Non-Recursive Wavelet Transforms in $\ell^2(\mathbb{Z}_+^\infty)$

Li Xiaoxin\textsuperscript{1}, Qi Deyu\textsuperscript{2}, Qian Zhengping\textsuperscript{3}

\textit{School of Computer Science and Engineering, South China University of Technology, Guangzhou, Guangdong, 510640, China}

mordecai@163.com\textsuperscript{1}, qideyu@scut.edu.cn\textsuperscript{2}, cszpqian@scut.edu.cn\textsuperscript{3}

Abstract

Today, almost all of the implementations of the discrete wavelet transforms are based on the recursive way. However, non-recursive wavelet transforms (NRWT) are more effective and more flexible. We extend the NRWT theory in $\ell^2(\mathbb{Z})$ and propose a new NRWT theory based on 6 different downsampling modes in $\ell^2(\mathbb{Z}_+^\infty)$. This extending makes NRWT more practical and can be compatible with the traditional recursive wavelet transform. We study the properties of the NRWT under the 6 downsampling modes, $\mathcal{W}_{3:3:5:2}$, through the analysis of redundancy degree and point out that $\mathcal{W}_{-2}$ is optimal and the redundancy degrees of $\mathcal{W}_{-2}$ and $\mathcal{W}_0$ are identical. The analysis of redundancy degree offers a method to choose the NRWT mode.

1. Introduction

Discrete wavelet transform (DWT)\textsuperscript{[1][2]} has been studied deeply and widely. Many wavelet softwares have provided DWT implementation, such as Wavelet Toolbox, WaveLab, LastWave and etc. However, almost all the implementations are based on the recursive filter bank shown in Figure 1 (All of the notations in this article are consistent with [3] and the important notations are listed in the appendix). We define the wavelet transforms based on the recursive filter bank as \textbf{recursive wavelet transforms (RWT)}.

![Figure 1. The filter bank used by RWT](Image)

RWT have obvious low effectivity, which can be explained by expanding the detail $d_i$ and the approximation $a_1$ in Figure 1 as following:

$$d_1 = D(D(...D(D(z \ast \bar{u}) \ast \bar{u}) \ast ... \bar{u}) \ast \bar{u})$$

$$D_i \left( x \ast \left( \left( V_{n=1}^{l-1} [U^{n-1}(u)] \ast U^{l-1}(v) \right) \right) \right) = D_i \left( x \ast f_i \right)$$

$$a_i = D(D(...D(D(z \ast \bar{u}) \ast \bar{u}) \ast ... \bar{u}) \ast \bar{u})$$

$$D_i \left( x \ast \left( \left( V_{n=1}^{l-1} [U^{n-1}(u)] \right) \right) \right) = D_i \left( x \ast g_i \right)$$

where

$$f_i = V_{n=1}^{l-1} [U^{n-1}(u)] \ast U^{l-1}(v)$$

$$g_i = V_{n=1}^{l-1} [U^{n-1}(u)]$$

When we make RWT on many input signals, we have to repeat the computations of $f_i$ and $g_i$ although only the input signals are different. To avoid the repeated computations, we can compute $f_i$ and $g_i$ in advance and make wavelet transforms based on the non-recursive filter bank in Figure 2. We call this kind of wavelet transforms as \textbf{non-recursive wavelet transforms (NRWT)}.

![Figure 2. The filter bank used by L-stage NRWT](Image)

NRWT was first presented by Frazier\textsuperscript{[3]} in order to illustrate the wavelet transforms in $\ell^2(\mathbb{Z}_n)$, $\ell^2(\mathbb{Z})$ and $L^2(\mathbb{R})$ through linear algebra. Actually, for $\ell^2(\mathbb{Z}_n)$ and $\ell^2(\mathbb{Z})$, $f_i$ and its even shifts form the basis of the $i$th wavelet space and $g_i$ and its even shifts form the basis of the $i$th scaling space\textsuperscript{[3]}; for $L^2(\mathbb{R})$, $lim_{n\rightarrow \infty} f_i$ approximates the wavelet function $\psi$ in the multiresolution space $W_0$ and $lim_{n\rightarrow \infty} g_i$ approximates the scaling function $\phi$ in the multiresolution space $V_0$\textsuperscript{[3][4]}.

