Rapid Upper Bounding of Margin Loss in CP Digital Systems Due to Rain-Induced XPI

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ABSTRACT

Simple engineering approximations are given for the loss in margin incurred when circularly polarized digital signals propagate through rain and some of the signal on each polarization "leaks" into the channel of opposite polarization to produce cross polarization interference. Good agreement with laboriously derived prior results for typical PSK and QAM signals is shown.

INTRODUCTION

When two M-PSK or M-QAM signals on the same (or nearly the same) carrier frequency are transmitted through rain using left and right circular polarization to obtain a frequency reuse capability, each signal will undergo an amount of rain attenuation which depends upon the rainfall rate and other factors. The rain attenuation reduces the value of $E_b/N_0$ available in the receiver, and thus may degrade the performance to an unacceptable level unless sufficient rain margin has been allowed for in the link calculations.

The presence of the rain also allows a certain amount of each polarized signal to "leak" into the channel of opposite polarization, thus generating an interfering signal which will further degrade the performance; one way of looking at the situation would be to say that the rain attenuation (or the rain margin loss) is effectively increased over the value that would apply if the signal of opposite polarization were not present.

Given knowledge of the rain attenuation, operating frequency (assumed to be 10 GHz or greater), and path elevation angle, the semi-empirical formula of Chu (Reference 1) may be used to estimate the cross-polarized interference (XPI) level with respect to that of the desired signal. This level then becomes a parameter in the analyses of Prabhu (References 2,3) and Javed and Tetarenko (Reference 4) who have studied the problem of interference between a desired M-ary signal and an undesired statistically independent M-ary interferer of the same type. We will obtain simple upper bounds to the published error probability formulas, and from these obtain formulas showing by what maximum amount the value of $E_b/N_0$, in the presence of interference, must be increased to obtain the same value of error probability.
that would be gotten if the interference were not present. Comparison of our easily obtained results with the typical detailed Prabhu results at an error probability of $10^{-6}$ shows that our formulas overestimate, for $M \leq 16$, the required increases in $E_b/N_0$ by less than 2 dB, which is acceptable for preliminary design purposes, particularly in view of the fact that the Chu model is only an approximation anyway. In both this and the referenced analyses, timing errors, LO phase noise, ISI, and other sources of performance degradation are neglected, but could be accounted for later if a more detailed analysis were undertaken.

**ANALYSIS**

According to Chu, the cross-polarization discrimination for circular polarization above 10 GHz is given in dB by the approximation

$$X_{PD} = 11.5 + 20 \log_{10} \frac{a}{A_d} - 20 \log_{10} \rho - 40 \log_{10} \cos \phi$$

where $X_{PD}$ is the cross-polarization discrimination, $a$ is the rain attenuation in dB, and the other symbols are self-explanatory. From (1) it is easily shown that the voltage interference to signal ratio will be given by

$$\frac{(I/S)^{3/2}}{X_{PD}} = \frac{A_d}{a} (\cos \phi)$$

and is in effect a power signal to interference ratio (S/I) expressed in dB. $A_d$ is the rain attenuation in dB, and the other symbols are self-explanatory. From (1) it is easily shown that the voltage interference to signal ratio will be given by

$$\frac{I}{S} = \frac{(I/S)^{3/2}}{X_{PD}} = \frac{A_d}{a} (\cos \phi)$$

Now from page 756 of Reference 2 we find that the conditional probability of error for an $M$-PSK signal is

$$P_e(\lambda) = \frac{1}{2} \text{ERFC} \left[ \frac{1}{\sqrt{2}} \left( \frac{E_b}{N_0} \log_2 M \right)^{3/2} \right]$$

where we have set, in the notation of Reference 2,

$$\rho^2 = \frac{(E_b}{N_0} \log_2 M$$

and

$$R_e = (I/S)^{3/2}$$

The parameter $\lambda$ is given by

$$\lambda = \left( \frac{N_r}{N_f} \right) \rho - \theta_r - \theta_s + \mu$$

where the first term involves the carrier frequency difference (if any), $\theta_r$ and $\theta_s$ are the phase states of the interfering and desired $M$-PSK signals, and $\mu$ is a phase angle.

