$ and $\forall \gamma \geq \delta(t) : M_2(t, \gamma) = 0$. □

We will call M_1 the *current marking* (here *marking* has the ordinary meaning), and we say that M_2 is the *multiset of pending transitions*. We have considered multisets of pending transitions instead of just sets, in order to allow the overlapped firing of any kind of transitions. Thus we have the maximal generality, and we are free to introduce restrictions when defining the firing rule, if we desire to introduce any kind of limitation.

Contrary to what happens in ordinary nets, at a concrete instant a Timed Petri Net may have a determined current marking, and in the next instant this marking will change, even though no new transition has been fired, by effect of the termination of some transition (or several) which were in execution.

On the other hand, if no new transition is fired from a certain instant on, the marking of the net will only suffer variations for a finite time, stabilizing itself at certain instant. These stable markings are those such that $M_2 = \emptyset$. Naturally, initial markings should be stable. Thus, we say that a marking $M = (M_1, M_2)$ *can be initial* iff $M_2 = \emptyset$.

Definition 3 (Firing Rule)
Let $N = (P, T, F, W, \delta)$ be a TPN and $M = (M_1, M_2)$ the marking of it at some instant $\beta \in \mathbb{N}$. We say that a multiset of transitions R *is enabled* at the instant β iff

$$M_1(p) \geq \sum_{t \in T} R(t) \cdot W(p, t), \quad \forall p \in P$$

If any multiset of transitions R is enabled at an instant $\beta \in \mathbb{N}$, and we fire them at that instant, the reached marking at the instant $\beta + 1$ is the marking $M' = (M'_1, M'_2)$ defined by:

$$M'_1 = M_1 - \sum_{t \in C_0} R(t) \cdot W(-, t) +$$
$$\sum_{t \in C_1} R(t) \cdot W(t, -) + \sum_{(t,1) \in C_2} M_2(t, 1) \cdot W(t, -)$$

where
$$C_0 \doteq \{t \in T | R(t) > 0\}$$
$$C_1 \doteq \{t \in T | R(t) > 0 \wedge \delta(t) = 1\}$$
$$C_2 \doteq \{(t, 1) \in T \times \mathbb{N} | M_2(t, 1) > 0\}$$

$M'_2 : T \times \mathbb{N}^+ \longrightarrow \mathbb{N}$ with

$$M'_2(t, \beta') \doteq \begin{cases} R(t), & \text{if } \beta' = \delta(t) - 1 \wedge R(t) > 0 \\ M_2(t, \beta' + 1), & \text{otherwise} \end{cases}$$

The described step is denoted by the notation $M[R\rangle M'$. □

A particular case of the previous definition corresponds to the case $R = \emptyset$, reflecting the passage of time on the execution of pending transitions.

From this definition it is deduced that, as happens in ordinary nets, if a multiset of transitions R is enabled at an instant β, then it is also enabled at any later instant, at least as far as no new transition will be fired, because the effect of passage of time can only be positive, in the sense that the current marking can only gain new tokens, each time the execution of a pending transition terminates.

The previous definition allows us to associate to TPN's a (general) step semantics. We also will consider two other more restrictive semantics.

Definition 4 (Timed Step Sequences)
Let $N = (P, T, F, W, \delta)$ be a TPN and M_0 a marking for it; we say that $\sigma = M_0[B_0\rangle \ldots M_{n-1}[B_{n-1}\rangle M_n$, is a *finite timed step sequence* of (N, M_0) iff:

1. $\forall i \in \{0, \ldots, n-1\}$: B_i is a multiset of transitions in T (which can be empty)
2. $\forall i \in \{1, 2, \ldots, n\}$: $M_{i-1}[B_{i-1}\rangle M_i$, where M_{i-1} and M_i are markings of N.

Note that from the initial marking M_0, the duration of the sequence, and the non-empty steps B_i along it, with their firing times, we can construct the entire corresponding step sequence. This will be denoted by $M[\sigma^{(n)}\rangle M_n$, where $\sigma = R_1^{(\beta_1)} \ldots R_r^{(\beta_r)}$, with $\beta_1 < \beta_2 < \ldots < \beta_r$, and each R_j is a non-empty multiset of transitions in T.

We denote by $P(N, M_0)$ the set of timed step sequences, defining the step semantics of the net N.

$$P(N, M_0) = \{\sigma | \sigma \text{ is a finite timed step sequence for } N \text{ from } M_0\}$$

□

The other two more restrictive semantics are defined by restricting the firing rule in adequate ways:

Definition 5 (Restrictive Firing Rules)
Let $N = (P, T, F, W, \delta)$ be a TPN and $M = (M_1, M_2)$ the marking of it at some instant $\beta \in \mathbb{N}$.

- We say that a multiset of transitions R *is enabled without overlapping of executions of the same transition* (*wo-enabled*) at the instant β iff it is enabled without restrictions, and $\forall t \in T$ we have:
  1. If $\exists \gamma \in \mathbb{N},\ M_2(t, \gamma) > 0$ then $R(t) = 0$
  2. $R(t) \leq 1$
- We say that a multiset of transitions R *is enabled without the simultaneous firing of several instances of the same transition* (*wsf-enabled*) at the instant β iff it is enabled without restrictions, and $\forall t \in T$ we have $R(t) \leq 1$.

□

Definition 6 (Restrictive Timed Step Sequences and Semantics) Let $N = (P, T, F, W, \delta)$ be a TPN and M_0 a marking for it; we say that $\sigma = M_0[B_0\rangle \ldots M_{n-1}[B_{n-1}\rangle M_n$, is a *wo-finite* (resp. *wsf-finite*) *timed step sequence* of (N, M_0) iff it is an ordinary timed step sequence, and all the steps along it are *wo-enabled* (resp. *wsf-enabled*) at the corresponding markings. We denote by $S(N, M_0)$ (resp. $D(N, M_0)$) the set of *wo-finite (resp. wsf-finite) timed step sequences* of N from M_0. □

Proposition 1 Timed Petri Nets such that all their transitions have a duration of one unit of time are (naturally) equivalent, under the timed step sequence semantics to the associated ordinary net with its (ordinary) step semantics. □

3 The Reachability Problem and other related problems for Timed Petri Nets

3.1 The Reachability Problem

Besides the usual notion of reachability we also will define the strict reachability, which has as goal both a marking and the time at which we want to get it.

