Efficient Algorithms in Irregular Sampling of Band-Limited Functions

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Abstract

We discuss some recent algorithms for the iterative reconstruction of band-limited signals from irregularly sampled values. The emphasis is on quantitative aspects: we give explicit estimates for the required sampling density and for the rate of convergence of the iteration algorithm.

1 Introduction

It is well known that a band-limited signal $f$ is completely determined by its irregularly sampled values $f(x_i)$, provided that $f$ is sampled above the Nyquist rate (e.g., Landau or Beurling [1, 12]. Under this condition the sampling is also stable. Because of the lack of efficient computational methods statements that error-free recovery of a signal from its irregularly sampled values is possible in principle have had little impact in the engineering sciences.

For effective numerical work with irregular sampling one needs

1. an implementable algorithm for the reconstruction or approximation of the signal from its irregular samples,
2. information on how the sampling geometry or density affect the performance of the algorithm,
3. sharp estimates on the accuracy of the approximation or on the rate of convergence of the algorithm.

Currently several iterative reconstruction methods are used which meet these requirements to various degrees. For an overview and more references see [13, 4, 5].

The classical frame method of R. Duffin and A. Schaeffer [3] and its variations [2, 14, 17] work well when the sampling set is a perturbation of the regular oversampling. However, the convergence of this method is slow for random samplings where the local density is allowed to vary. Quantitative estimates for the rate of convergence are unknown.

The POCS method – projections onto convex sets – has been applied recently to signal and image restoration, i.e., approximation of a signal from a finite number of irregular samples [18, 16]. Estimates for the degree of the approximation or the speed of convergence are unknown.

A third class of iterative methods is based on the fact that every band-limited function $f$ satisfies a reproducing formula $f = f * g$ for an appropriate function $g$ [4, 5, 6]. In theory, these algorithms have all properties that are desired from an efficient algorithm – stability, locality, detailed error analysis. In numerical experiments these algorithms perform well or better than other methods, cf. [8] and H. Feichtinger’s contribution in this volume. On the other hand, our work so far has been merely qualitative. The size of the constants involved is not known, therefore it is not clear, for which sampling rates and how fast these algorithms converge.

This article is intended to vindicate the qualitative theory of [4, 5, 6]. We show that a simplified version of these algorithms allows for a quantitative theory of irregular sampling. In contrast to previous trial-and-error experiments which have to be carried out with most methods, we show that this iteration algorithm converges and yields a complete reconstruction of a band-limited signal from a randomly distributed sampling sequence, provided that the distance between adjacent sampling points is at most the Nyquist distance. We give an a priori estimate on the required number of iterations required to achieve a certain accuracy of the approximation to the original signal.

Section 1 presents this simple algorithm and investigates into its convergence properties. In Section 2 some variations are discussed without proof: a discrete implementation of the algorithm, increasing the speed...
of convergence, and a multivariate version of irregular sampling. Section 3 is devoted to the proof of the statements of Section 1.

2 The Reconstruction Algorithm

Let \( L^2(\mathbb{R}) \) denote the Hilbert space of square-integrable functions on \( \mathbb{R} \) with norm \( \|f\| = (\int_{-\infty}^{\infty} |f(x)|^2 dx)^{1/2} \). The Fourier transform is defined by \( \hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x} dx \).

The Hilbert space of band-limited functions of finite energy with spectrum \( \mathcal{S} = \text{support of the Fourier transform} \) in the interval \( [-\omega, \omega] \) is denoted by \( B^\omega \), the orthogonal projection from \( L^2(\mathbb{R}) \) onto \( B^\omega \) is given by \( (Pf)(\xi) = \chi_{[-\omega,\omega]}(\xi)\hat{f}(\xi) \) a.e., where \( \chi_S \) is the characteristic function of the set \( S \).

A sequence \( X = (x_i)_{i \in \mathbb{Z}} : x_{i-1} < x_i < x_{i+1} < \ldots \) is \( \delta \)-dense, if \( \sup_i (x_{i+1} - x_i) \leq \delta \). It is not required that the \( x_i \)'s are separated from each other by a minimal distance. This density definition is slightly weaker than Landau's \([12]\), which allows arbitrary gaps in the sampling sequence, but it is sufficient for most purposes. The Nyquist rate for \( B^\omega \) is \( \delta = \pi/\omega \).

Denote the midpoints by \( y_i = (x_{i+1} + x_i)/2 \) and set \( \chi_i = \chi_{[y_i, y_{i+1}]}(x) \). Then the \( \chi_i \)'s form a partition of unity, i.e. \( \sum_{i=\infty}^{\infty} \chi_i(x) = 1 \) for all \( x \).