Frazier did not study deeply the implementation of NRWT in [3]. Actually, it’s difficult to make the NRWT in reality for 3 reasons. First, the NRWT in $\ell^2(\mathbb{Z})$ are made on the region $(-\infty, +\infty)$ in theory while, in reality, we can only deal with the case where the length of the length of
filters and the input signals are finite. Second, the NRWT in $ℓ^2(\mathbb{Z})$ may be noncausal filtering but noncausal filtering cannot happen in reality\cite{9}\cite{10}. Third, the downsampling modes are different for different stages of NRWT, which may result in the results of NRWT are different from RWT.

We can solve the first problem by dealing with the borders of the input signal with finite length\cite{7}\cite{8}\cite{9} and solve the second problem by making the noncausal filtering causality\cite{9}\cite{10}. However, the third problem has not been studied deeply yet. This problem can be divided into 3 aspects: (1) how many modes of downsampling can be used in NRWT; (2) which mode is optimal; (3) how can we keep the results of NRWT consistent with the RWT.

We study NRWT under multi-downsampling in $ℓ^2(\mathbb{Z}_c)$ and $ℓ^2(\mathbb{Z}_c^+)$. In the following section, we first give the descriptions of $ℓ^2(\mathbb{Z}_c)$ and $ℓ^2(\mathbb{Z}_c^+)$ and their operators. For convenience, we only study the compact support and orthogonal NRWT.

2. $ℓ^2(\mathbb{Z}_c)$, $ℓ^2(\mathbb{Z}_c^+)$ and Their Operators

Because we can only make the wavelet transform in the stable system\cite{5}, where the length of filters and the input signals are all finite, we need to define a new subspace of $ℓ^2(\mathbb{Z})$, $ℓ^2(\mathbb{Z}_c) = \{ z = (z(n))_{n_1 < n < n_2} : z(n) \in \mathbb{C}, -\infty < n_1 \leq n_2 < +\infty \}$. In addition, because we can only make causal filtering, we define the subspace of $ℓ^2(\mathbb{Z}_c)$, $ℓ^2(\mathbb{Z}_c^+)$, as: $ℓ^2(\mathbb{Z}_c^+) = \{ z = (z(n))_{n_1 < n_2} : z(n) \in \mathbb{C}, 1 \leq n_2 < +\infty \}$.

If $z \in ℓ^2(\mathbb{Z}_c^+)$, then $z \in ℓ^2(\mathbb{Z}_c)$. Otherwise, if $z \in ℓ^2(\mathbb{Z}_c) \setminus ℓ^2(\mathbb{Z}_c^+)$, then $z$ may not belong to $ℓ^2(\mathbb{Z}_c)$. So, $ℓ^2(\mathbb{Z}_c^+) \subset ℓ^2(\mathbb{Z}_c) \subset ℓ^2(\mathbb{Z})$. Since the length of $z \in ℓ^2(\mathbb{Z}_c)$ is finite, let $\Lambda_z$ denote the length of $z$, $E_z$ denote the region of $z$ and $E_z^2 = \min(E_z)$, $E_z^2 = \max(E_z)$, hence $\Lambda_z = E_z^2 - E_z^2 + 1$. In order to study the NRWT in $ℓ^2(\mathbb{Z}_c)$ and $ℓ^2(\mathbb{Z}_c^+)$, we need to define some of their operators.

Define downsampling operator in $ℓ^2(\mathbb{Z}_c)$ and $ℓ^2(\mathbb{Z}_c^+)$ as follows:

**Definition 1** Define $D_k^c : ℓ^2(\mathbb{Z}_c) \rightarrow ℓ^2(\mathbb{Z}_c)$ by setting, for $z \in ℓ^2(\mathbb{Z}_c)$,

$$D_k^c(z)[n] = \begin{cases} z[2^k n] & \text{if } 2^k n \in E_z \\ 0 & \text{otherwise} \end{cases}$$

Define $D_k^c : ℓ^2(\mathbb{Z}_c^+) \rightarrow ℓ^2(\mathbb{Z}_c^+)$ by setting, for $z \in ℓ^2(\mathbb{Z}_c)$,

$$D_k^c(z)[n] = \begin{cases} z[k + 2^k (n - 1)] & \text{if } k \leq 2^k \leq n \end{cases}$$

where $k \in \mathbb{Z}^+, 1 \leq k + 2^k (n - 1) \leq \Lambda_z$.

$D_k^c$ and $D_k^c$ are called the downsampling operator.