Definition 7 (Reachability Problems for TPN's)
Let N be a TPN, M a marking for it and $\beta \in \mathbb{N}$. We say that M is reachable (resp. strictly reachable at the instant β) in N, which will be denoted by $M \in [M_0\rangle$, (resp. $M \in [M_0\rangle_\beta$) iff there exists a finite step sequence σ such that $M_0[\sigma\rangle M$ (resp. $M_0[\sigma^{(\beta)}\rangle M$). □

Theorem 1 (Decidability of the Reachability Problem for TPN's) Let $N = (P, T, F, W, M_0)$ be a TPN and M a marking for it. We can decide if $M \in [M_0\rangle$.

Proof: We must distinguish two cases:

1. If M is stable, then M is reachable in N with regard to the timed step semantics iff it is reachable in the ordinary net associated to N, which is obtained by ignoring the durations of the transitions in it. This is because we can always fire the transitions more slowly, getting an equivalent sequence without overlappings.
2. If M is not stable, we can consider the stable marking M' obtained from the marking M by adding to it the tokens substracted by the firings of the transitions which are in execution in M. Then if M is reachable, M' will be too, because if we take the step sequence leading us to M, and we avoid the execution of the transitions in execution in M, we obtain a step sequence leading us to M'. But it is clear that in M' we can fire (even together) all those transitions, and in particular we can fire them in the adequate way to reach M. Thus if M' is reachable, M will be too.

□

When the given Timed Petri Net does not contain any autonomous transition, that is, when there is no transition without preconditions, the strict reachability problem is trivially decidable, as we can (finitely) enumerate the reachable markings at the instant β, because at any instant we can only fire a finite number of multisets of transitions.

Theorem 2 (Decidability of the Strict Reachability Problem for TPN's without autonomous transitions) Let $N = (P, T, F, W, M_0)$ be a TPN without autonomous transitions, M a marking for it, and $\beta \in \mathbb{N}$. We can decide if $M \in [M_0\rangle_\beta$. □

Later we will prove, using the codification by Ordinary Petri Nets, that the Strict Reachability Problem is also decidable for TPN's with autonomous transitions.

3.2 Other related problems

The following problems can be posed on Timed Petri Nets, and can be solved, and thus be proved as decidable, by means of the codification that we will present in Section 4.

Definition 8 Let N be a TPN, $s \in P$ and $k, \beta \in \mathbb{N}$.
i) We say that N is s, k*-linearly unbounded* iff there exists some $\gamma \geq k$, and some marking M such that $M \in [M_0\rangle_\gamma$ and $M(s) \geq \gamma$.

ii) We say that N is *uniformly s-linearly unbounded* iff for all $M \in [M_0\rangle$ there exists some $\gamma \in \mathbb{N}$ and some marking $M' \in [M\rangle_\gamma$ such that $M'(s) \geq \gamma$.
iii) We say that t is *β-live* iff for any reachable marking M there exists a marking $M' \in [M\rangle_\beta$ enabling the transition t. We say that N is β-live iff all its transitions are β-live.
iv) We say that a marking M of N is *dead* iff there is no transition enabled at M. We say that N *can β-deadlock* iff there is some dead marking $M' \in [M_0\rangle_\beta$.
□

We could also extend the usual concepts of liveness and deadlock freeness to TPN's in an straightforward way. These properties can be proved to be decidable in an easy way, by repeating the reasoning followed in our previous proof of decidability of the Reachability Problem for TPN's.

4 Simulation of TPN's with Ordinary Petri Nets

To obtain this simulation we are going to split the transitions of the timed net into pieces of duration 1, and then to apply prop.1.

The construction is presented in a gradual way. First the wo-finite timed step sequence semantics is considered, getting a simulation which codifies the wo-finite timed step sequences of the original net by the occurrence sequences of the constructed net. In a second step, the construction is extended to cover the wsf-finite timed step sequence semantics; and finally, the general firing rule is also studied.

In this construction we do not represent the passage of time by means of an special transition; instead we have an implicit clock, represented by the number of tokens over a distinguished place. Each token added to this place represents the passage of one unit of time (that is to say the execution of one step) in the original Timed Net.

We begin by introducing a definition borrowed from [8], which allows us to model any finite state automaton by means of an Ordinary Petri Net.

Definition 9 (Petri Net modelling a Finite State Automaton) Let $\mathcal{A} = (Q', \Sigma', \Theta', \delta', \Gamma', q_0)$ be a finite state automaton, where

Q' : Finite set of *states*
Σ' : *Input* alphabet
Θ' : *Output* alphabet
δ' : State *transition function* : $\delta' : Q' \times \Sigma' \longrightarrow Q'$
Γ' : *Output function* : $\Gamma' : Q' \times \Sigma' \longrightarrow \Theta'$
q_0 : *Initial state*, $q_0 \in Q'$

verifying that Q', Σ' and Θ' are disjoint each other. Then we define the Ordinary Petri Net $N(\mathcal{A}) = (P, T, F, W, M_0)$ modelling the behaviour of $\mathcal{A}$ as follows:

$$P = Q' \cup \Sigma' \cup \Theta'$$
$$T = \{(q,\sigma) | q \in Q' \wedge \sigma \in \Sigma'\}$$
$$F = \{(q,t) | q \in Q' \wedge t = (q,\sigma) \in T\} \cup \{(\sigma,t) | \sigma \in \Sigma' \wedge t = (q,\sigma) \in T\} \cup \{(t,q') | q' = \delta'(q,\sigma) \in Q' \wedge t = (q,\sigma) \in T\} \cup \{(t,\theta) | t = (q,\sigma) \in T \wedge \theta = \Gamma'(q,\sigma) \in \Theta'\}$$
$$W(f) = 1,\ \forall f \in F$$
$$M_0(p) = \begin{cases} 1 & \text{if } p = q_0 \\ 0 & \text{otherwise} \end{cases}$$
□

4.1 Case 1: wo-finite timed step sequence semantics

The construction is based on a finite state automaton controlling the system evolution.

Definition 10 (Non-Timed Petri Net associated to a TPN) Let $N = (P, T, F, W, \delta, M_0)$ be a TPN. The *Non-Timed Petri Net associated to* N is the net $N'' = (P'', T'', F'', W'', M_0'')$ obtained by applying the following construction.