Theorem 1 If \( \delta < \pi/\omega \), i.e., for a sampling rate of \( X \) higher than the Nyquist rate, every \( f \in B^\omega \) can be completely reconstructed from the sampling values \( f(x_i) \) on an arbitrary \( \delta \)-dense sampling sequence \( X \) by the following algorithm:

\[
\phi_0 = P(\sum_{i \in \mathbb{Z}} f(x_i) \chi_i) \tag{1}
\]

\[
\phi_{n+1} = P(\phi_n - \sum_{i \in \mathbb{Z}} \phi_n(x_i) \chi_i) \tag{2}
\]

and

\[
f = \sum_{n=0}^{\infty} \phi_n \tag{3}
\]

where all sums converge in \( L^2 \).

Remarks: 1. Since \( A^{-1} \) is bounded on \( B^\omega \), the stability of the reconstruction follows from (34): The sampling from \( X \) is stable, i.e.,

\[
\|f\| \leq C \| \sum_{i \in \mathbb{Z}} f(x_i) \chi_i \|_2 \tag{4}
\]

If \( X \) is separated, i.e. \( |x_i - x_j| \geq \alpha > 0 \), then obviously

\[
\|f\| \leq C' \left( \sum_{i \in \mathbb{Z}} |f(x_i)|^2 \right)^{1/2} \tag{5}
\]

2. Set \( e_i = A^{-1} P \chi_i \). Then the \( e_i \)'s are a bounded sequence in \( B^\omega \) and every \( f \in B^\omega \) has the representation

\[
f(x) = A^{-1} P(\sum_{i} f(x_i) \chi_i) = \sum_{i} f(x_i) e_i \tag{6}
\]

with convergence of the partial sums in the \( L^2 \)-norm.

3. It should be emphasized that the only condition on the sampling set \( X \) is its density. In contrast to the other constructive methods, neither separation nor any other structure of \( X \) is required. The convergence of the reconstruction of Duffin and Schaeffer \([3]\) by means of the "frame operator" and all methods based on it seem to be very slow, when the sampling density varies locally. The algorithm of Theorem 1 balances the local variation of the sampling density with the partition of unity \( \chi_i \) and thus adapts to local changes of the sampling density.

For more details and discussion we refer to \([10]\), for the error analysis of the algorithm see \([9]\).

Rate of Convergence of the Algorithm

Theorem 2 Let \( f_n = \sum_{k=0}^{n} \phi_k \) be the resulting approximation of \( f \) after \( n \) iterations of (2). Then

\[
\|f - f_n\| \leq \left( \frac{\delta \omega}{\pi} \right)^n + \frac{\pi + \delta \omega}{\pi - \delta \omega} \|f\|_2 \tag{7}
\]

Thus (7) allows to determine the number of iterations required for a certain accuracy. For instance, in order to achieve an accuracy of \( \|f - f_n\|/\|f\| < 0.001 \) with fourfold oversampling, i.e. \( \delta \omega = \pi/4 \), about 6 iterations are necessary. The geometric decay of the relative error has also been verified numerically \([8]\).

3 Variations

In this section we consider some variations of the algorithm of Section 1, which could be useful in applications. Since the ideas and proofs are similar to those of Section 1, we refer to \([10]\) for the precise details.

2.1 The Discrete Implementation

A numerical implementation requires a discrete model of the algorithm (1)-(3). Thus instead of a band-limited signal \( f \) living on \( \mathbb{R} \) we consider now a finite sequence \( s = (s(n))_{n=0,...,N-1} \) which is sampled at the
irregularly spaced points 0 \leq k_1 < k_2 < \ldots < k_m < N, k_i \in Z. Extending s periodically with period N, s(n + lN) = s(n) for all n, l \in Z, we interpret s as a function on the cyclic group \( Z_N \).

Since the discrete Fourier transform (DFT) on \( Z_N \) is available - given by 
\[
\hat{s}(k) = \frac{1}{N} \sum_{n=0}^{N-1} s(n) e^{-2\pi i nk/N},
\]
where \( e^{2\pi i/N} \) - a subspace of discrete band-limited signals can be defined similar to the continuous case as follows:
\[
B_\omega = \{ s : \hat{s}(k) = 0 \text{ for } |k| > \omega \}
\]
with norm \( \|s\| = (\sum_{n=0}^{N-1} |s(n)|^2)^{1/2} \). The orthogonal projection \( P \) from \( P(Z_N) \) onto \( B_\omega \) is given by 
\[
(Ps)(k) = \hat{s}(k) \text{ for } |k| \leq \omega \text{ and } (Ps)(k) = 0 \text{ for } |k| > \omega.
\]

Set \( k_{m+1} = N + k_1 \) and \( l_i = \lfloor (k_{i+1} + k_i)/2 \rfloor, i = 1, \ldots, m \) and let \( \chi_i \) be the characteristic function of the segment \( \{ l_{i-1}, \ldots, l_i \} \).