In the Prabhu M-PSK analysis for the case of a single interferer, $\lambda$ was assumed to be a random variable uniformly distributed over a range of 360 degrees, and the average error probability was computed according to the formula

$$P_e = \frac{1}{2 \pi} \int_{-\pi}^{\pi} P_e(\lambda) d\lambda$$

For our purposes we dispense with the difficult averaging process, and recognize instead from (3) and the monotonic properties of the ERFC function that we must have
\[ P_e < \text{ERFC}\left[\frac{\left(E_b/N_0\right) \log_2 M}{\left(SIN(\pi/M) - (I/S)^{M}\right)}\right] \]  
\[ (8) \]

regardless of the value of \( \lambda \). Now we suppose \( P_e \) to be held constant, and so as \( (I/S)^{M} \) increases the value of \( E_b/N_0 \) must also be increased to hold the argument constant. Thus for a constant \( P_e \) we have the simple equation

\[ \frac{(E_b/N_0)_{I=0}}{(E_b/N_0)_{I=0}} = \left[ 1 - \frac{(I/S)^{M}}{\text{SIN}(\pi/M)} \right]^{-1} \]  
\[ (9) \]

and thus the maximum dB increase in required \( E_b/N_0 \) (or the maximum margin loss) as a result of the cross-polarized interference will be given by the inequality

\[ (\Delta E_b/N_0)_{\text{M-MSK}} = -20 \log \left[ 1 - \frac{(I/S)^{M}}{\text{SIN}(\pi/M)} \right] \]  
\[ (10) \]

and some typical results obtained using this formula are shown later in Table 1, together with more accurate results obtained by Prabhu.

In the case of M-QAM signals, the analysis in Reference 4 and Section 2.2 of Reference 3 indicate that the conditional error probability will be given by

\[ P_e(\Phi) = \frac{4K_1}{M} \sum_{j=0}^{K_2} \sum_{k=1}^{K_2} \text{ERFC}[K_2(E_b/N_0)(1-(I/S)^{M}(j^2+k^2)^{M} \cos \Phi))} \]  
\[ (11) \]

where \( K_1 \) and \( K_2 \) are functions of \( M \) only, \( k \) and \( j \) can each take any of the values 1, 3, ---- \( M^2-1 \), and \( \Phi \) is a random phase angle uniformly distributed over 360 degrees. Each of the \( M/4 \) terms in (11) is upper bounded by the error function having the smallest argument, which will occur for

\[ j = k = M^2 - 1 \]  
\[ \phi = 0 \]  
\[ (12) \]

and so we find

\[ P_e \leq 4 \text{ERFC}[K_2(E_b/N_0)(1-(2I/S)^{M}(M^2-1))] \]  
\[ (13) \]

and by using the same argument as was employed in the M-PSK case we find

\[ \frac{(E_b/N_0)_{I=0}}{(E_b/N_0)_{I=0}} = 1 - (2I/S)^{M}(M^2-1) \]  
\[ (14) \]

and thus the maximum dB increase in required \( E_b/N_0 \) will be given by the inequality

\[ (\Delta E_b/N_0)_{\text{M-QAM}} \leq -20 \log [1 - (2I/S)^{M}(M^2-1)] \]  
\[ (15) \]

RESULTS

Using (10) and (15), selected results are tabulated in Table 1, together with comparable results obtained by Prabhu and given in Table 1 of Reference 3 and the graphs in Reference 2 for an error probability of 10^-6. The comparisons are seen to be quite favorable for modest I/S values and M equal to 16 or less; the bounds computed for larger values of M might, however, be expected
to be somewhat looser. A comparison of the results for 16-PSK and 16-QAM indicates that the QAM signal has a higher immunity to XPI; whether this is true for higher values of \( M \) is not known at this time.

**REFERENCES**


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Table 1. Comparison of (10) and (15) with Prabhu Results