First we define the automaton $\mathcal{A} = (Q', \Sigma', \Theta', \delta', \Gamma', q_0)$ that controls the evolution of N, as follows:

$$Q' = \{(q_1, \ldots, q_n) \in \mathbb{N}^n | n = |T| \wedge 0 \leq q_j \leq \delta(t_j) - 1\}$$
$$q_0 = (0, \ldots, 0)$$
$$\Sigma' = \{(A, in) | A \in \mathcal{P}(T)\}$$
$$\Theta' = \{(out, A) | A \in \mathcal{P}(T)\}$$
$\delta' : Q' \times \Sigma' \longrightarrow Q'$, $\Gamma' : Q' \times \Sigma' \longrightarrow \Theta'$, defined by
(i) $\delta'(q, (A, in))$ is defined iff $\forall t_j \in A,\ q_j = 0$.
In such a case $\delta'(q, (A, in)) \doteq q'$, where:

$$q'_i \doteq \begin{cases} q_i \dot{-} 1 & \text{if } t_i \notin A \\ \delta(t_i) - 1 & \text{if } t_i \in A \end{cases}$$

where $\dot{-}$ represents the *corrected substraction*, defined by $x \dot{-} y = Max\{0, x - y\}$
(ii) $\Gamma'(q, (A, in)) \doteq (out, C)$, where

$$C = \{(t_j \in T | q' = \delta(q, (A, in)) \wedge ((q_j > 0 \wedge q'_j = 0) \vee (q_j = 0 \wedge t_j \in A \wedge \delta(t_j) = 1))\}$$

The states of this automaton represent the transitions in execution in the markings of the original Timed Net. Each component of a state tells us if there is some executing instance of each transition, and in such a case the time left till the conclusion of its execution. In particular, the initial state corresponds to any stable marking. The inputs of the automaton represent the steps to be executed at each moment, while the outputs indicate the sets of transitions terminating at each instant. In particular, the empty set in Σ' corresponds to the passage of time, that is to say to empty steps. Finally functions δ' and Γ' codify the firing rule of the original Timed Net.

Let $N(\mathcal{A}) = (P', T', F', W', M_0')$ be now the Non-Timed Petri Net modelling the automaton that we have just defined. We define the Non-Timed Petri Net $N'' = (P'', T'', F'', W'', M_0'')$ representing (under the current restriction) the Timed Petri Net in the following way:

$$P'' = P \cup P' \cup \{clock\}$$
$$T'' = T'$$
$F'' = F_1'' \cup F_2''$, where:
$$F_1'' = \{f' \in F' | f' = (q', t') \vee f' = (t', q'),\ \text{where } q' \in Q',\ t' \in T'\}$$
$$F_2'' = \{(p, t') | p \in P \wedge t' \in T' \wedge t' = (q', (A, in)) \wedge t \in A \wedge t \in p^\bullet\} \cup \{(t', p) | p \in P \wedge t' \in T' \wedge$$

$$(t',(out,A)) \in F' \wedge p \in t^\bullet \wedge t \in A\} \cup \{(t',clock) | t' \in T'\}$$

$$W''(f) = \begin{cases} 1 & \text{if } f = (t',clock) \vee f \in F_1'' \\ \sum_{t_j \in A \cap p^\bullet} W(p,t_j) & \text{if } f = (p,t') \in F_2'' \wedge t' = (q',(A,in)) \\ \sum_{t \in T_A} W(t,p) & \text{if } f = (t',p) \in F_2'', \text{ where } T_A = \{t \in A | p \in t^\bullet\} \text{ with } (t',(out,A)) \in F' \end{cases}$$

$$M_0''(p) = \begin{cases} 0 & \text{if } p = clock \\ 1 & \text{if } p \text{ corresponds to the initial state of the automaton} \\ M_0(p) & \text{if } p \in P \\ 0 & \text{otherwise} \end{cases}$$

□

Definition 11 Let N be a TPN and N'' the associated Non-Timed Petri Net. We denote by $\mathcal{M}$ the set of markings of N, and by $\mathcal{M}'$ the set of markings of N''. For any marking $M = (M_1, M_2)$ in $\mathcal{M}$ let $\mathcal{S}(M) = (q_1, \ldots, q_n)$ be the associated state in Q', which is defined as follows:

$$q_j = \begin{cases} 0 & \text{if } \nexists \gamma \text{ such that } M_2(t_j,\gamma) > 0 \\ \gamma & \text{if } \exists \gamma \text{ such that } M_2(t_j,\gamma) > 0 \end{cases}$$

Finally, we define the *marking correspondence function* $\varphi_N^\beta : \mathcal{M} \longrightarrow \mathcal{M}''$ associated to N for an instant β, in the following way:

$$\varphi_N^\beta(M)(p) \doteq \begin{cases} \beta & \text{if } p = clock \\ 1 & \text{if } p = \mathcal{S}(M) \\ M_1(p) & \text{if } p \in P \\ 0 & \text{otherwise} \end{cases}$$

□

Theorem 3 Let $N = (P,T,F,W,\delta,M_0)$ be a TPN and $N'' = (P'',T'',F'',W'',M_0'')$ the associated Non-Timed Net. Then for any two markings of N, M_1, M_2, and for any set of transitions $R \subseteq T$ we have:

$$M_1[R\rangle M_2 \text{ iff } \varphi_N^\beta(M_1)[(\mathcal{S}(M_1),(R,in))\rangle\varphi_N^{\beta+1}(M_2) \quad (\forall \beta \in \mathbb{N})$$

Proof: It is an immediate application of Def. 3. □

Definition 12 Let $N = (P,T,F,W,\delta,M_0)$ be a TPN and $N'' = (P'',T'',F'',W'',M_0'')$ the associated Non-Timed Petri Net according to the wo-finite timed step sequence semantics. The function relating both semantics, $\psi : S(N,M_0) \longrightarrow L''(N'',M_0'')$, is defined for each $\sigma = M_0[B_0\rangle \ldots M_{n-1}[B_{n-1}\rangle M_n \in S(N,M_0)$ by

$$\psi(\sigma) = \varphi_N^0(M_0)t_0'\varphi_N^1(M_1)\ldots t_{n-1}'\varphi_N^n(M_n)$$

where $t_j' = (\mathcal{S}(M_j),(B_j,in))$, for $j = 0,\ldots,n-1$. □

Corollary 1 Let $N = (P,T,F,W,\delta,M_0)$ be a TPN, N'' the associated Non-Timed Net according to the wo-finite timed step sequence semantics and $\sigma = B_1^{(\beta_1)} \ldots B_r^{(\beta_r)}$, with $B_j \subseteq T$, $B_j \neq \emptyset$, $\beta_j \in \mathbb{N}$, $\forall j \in \{1,\ldots,r\}$ and $\beta_1 < \ldots < \beta_r$. Then:

$M_0[\sigma^{(n)}\rangle M_n$ if and only if $\varphi_N^0(M_0)[\psi(\sigma)\rangle\varphi_N^n(M_n)$ □

Next, using this restricted semantics we study the decidability of the properties that we have introduced in section 3. Firstly, since we cannot have overlapped executions of each transition, the computations tree is finitary, and thus the strict reachability problem, the β-liveness problem, and the β-deadlock freeness problem, are trivially decidable, just by exploring that tree up to depth β.