For this discrete model of band-limited signals the following irregular sampling theorem can be proved along the lines of Theorems 1 and 2.

**Theorem 3** Assume that \( \sup_{i=1, \ldots, m}(k_{i+1} - k_i) = d \) and that \( \omega d < N/2 \). Then any discrete band-limited signal \( s \in B_\omega \) is uniquely determined by the samples \( s(k_i), i = 1, \ldots, m \). It can be reconstructed by the following algorithm:

\[
\sigma_0 = P(\sum_{i \in \mathbb{Z}} s(k_i) \chi_i)
\]
\[
\sigma_{n+1} = P(\sigma_n - \sum_{i \in \mathbb{Z}} \sigma_n(k_i) \chi_i)
\]
\[
s = \sum_{n=0}^{\infty} \sigma_n
\]

If \( s_n = \sum_{k=0}^{n} \sigma_k \) and \( \gamma = \frac{\sin \frac{\pi}{N}}{\sin \frac{\pi}{2N}}, \) then
\[
\|s - s_n\| \leq \gamma^{n+1} \frac{1 + \gamma}{1 - \gamma}\|s\|
\]

It was in this form that we did the first numerical experiments with the reconstruction algorithm. The projection \( P \) is easily implemented by means of the DFT and its inverse. If \( N \) is a power of 2, then with the FFT each iteration can be calculated quite fast.

An alternative approach to compute uses the observation that each iteration \( \sigma_n \rightarrow \sigma_{n+1} \) is a linear transformation and thus can be realized by a matrix multiplication.

\( s \in B_\omega \) can be identified with a trigonometric polynomial of order \( 2\omega + 1 \). Since the number of samples is \( m \geq N/d \geq 2\omega + 1 \), the uniqueness of \( s \) is clear and one might as well use an interpolation method for the reconstruction. We do not know how this algorithm compares with interpolation methods, but we expect it to be more robust, especially for large values of \( N \) and \( \omega \).

### 2.2 Improving the Convergence Rate

The speed of convergence of the iteration can be increased, if piecewise linear interpolation instead of step functions are used in each iteration step.

If \( \delta < \pi/\omega \), then \( f \in B_\omega^2 \) can be reconstructed from any \( \delta \)-dense sampling sequence \( X = (x_i)_{i \in \mathbb{Z}} \), by means of the iteration procedure

\[
\phi_0 = \Phi \left( \sum_i f(x_i) \right)
\]
\[
\phi_{n+1} = \phi_n - \frac{1}{\pi} \left( \int_{x_0}^{x_{n+1}} f(x) \, dx - \int_{x_i}^{x_{i+1}} f(x) \, dx \right) \chi_{[x_i, x_{i+1}]}
\]

and

\[
f = \sum_{n=0}^{\infty} \phi_n
\]

where all sums converge in \( \mathbb{L}_2 \). Setting \( f_n = \sum_{k=0}^{2\omega} \phi_k \), the rate of convergence is

\[
\|f - f_n\| \leq \left( \frac{\delta \omega}{\pi} \right)^{2n+1} \frac{\pi^2 + \delta^2 \omega^2}{\pi^2 - \delta^2 \omega^2} \|f\|
\]

Consequently even for rates of oversampling this algorithm converges quite fast to the reconstruction. The version stated here is slightly sharper the one in [10], but is proved in a similar fashion.

### 2.3 Complete Reconstruction from Averages

In the physical world only a local average of the signal at \( x_1 \) can be measured, rather than the value of \( f \) at precisely \( x_1 \). The input for a signal reconstruction from sampled values is therefore local averages

\[
f_i = \int f(x) u_i(x) \, dx = \langle u_i, f \rangle
\]

Here the \( u_i, i \in \mathbb{Z} \), are a collection of averaging functions and satisfy the properties

\[
\sup u_i \leq \left[ x_i - \delta/2, x_i + \delta/2 \right], \quad 0 \leq u_i(x) \leq 1,
\]
\[ \int u_i(x)dx = 1 \]  

(18)

Observe that the averaging procedure may vary from point to point.