Also, s,k-linear unboundness and uniform s-linear unboundness properties can be decided using the given simulation.

Proposition 2 Let $N = (P,T,F,W,\delta,M_0)$ be a Timed Petri Net, and $N' = (P',T',F',W',M_0')$ the associated Ordinary Petri Net, following Def. 10. We have:

i) N is s,k-linearly unbounded iff we have some $M' \in [M_0'\rangle$ with $M'(s) \geq M'(clock) \geq k$.

ii) N is uniformly s-linearly unbounded iff for all $M' \in [M_0'\rangle$ there exists some $M'' \in [M'\rangle$ with $M''(s) \geq M''(clock)$.

Proof: Immediate consequences of Cor. 1. □

Corollary 2 The s,k-linear unboundness property, and the uniform s-linear unboundness property of Timed Petri Nets with their wo-finite timed step sequence semantics, are both decidable.

Proof: As concerns the first property, we can decide the property of Ordinary Petri Nets to which we have reduce it, by adding to the net a new transition firable when time k has ellapsed, and whose effect is the removal of one token both from s and *clock*. Then we have that N' has the desired property iff a marking with no tokens in the place *clock* is reachable.

For the second property, we add to N' the same transition as in the former case, and we have that it has the desired property iff the set of markings with no tokens in the place *clock* is a home space of the net, property that has been proved to be decidable in [2,3]. □

4.2 Case 2: wsf-finite timed step semantics

Now our aim is to generalize the construction presented in the previous section, to wsf-finite timed step semantics.

The necessary modifications are not too deep, because although we can have several instances of the same transition executing simultaneously, we cannot have two such executions beginning at the same time. Thus the differences between the previous construction and the current one will be based on the substitution of tuples indicating if there is some pending execution of each transition, by sets of pairs (t,n), whose elements indicate that there is some pending execution of t that will terminate after time n.

Definition 13 (Non-Timed Petri Net associated to a TPN) Let $N = (P,T,F,W,\delta,M_0)$ be a TPN. We define the automaton $\mathcal{A}$ controlling the system evolution corresponding to the wsf-finite timed step sequence semantics of N as follows:

$$\mathcal{A} = (Q', \Sigma', \Theta', \delta', \Gamma', q_0)$$

where:

$Q' = \mathcal{P}(\{(t,n) | t \in T \ \wedge \ n \in \{1, \ldots, \delta(t)-1\}\})$
$q_0 = \emptyset$
$\Sigma' = \{(A, in) | A \in \mathcal{P}(T)\}$
$\Theta' = \{(out, A) | A \in \mathcal{P}(T)\}$
$\delta' : Q' \times \Sigma' \longrightarrow Q'$, $\Gamma' : Q' \times \Sigma' \longrightarrow \Theta'$, defined by
(i) $\delta'(q, (A, in)) \doteq \{(t,n) | n \geq 1 \wedge (t, n+1) \in q\} \cup \{(t, \delta(t)-1) | t \in A \wedge \delta(t) > 1\}$
(ii) $\Gamma'(q, (A, in)) \doteq \{t \in T | (t, 1) \in q \vee (t \in A \wedge \delta(t) = 1)\}$

Again, the desired Non-Timed Petri Net is obtained by applying first the construction in Def.9 to the automaton just defined, and then constructing the corresponding net N'' exactly as in Def.10. □

The algorithms to decide the properties in Section 3 are analogous to the corresponding ones for the case of wo-finite step sequence semantics. In particular, the computations tree is still finitary, and thus most (the same that in the former case) of the properties can be decided just by exploring it up to the given depth.

4.3 General construction

In this section we will study TPN's with their unrestricted step semantics. In this case the simulation cannot follow the same procedure as in the preceding cases, since in general we would obtain a net having an infinite number of places, if doing so. Thus to develop the construction we will follow a rather different idea. The first step will be to consider Non-Timed Nets, with their step semantics, showing that this semantics can be simulated by the occurrence sequence semantics of an associated Non-Timed Petri Net.

Definition 14 Let $N = (P, T, F, W, M_0)$ be a Non-Timed Petri Net, M a marking for it, $\beta, k \in \mathbb{N}$ and $s \in P$.
i) We say that M *is reachable in* N *in* β *steps*, which will be denoted by $M \in [M_0\rangle_\beta$, iff there exists a sequence σ of non-empty steps of length β such that $M_0[\sigma\rangle M$. Sometimes we will also accept empty steps, which is of course equivalent to allow sequences (of non-empty steps) of length less than or equal to the given number of steps β. The corresponding reachability relation will be denoted by writing $M \in [M_0\rangle_\beta^{\leq}$.
ii) We say that N is *s, k-linearly unbounded* iff there exists some $\gamma \geq k$, and some marking M such that $M \in [M_0\rangle_\gamma$ and $M(s) \geq \gamma$.
iii) We say that N is *uniformly s-linearly unbounded* iff for all $M \in [M_0\rangle$ there exists some $\gamma \in \mathbb{N}$ and some marking $M' \in [M\rangle_\gamma$ such that $M'(s) \geq \gamma$.
iv) We say that $t \in T$ is *β-live* iff for any reachable marking M there exists another one M' such that $M' \in [M\rangle_\beta$ enabling t. We say that N is *β-live* iff all its transitions are β-live.
v) We say that a marking M of N is *dead* iff there is no transition enabled at M. We say that N *can β-deadlock* iff there is some dead marking $M' \in [M_0\rangle_\beta$.
□

To prove the decidability of these problems, we will construct for each net another one that simulates (taking into account the length of sequences) the step semantics of the first one, by means of the ordinary occurrence sequence semantics of the constructed net.

Definition 15 (Construction 1)
Let $N = (P, T, F, W, M_0)$ be a Non-Timed Petri Net. We define the following sets of (new) places and transitions:

$$\bar{P} = \{\bar{p} | p \in P\}$$
$$T_p^1 = \{t_p^1 | p \in P\}$$
$$T_p^2 = \{t_p^2 | p \in P\}$$

Then we define the *net* N' *associated to* N, by $N' = (P', T', F', W', M_0')$, where

$P' = P \ \cup \ \bar{P} \ \cup \ \{f_{step}, c_{step}, clock, \bar{p}_s\}$
$T' = T \ \cup \ T_p^1 \ \cup \ T_p^2 \ \cup \ \{n_{step}, t^1, t^2, r\}$
$F' = \{(t, \bar{p}), (t_p^1, p), (t_p^2, p), (\bar{p}, t_p^1), (\bar{p}, t_p^2), (f_{step}, t_p^1), (c_{step}, t_p^2), (t_p^2, c_{step}) | (t,p) \in F\} \cup \{(f_{step}, t), (t, f_{step}) | t \in T\} \cup \{(p,t) | (p,t) \in F\} \cup \{(c_{step}, n_{step}), (n_{step}, f_{step})\} \cup \{(t_p^1, clock), (t_p^1, c_{step}) | p \in P\} \cup \{(t, \bar{p}_s) | t \in T \wedge t^\bullet = \emptyset\} \cup \{(t^1, c_{step}), (t^2, c_{step}), (c_{step}, t^2), (f_{step}, t^1), (\bar{p}_s, t^1), (t^1, clock), (\bar{p}_s, t^2), (r, clock)\}$

$$W(f) = \begin{cases} W(t,p) & \text{if } f = (t, \bar{p}),\ p \in P \\ W(p,t) & \text{if } f = (p,t) \in F \\ 1 & \text{otherwise} \end{cases}$$

$$M_0'(p) = \begin{cases} M_0(p) & \text{if } p \in P \\ 1 & \text{if } p = f_{step} \\ 0 & \text{otherwise} \end{cases}$$

□

The construction is illustrated by figure 1.

Definition 16 Let $N = (P, T, F, W, M_0)$ be a Non-Timed Petri Net and $N' = (P', T', F', W', M_0')$ the associated net according to the previous construction. For each $\beta \in \mathbb{N}$ we define the *marking correspondence function* $\varphi_N^\beta : \mathcal{M} \longrightarrow \mathcal{M}'$ as follows:

$$\varphi_N^\beta(M)(p') \doteq \begin{cases} M(p) & \text{if } \ p' = p \in P \\ 1 & \text{if } \ p' = f_{step} \\ \beta & \text{if } \ p' = clock \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{M}$ and $\mathcal{M}'$ are the sets of markings of N and N' respectively. □

Definition 17 (Multiset associated to an occurrence sequence) Let N be an ordinary Petri Net and σ an occurrence sequence of it. We define the multiset associated to σ $B(\sigma) : T \longrightarrow \mathbb{N}$ as follows:

$$B(\sigma)(t) \doteq \begin{cases} 0 & \text{if } \ \sigma = \langle\rangle \\ B(s)(t) + 1 & \text{if } \ \sigma = \langle s \rangle \circ t \\ B(s)(t) & \text{if } \ \sigma = \langle s \rangle \circ t',\ t' \neq t \end{cases}$$

□

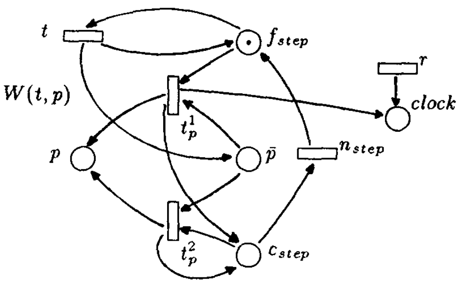

Figure 1: Construction 1

Theorem 4 Let $N = (P, T, F, W, M_0)$ be an Ordinary Petri Net and N' the associated net according to Def.15. Then, given two markings of N, M, M_1, and given a multiset B of transitions of T, we have $M[B\rangle M_1$ if and only if there exists an occurrence sequence σ in N' such that:

1. $\varphi_N^\beta(M)[\sigma\rangle\varphi_N^{\beta+1}(M_1), \quad \forall\beta \in \mathbb{N}$
2. $B(\sigma)(t) = B(t) \quad \forall t \in T$

Proof: Let us suppose that $M[B\rangle M_1$. Then, if B is empty it is sufficient to take as σ the transition r in N'. If B is not empty, then since we can fire together in N all its elements from the marking M, then in N' we can fire them in a row in any order, from the marking $\varphi_N^\beta(M)$. After that we distinguish two cases, depending on if there is some transition in B, t, with $t^\bullet \neq \emptyset$. If there is some such transition, we take any of them t, selecting any of its output places s, and firing the transition t_s^1. Next, we fire as many times as possible all the transitions in $\{t_p^2 | p \in P\} \cup \{t^2\}$, until all the places in $\{\bar{p} | p \in P\} \cup \{\bar{p}_s\}$ are empty. Finally we fire the transition n_{step} to recover the token over f_{step}, which allows the firing of a new bag of transitions. If there is no transition t in B with $t^\bullet \neq \emptyset$ we first fire t^1 to add a new token to the clock, and after t^2 until $\bar{p}_s$ becomes empty.

For the converse, we have that either σ is r, and then it corresponds to an empty step in N, or $\sigma = t_1 t_2 \ldots t_s t_{p_1}^1 t_{p_2}^2 \ldots t_{p_r}^2 n_{step}$, with $s > 0$. Since the firing of transitions from T in N' remove from their preconditions the same number of tokens that the corresponding firings in N, but it puts no tokens over the original places in P, but over the associated ones in $\bar{P}$, then it is clear that we can fire in N at M the multiset of transitions $B(t_1 t_2 \ldots t_s)$, getting some marking M_2. But, by hypothesis after firing σ we get some marking $\varphi_N^{\beta+1}(M_1)$, and this implies that the transitions in $\{t_{p_1}^1\} \cup \{t_{p_j}^2 | j = 2, \ldots, r\}$ exactly correspond to the displacement to the original places in P of the tokens over the places in $\bar{P}$, generated by the firing of the transitions in $t_1 t_2 \cdots t_s$. As a consequence M_2 must be equal to M_1, which concludes the proof. □

A bit surprisingly, this result cannot be generalized to sequences of steps. This is due to the fact that the corresponding first condition for the desired occurrence sequence σ, would be

$$\varphi_N^\beta(M)[\sigma\rangle\varphi_N^{\beta+n}(M_1), \quad \forall\beta \in \mathbb{N}$$

where n is the length of the given step sequence. The problem is that nothing is said about the intermediate markings, so that it is perfectly possible to increase the number of tokens on the *clock* place without firing any transition from the original net. This is not only due to the existence of the transition r. Even if we remove it the problem remains, because we are not obliged to return all the tokens over the places in $\{\bar{p} | p \in P\}$, before the beginning of the next *cycle* of a computation of N'. But in such a case, this next cycle can begin with the execution of a t_p^1 transition, and so a new token would be added to the place *clock* without having executed any new t transition simulating those in N.