It is quite surprising that band-limited functions are completely determined by their local averages:

Assume that \( S = (x_i)_{i \in \mathbb{Z}} \) is a dense sampling set and that \( U_{\delta}(x) \) is a collection of averaging functions with properties (18). If \( \delta < (\sqrt{2} \omega)^{-1} \), then every \( f \in B_2^\omega \) is uniquely determined by the local averages \( (u_i, f) \) around \( x_i \). Moreover, \( f \) can be reconstructed by the following iteration scheme:

\[ \phi_0 = P \left( \sum_i (u_i, f) x_i \right) \]

(19)

\[ \phi_{n+1} = P \left( \phi_n - \sum_{i \in \mathbb{Z}} (u_i, \phi_n) x_i \right) \]

(20)

and

\[ f = \sum_{n=0}^{\infty} \phi_n \]

(21)

where all sums converge in \( L^2 \).

2.4. Irregular Sampling in Higher Dimensions

The algorithm of Section 1 can be adapted to higher dimensions, but the numerical constants are not yet optimal.

Let \( \Omega \) be the cube \( \Omega = \prod_{i=1}^{n} (\omega_i, \omega_i) \subseteq \mathbb{R}^n \) and \( B_2^\omega = \{ f \in L^2(\mathbb{R}^n) : f(\xi) = 0 \text{ for } \xi \not\in \Omega \} \) subspace of band-limited function with spectrum in \( \Omega \). The projection from \( L^2(\mathbb{R}^n) \) onto \( B_2^\omega \) is given by \( (Pf) = \chi_\Omega f \).

A sampling set \( X = (x_i)_{i \in \mathbb{Z}} \) in \( \mathbb{R}^n \) is said to be \( \delta \)-dense, if \( U_{\delta}(x_i) = \mathbb{R}^n \), where \( U_{\delta}(x) \) denotes the cube \( \prod_{i=1}^{n} (x_i - \delta_i/2, x_i + \delta_i/2) \) centered at \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \). (Note that for dimension \( n = 1 \) this definition differs slightly from the density used in the previous sections.) For every \( \delta \)-dense set \( X \) chose a partition of unity \( (\psi_i)_{i \in \mathbb{Z}} \) with the properties: \( \text{supp} \psi_i \subseteq B_\delta(x_i), 0 \leq \psi_i(x) \leq 1 \) and \( \sum_i \psi_i(x) \equiv 1 \). In analogy with Section 1 the partition of unity may consist of the characteristic functions \( \chi_{V_i} \) of the Voronoi regions \( V_i = \{ x \in \mathbb{R}^n : |x - x_i| \leq |x - x_j| \text{ for } j \neq i \} \) or the triangular tessellation as in Method 2 of [10].

Given \( \Omega \), choose \( \delta = (\delta_1, \ldots, \delta_n) \), such that \( \delta \cdot \omega = \sum_{i=1}^{n} \delta_i \omega_i < \ln 2 \). If \( X = (x_i)_{i \in \mathbb{Z}} \) is a \( \delta \)-dense sampling set in \( \mathbb{R}^n \), then every \( f \in B_2^\omega \) can be reconstructed from its sampled values \( f(x_i) \) by means of the following iteration procedure:

\[ \phi_0 = P \left( \sum_i f(x_i) \psi_i \right) \]

(22)

\[ \phi_{n+1} = P \left( \phi_n - \sum_i \phi_n(x_i) \psi_i \right) \]

(23)

and

\[ f = \sum_{n=0}^{\infty} \phi_n \]

(24)

where all sums converge in \( L^2(\mathbb{R}^n) \).

If \( \Omega = [-\omega_0, \omega_0]^n \) and \( \delta_1 = \ldots = \delta_n = \delta_0 \), then the density has to satisfy

\[ \delta_0 < \frac{\ln 2}{n \omega_0} \]

(25)

With \( n \) increasing, this is far from optimal, but currently it seems to be the only available estimate. For other, mostly qualitative approaches to multivariate irregular sampling see [2, 16, 5, 6].

4 Appendix — Proofs of Theorems 1 and 2

For the proof of Theorem 1 the following well-known inequalities are needed.

**Lemma 1 (Wirtinger's inequality)**

If \( f, f' \in L^2(a,b) \) and either \( f(a) = 0 \) or \( f(b) = 0 \), then

\[ \int_a^b |f(x)|^2 dx \leq \frac{4}{\pi^2} \frac{(b-a)^2}{a} \int_a^b |f'(x)|^2 dx \]

(26)

The Lemma follows from [11], p. 184, by a change of variables.