This implies that in the net N' (even with the removal of the transition r) the firing of transitions in T can occur *more slowly* than in N'. To be exact, if we have in N' without r that $\varphi_N^\beta(M)[\sigma\rangle\varphi_N^{\beta+n}(M_1)$, then we have in N, $M_1 \in [M_0\rangle_n^{\leq}$. But the converse would not be true without the introduction of the transition r that allows us to leave the *time* pass whenever we desire to do it. That is why we introduced it, arriving at the following theorem, which needs a previous definition.

Definition 18 (Step sequence of N corresponding to an occurrence sequence of N') Let $N = (P, T, F, W, M_0)$ be an Ordinary Petri Net and N' the associated net following Def.15. If for two markings of N, M and M' we have $\varphi_N^\beta(M)[\sigma\rangle\varphi_N^{\beta+n}(M_1)$, where $\sigma = \sigma_1 \ldots \sigma_s$ is an occurrence sequence in N', we define

1. The *tic subsequence* of σ, $tic(\sigma) = \sigma_{i_1} \cdots \sigma_{i_n}$, as that constituted by the transitions along it in the set $\{t_p^1 | p \in P\} \cup \{t^1, r\}$
2. The *step guided decomposition* of σ is that defined by $\sigma = \sigma^1 \circ \cdots \circ \sigma^n \circ \sigma^{n+1}$ where for each $j \in \{1, \ldots, n\}$ we have $\sigma^j = \sigma_{i_{j-1}+1} \cdots \sigma_{i_j}$, taking $i_0 = 0$, and $\sigma^{n+1} = \sigma_{i_n+1} \ldots \sigma_s$.
3. The *step sequence* of N (possibly containing empty steps) *simulated* by σ is that defined by $BS(\sigma) = B(\sigma^1) \cdots B(\sigma^n)$

□

Theorem 5 Let $N = (P, T, F, W, M_0)$ be an Ordinary Petri Net, N' the net associated to it following Def. 15, and $n \in \mathbb{N}$. If for two markings of N, M and M' we have $\varphi_N^\beta(M)[\sigma\rangle\varphi_N^{\beta+n}(M_1)$, then in N we have $M[BS(\sigma)\rangle_n^{\leq} M_1$.

Proof: (sketch) As we said before, it is possible that the displacement of the tokens on the places in $\{\bar{P} | p \in P\}$ to the corresponding original places, will not be done in the same "cycle" in which they were generated, but later on. But it is easy to check that all these displacements can be moved forward to the corresponding cycles that generated them, and then we can apply Th. 4, to conclude the proof. □

Corollary 3 Let $N = (P, T, F, W, M_0)$ be an Ordinary Petri Net, M a marking of it, $s \in P$, and $\beta \in \mathbb{N}$.

1. We can decide the Strict Reachability Problem when we allow empty steps.
2. We can decide the s, k-linear unboundness property and the uniform s-linear unboundness property when we allow empty steps.

□

Theorem 6 Let $N = (P, T, F, W, \delta, M_0)$ be an Ordinary Petri Net, and $\beta \in \mathbb{N}$. We can decide if N is β-live.

Proof: We consider the net N' associated to N by applying Def. 15. Then let $t \in T$; the β-liveness of t in N can be decided by adequating the process followed in [2] to prove the decidability of (usual) liveness for Ordinary Petri Nets.

In that paper it is shown that for any net N, we can obtain a finite set of reachable markings of N $\{M_B^1, \ldots, M_B^b\}$ verifying that for any reachable marking M' there is some $i \in \{1, \ldots, b\}$ such that M' covers M_B^i (which means $\forall p \in P\ M'(p) \geq M_B^i(p)$ and is denoted by $M' \geq M_B^i$).

We apply this result to our net N'_β, getting the set $\{M_B^1, \ldots, M_B^b\}$. Then, we consider the marking M^t associated to t in N'_β, defined by: $M^t(p) = W'(p, t)$, $\forall p \in P'$.

We have to test, if for all $i \in \{1, \ldots, b\}$ there is some $M'' \in [M_B^i\rangle$ such that $M''(clock) = M_B^i(clock) + \beta$ and $M^t \geq M''$.

For we define a net $N'^i_\beta = (P', T', F'', W'', M'^{\beta,i}_0)$ associated to N, which is obtained from N' by *reversing* time, which means to reverse all the arrows in F' reaching the place *clock*, initializing this place with β tokens, and the rest of the places with their number of tokens in M_B^i. Then we test if there is some $M'' \in [M'^{\beta,i}_0\rangle$ such that $M^t \geq M''$. This can be done by studying the coverability tree of N'^i_β.

If any of these tests fails, then t is not β-live in the original net N. Otherwise, it is, because for any reachable marking M_1 in N there is some $i \in \{1, \ldots, b\}$ such that $M_B^i \geq M_1$. Then, we have some reachable marking M'' from this marking M_B^i such that $M''(p) \geq M^t(p)$, for all $p \in P$ and $M''(clock) = M_B^i(clock) + \beta$. Now the occurrence sequence σ such that $M_B^i[\sigma\rangle M''$ can be fired from M_1, as it covers M_B^i, leading us to a marking covering M'', and with β tokens more over the place *clock* than the original marking. Thus the corresponding marking can be reached in N by an step sequence of length β, and t will be enabled at this marking, thus proving that the transition t is β-live in N. □

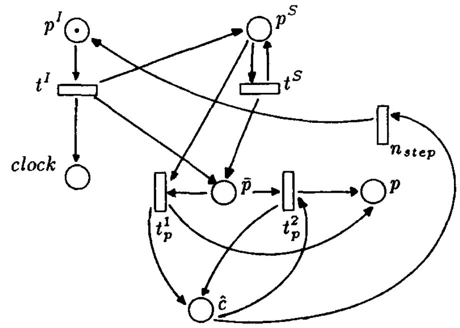

Figure 2: Construction 2

Note that we cannot solve, at least in a direct way, the β-deadlock freeness problem with this first construction, because the transition r is autonomous, and thus the associated net cannot get deadlocked.

Next we present a second construction that eliminates the discussed problem, allowing us to simulate correctly the step semantics of a given net, without the allowance of empty steps.

Definition 19 (Construction 2)
Let $N = (P, T, F, W, M_0)$ be a Non-Timed Petri Net. We consider the following sets of (new) places and transitions:

$$\bar{P} = \{\bar{p} | p \in P\}$$
$$T^I = \{t^I | t \in T\}$$
$$T^S = \{t^S | t \in T\}$$
$$T_p^k = \{t_p^k | p \in P\} \ \ for\ k = 1, 2$$

We define then the associated net $N' = (P', T', F', W', M'_0)$, as follows:

$$P' = P \cup \bar{P} \cup \{\hat{c}\} \cup \{p^I, p^S, clock, \bar{p}_s\}$$
$$T' = T^I \cup T^S \cup T_p^1 \cup T_p^2 \cup \{n_{step}, t^1, t^2\}$$
$$F' = \{(p^S, t^S), (t^S, p^S), (p^I, t^I), (t^I, p^S) | t \in T\} \cup \{(t^I, \bar{p}), (t^S, \bar{p}) | (t, p) \in F\} \cup \{(p^S, t_p^1), (\bar{p}, t_p^1), (\bar{p}, t_p^2), (t_p^1, \hat{c}), (\hat{c}, t_p^2), (t_p^2, \hat{c}), (t_p^1, p), (t_p^2, p) | p \in P\} \cup \{(t^I, clock) | t \in T\} \cup \{(p, t^I), (p, t^S) | (p, t) \in F\} \cup \{(t^I, \bar{p}_s), (t^S, \bar{p}_s) | t \in T, t^\bullet = \emptyset\} \cup \{(\bar{p}_s, t^1), (t^1, \hat{c}), (\hat{c}, t^2), (t^2, \hat{c}), (\bar{p}_s, t^2), (p^S, t^1), (\hat{c}, n_{step}), (n_{step}, p^I)\}$$

$$W'(f) = \begin{cases} W(p,t) & \text{if } f = (p, t^I) \vee f = (p, t^S), \\ & t \in T \\ W(t,p) & \text{if } f = (t^I, \bar{p}) \vee f = (t^S, \bar{p}) \\ 1 & \text{otherwise} \end{cases}$$

$$M'_0(p) = \begin{cases} M_0(p) & \text{if } p \in P \\ 1 & \text{if } p = p^I \\ 0 & \text{otherwise} \end{cases}$$

□

This construction is illustrated by figure 2. The key idea to understand it is that each step B of the original net N is simulated by the following sequence of firings: first an arbitrary transition $t \in B$ is selected, and the corresponding transition t^I fired, which updates the clock to reflect the execution of the step. The role of places in $\{\bar{p}|p \in P\}$ is analogous to that of the same set in construction 1. So, after the firing of t^I it is only possible to fire a sequence of transitions from T^S that can be executed together at the same time that the fired t (in particular any sequence constituted by the transitions in $B - \{t\}$) until some t_p^1 transition will be executed, which disallows the firing of transitions Tt^S, and thus implies that the next execution of a transition associated to those of the original net will correspond to a new step.

Observe that since time only passes when some transition from T^I is executed, empty steps are not possible at all.

By means of this net we can simulate again the step semantics of N, obtaining the corresponding versions of theorems 4 and 5, thus concluding the following corollary.

Corollary 4 We can decide the strict reachability problem, the s, k-linear unboundness property and the uniform s-linear unboudness property. □

However, this second construction raises a new problem, which is the possible introduction of new deadlocks in the constructed net. As a consequence, liveness can be lost. The reason being that if the places in $\bar{P}$ are not completely empty before the firing of transition n_{step}, then it is possible that there will not be in the original places of the net enough tokens to fire any transition in T, thus becoming blocked.

As a consequence β-deadlock freeness and β-liveness cannot be decided, at least in a direct way, by using this simulation. Fortunately we have a third proposal, which has neither this problem, nor the one commented on in our first construction.

Definition 20 (Construction 3)
Let $N = (P, T, F, W, M_0)$ be an Ordinary Petri Net, $\bar{P}$, T^I, T^S, T_p^1, T_p^2 the sets introduced in Def. 19 for this net and N' the net associated to N according to that construction. Now we consider a new set, $T_p^3 = \{t_p^3|p \in P\}$, and we define the *net N'' associated to N* as follows:

$$N'' = (P', T' \cup T_p^3 \cup \{t^3\}, F' \cup \tilde{F}, W'', M_0')$$

where:

$$\tilde{F} = \{(\bar{p}, t_p^3), (p^I, t_p^3), (t_p^3, p^I), (t_p^3, p)|p \in P\} \cup \{(p^I, t^3), (t^3, p^I) (\bar{p}_s, t^3)\}$$

$$W''(f) = \begin{cases} W'(f) & \text{if } f \in F' \\ 1 & \text{otherwise} \end{cases}$$

□

The only essential difference with respect to the previous construction is the introduction of a new transition for each place of the original net t_p^3, to allow us to empty the places $\bar{p}$, when they have some token after a simulation of a step. It can be only done when p^I is marked, that is, when there is no step in execution.

Again, the corresponding versions of Th. 4 and Th. 5 can be proved, which allows us to conclude the decidability of the β-liveness and β-deadlock freeness properties.

Corollary 5 The β-liveness property and the β-deadlock freeness property are both decidable.