**Lemma 2 (Bernstein's inequality)**

If \( f \in B_2^\omega \), then \( f' \in B_2^\omega \) and

\[ ||f'|| \leq \omega ||f|| \]

(27)

**Proof of Theorem 1:** Define

\[ Af = P \left( \sum_{i \in \mathbb{Z}} f(x_i) x_i \right) \]

(28)

It is easily seen that \( A \) is a bounded linear operator from \( B_2^\omega \) into \( B_2^\omega \) (see also (32) below). The iteration step (2) requires an estimate on \( ||f - Af|| \) for general
By writing $f \in B_2^E$ as $f = Pf = P(\sum f(x_i))$, one obtains

$$\|f - Af\|^2 = \|P(\sum_i (f - f(x_i))\chi_i)\|^2 \leq$$

$$\leq \|\sum_i (f - f(x_i))\chi_i\|^2 = \int_R \|\sum_i (f(x) - f(x_i))\chi_i\|^2 dx$$

$$= \int \sum_i \|f(x) - f(x_i)\|\chi_i|^2 dx$$

Since the $\chi_i$'s are characteristic functions and have mutually disjoint support, the last expression equals

$$\int \sum_i \|f(x) - f(x_i)\|^2 \chi_i(x) dx = \int \sum_i \|f(x) - f(x_i)\|^2 \chi_i(x) dx$$

Next one applies Wirtinger's inequality to each term:

$$\int_y \|f(x) - f(x_i)\|^2dx = \int_{y_i}^{y_{i-1}} \cdots \int_{x_i}^{x_{i-1}} \cdots \leq$$

$$\leq \frac{4}{\pi^2}(y_i - y_{i-1})^2 \int_{y_i}^{y_{i-1}} \|f'(x)\|^2 dx +$$

$$+ \frac{4}{\pi^2}(y_i - y_{i-1})^2 \int_{x_i}^{x_{i-1}} \|f'(x)\|^2 dx$$

Since $y_i - x_i \leq \delta/2$ and $x_i - y_{i-1} \leq \delta/2$, this implies

$$\int_y \|f(x) - f(x_i)\|^2dx \leq \frac{\delta^2}{\pi^2} \int_{y_i}^{y_{i-1}} \|f'(x)\|^2 dx$$

Summing over $i$ and using Bernstein's inequality, one obtains

$$\|f - Af\|^2 \leq \frac{\delta^2}{\pi^2} \sum_i \int_y \|f'(x)\|^2 dx =$$

$$= \frac{\delta^2}{\pi^2} \|f'\|^2 \leq \frac{\delta^2}{\pi^2} \|f\|^2$$

Thus we have obtained the basic estimate

$$\|f - Af\| \leq \frac{\delta\omega}{\pi} \|f\| \text{ for all } f \in B_2^E$$

(32)

In other words, for $\delta\omega/\pi < 1$ the operator $Id - A$ is invertible on $B_2^E$ with an inverse in form of a Neumann series

$$A^{-1} = \sum_{n=0}^{\infty} (Id - A)^n$$

(33)

and

$$f = A^{-1}Af = \sum_{n=0}^{\infty} (Id - A)^nAf$$

(34)

Setting $\phi_0 = Af$ and

$$\phi_n = (Id - A)^nAf = \phi_{n-1} - A\phi_{n-1}$$

yields the algorithm (1)-(3). Since the start of the iteration is $\phi_0$, the reconstruction contains indeed only the information on the samples $f(x_i)$.

**Proof of Theorem 2**: According to (3), (34) and (35)

$$f - f_n = \sum_{k=0}^{n} \phi_k = \sum_{k=n+1}^{\infty} (Id - A)^kAf$$

(36)

From (32) one deduces

$$\|(Id - A)^kAf\| \leq \left(\frac{\delta\omega}{\pi}\right)^k\|Af\|$$

(37)

and

$$\|Af\| \leq \|f\| + \|(Af - f)\| \leq (1 + \frac{\delta\omega}{\pi})\|f\|$$

(38)

Combining these estimates yields

$$\|f - f_n\| \leq \sum_{k=n+1}^{\infty} \left(\frac{\delta\omega}{\pi}\right)^k(1 + \frac{\delta\omega}{\pi})\|f\| =$$

$$= \left(\frac{\delta\omega}{\pi}\right)^{n+1} \frac{\pi + \delta\omega}{\pi - \delta\omega} \|f\|$$

(39)

**References**