Proof: The demonstration of the first one follows the same idea as those of the same results using the former constructions. The decidability of the β-deadlock freeness property can be solved by modifying the associated Ordinary Petri Net, adding a transition *life* which will be enabled forever whenever the place *clock* has (at least) β tokens. That is obtained by taking

$$W(clock, life) = \beta = W(life, clock)$$

Thus after time β we will have no deadlocks in the so modified net, and so it will have deadlocks iff the original net has β-deadlocks. □

4.4 Simulating Timed Petri Nets with their unrestricted semantics

The construction that we make to codify Timed Petri Nets with their general semantics, is based on the splitting of each transition t in $\delta(t)$ transitions, corresponding to each instant along the execution of t. Intuitively, in order that this codification will perfectly simulate the behaviour of timed nets, it seems that the application of the maximum parallelism hypothesis to the transitions corresponding to (original) transitions in execution is required. However, as we will see later, although we ignore this (in general impossible to be represented in the Ordinary Petri Net's world) restriction, the executions of transitions that do not obey this hypothesis can be seen as starting a little later, thus fulfilling the hypothesis, and so the simulation is meaningful.

Definition 21 (Net associated to a TPN)
Let $N = (P, T, F, W, \delta, M_0)$ be a TPN. For each $t \in T$ we will consider a set of (atomic) transitions $C_t = \{t_1, \ldots, t_{\delta(t)}\}$. Then we define the *Ordinary Petri Net associated to N*, $N' = (P', T', F', W', M_0')$, as follows:

$$P' = P \cup \bigcup_{t \in T} \{p_i^{(t)}|i = 1, \ldots, \delta(t) - 1\}$$

$$T' = \bigcup_{t \in T} C_t$$

$$\begin{aligned} F' = \ & \{(p, t_{\delta(t)})|p \in P \ \wedge \ t \in T \ \wedge \ (p, t) \in F\} \cup \\ & \{(p_i^{(t)}, t_i)|t \in T \ \wedge \ i \in \{1, \ldots, \delta(t) - 1\}\} \cup \\ & \{(t_1, p)|t \in T \ \wedge \ p \in P \ \wedge \ (t, p) \in F\} \cup \\ & \{(t_{i+1}, p_i^{(t)}|t \in T \ \wedge \ i \in \{1, \ldots, \delta(t) - 1\}\} \end{aligned}$$

$$W'(f) = \begin{cases} W(p, t) & \text{if } f = (p, t_{\delta(t)}) \\ W(t, p) & \text{if } f = (t_1, p) \\ 1 & \text{otherwise} \end{cases}$$

$$M_0'(p) = \begin{cases} M_0(p) & \text{if } p \in P \\ 0 & \text{otherwise} \end{cases}$$

□

Definition 22 Let $N = (P, T, F, W, \delta, M_0)$ be a TPN and $N' = (P', T', F', W', M_0')$ the associated Ordinary Petri Net. We define the *marking correspondence function* $\varphi_N : \mathcal{M} \longrightarrow \mathcal{M}'$ as follows:

$$\varphi_N(M)(p') \doteq \begin{cases} M_1(p) & \text{if } p' = p \in P \\ M_2(t,i) & \text{if } p' = p_i^{(t)}, \text{ for} \\ & 1 \le i \le \delta(t) - 1 \end{cases}$$

□

Definition 23 Let $N = (P, T, F, W, \delta, M_0)$ be a TPN and $N' = (P', T', F', W', M'_0)$ the associated net obtained according to the previous construction. We define the *step correspondence function* as follows:

$$\tau : \mathcal{M} \times \beta(T) \longrightarrow \beta(T')$$

$$\tau(M, B)(t') \doteq \begin{cases} B(t) & \text{if } t' = t_{\delta(t)},\ t \in T \\ M_2(t,\gamma) & \text{if } t' = t_\gamma,\ \gamma < t_{\delta(t)},\ t \in T \end{cases}$$

where $\beta(T)$ (resp. $\beta(T')$) is the set of all the multisets in T (resp. T'). □

Theorem 7 Let $N = (P, T, F, W, \delta, M_0)$ be a TPN and $N' = (P', T', F', W', M'_0)$ the associated net according to the construction in definition 21. Let M, M' be two markings of N and B a multiset of T. Then we have:

$M[B\rangle M'$ if and only if $\varphi_N(M)[\tau(M, B)\rangle\varphi_N(M')$ □

Corollary 6 Let $N = (P, T, F, W, \delta, M_0)$ be a TPN and N'' the net constructed over the net associated to N, N', by applying the construction of Def. 20. Then, $M = (M_1, M_2)$ is a reachable marking in N at the instant β if and only if there exists a reachable marking M' in N'' such that:

(1)	$M'(p) = M_1(p)$	$\forall p \in P$
(2)	$M'(p_i^{(t)}) = M_2(t, i)$	$\forall t \in T \wedge$ $\forall i : 1 \le i \le \delta(t) - 1\}$
(3)	$M'(clock) = \beta$	
(4)	$M'(p') = 0$	otherwise

Proof: The left to right implication is an immediate application of the equivalent theorems to Th.4 and Th.5 for the third construction. Let us look at the converse.

Let M' be a reachable marking in N'' verifying the conditions (1-4). For this marking there exists a marking $\tilde{M}$ in N' such that $\varphi_N^\beta(\tilde{M}) = M'$. Then, from the equivalent theorem to Th.5 for the third construction we obtain that $\tilde{M}$ is reachable in β steps. Besides, $\tilde{M} = \varphi_N(M)$. The step sequence in N' allowing us to reach $\tilde{M}$ in N' does not necessarily satisfy the maximum parallelism hypothesis restricted to the atomic transitions corresponding to the instants of transitions in execution. In consequence, we cannot apply Th.7 step by step to conclude that M is reachable in β steps in N. However, we can delay the activation times of the components of each execution of any (original) transition not satisfying the defined maximum parallelism condition. To be exact, we see when the last component of each execution has been fired, and we take as firing time for the original transition that time minus the duration of the transition plus one. Then we consider the firing in a row of the components of the transition from that instant, and so we obtain an equivalent step sequence satisfying this condition. Then we can apply Th.7 step by step, concluding that the marking M is reachable in β steps in N. □

Corollary 7 Let $N = (P, T, F, W, \delta, M_0)$ be a TPN, M a marking of it, and $\beta \in \mathbb{N}$. Then the following properties are decidable:

1. The strict reachability property.
2. The s-linear unboundness property and the uniform s-linear unboundness property.
3. The β-deadlock property
4. The β-liveness property.

The first sentence is an immediate consequence of the previous corollary. The other ones are obtained by considering that these properties are preserved by the codification to Ordinary Petri Nets with its step semantics, for which the decidability of these properties have been solved. □

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