It’s easy to prove that $D_k^c$ has the following properties:

**Proposition 1** For $z \in ℓ^2(\mathbb{Z}_c^+), \Lambda_{D_k^c(z)} = 2^k - 2^2 k + 1$.

**Proposition 2** For $z \in ℓ^2(\mathbb{Z}_c)$ and $0 \leq k \leq 2^k, D_k^c(z) = D_k^c(R_{-k}(z))$.

Define upsampling operators as follows:

**Definition 2** Define $U_k^c : ℓ^2(\mathbb{Z}_c) \rightarrow ℓ^2(\mathbb{Z}_c)$ by setting, for $z \in ℓ^2(\mathbb{Z}_c)$,

$$U_k^c(z)[n] = \begin{cases} z[n/2^k] & \text{if } 2^k n \in E_z \\ 0 & \text{otherwise} \end{cases}$$

Define $U_k^c : ℓ^2(\mathbb{Z}_c^+) \rightarrow ℓ^2(\mathbb{Z}_c^+)$ by setting, for $z \in ℓ^2(\mathbb{Z}_c^+)$,

$$U_k^c(z)[n] = \begin{cases} z[(n - 1)/2^k + 1] & \text{if } 2^k \leq n \\ 0 & \text{otherwise} \end{cases}$$

where $1 \leq n \leq 2^k(\Lambda_z - 1) + 1$.

$U_k^c$ is called the upsampling operator.

By the definition, $U_k^c$ has the following property:

**Proposition 3** $A_{U_k^c(z)} = 2^k(\Lambda_z - 1) + 1$

We often need to keep part of signals in the wavelet transform in $ℓ^2(\mathbb{Z}_c)$. Define cutoff operators as follows:

**Definition 3** Define $K_k^c : ℓ^2(\mathbb{Z}_c) \rightarrow ℓ^2(\mathbb{Z}_c)$ by setting, for $z \in ℓ^2(\mathbb{Z}_c)$,

$$K_{(j,l)}(z) = \begin{cases} z[i, j] & \text{if } l = j - i + 1 \\ 0 & \text{otherwise} \end{cases}$$

where $K_{(j,l)}$ is called the cutoff operator.

It’s easy to prove that $K_k^c$ or $K_{(j,l)}$ has the following property:

**Proposition 4** $K_k^c(R_k(z)) = R_k(K_{(j,k-l)}(z))$ or $K_{(j,l)}(R_k(z)) = R_k(K_{(j-k,l)}(z))$.

3. NRWT in $ℓ^2(\mathbb{Z}_c^+)$

In this section we first study the NRWT theory in $ℓ^2(\mathbb{Z}_c)$ and then derive the NRWT theory in $ℓ^2(\mathbb{Z}_c^+)$. By the definition of $U_k^c$ in section 2, equations (1) and (2) can be rewritten as follows:

$$f_j = \sum_{n=1}^{2^k-1} |u_{j-1}^{n-1}(w)| * U_{j-1}^{n-1}(w) \quad (3)$$

$$g_l = \sum_{n=1}^{2^k-1} |u_{j-1}^{n-1}(w)| \quad (4)$$

Suppose $x$ is the input signal, $x, f_j, g_l \in ℓ^2(\mathbb{Z}_c)$, and $\Lambda_x = N, \Lambda_u = \Lambda_v = \Lambda_f = \Lambda_g = \Lambda_l$. There exists the following relation between $\Lambda_l$ and $\Lambda_v$:

**Proposition 5** $\Lambda_v = (2^k - 1)(\Lambda - 1) + 1$

To prove Proposition 5 needs the following lemmas:

**Lemma 1** Suppose $z, w \in ℓ^2(\mathbb{Z}_c)$. Then $E_{z+w} = E_z + E_w$.

**Lemma 2** Suppose $u, v \in ℓ^2(\mathbb{Z}_c)$ and $E_u = E_v = [0, \Lambda - 1]$. Then $E_{u+v} = [0, (2^k - 1)(\Lambda - 1)]$.

It’s easy to prove Proposition 5 by Lemma 2.

3.1. NRWT in $ℓ^2(\mathbb{Z}_c)$

Frazier presented the NRWT theory in $ℓ^2(\mathbb{Z})$ in [3], which can be summarized as:

**Lemma 3** Suppose $x, g, f, f_{\text{isst}} \in ℓ^2(\mathbb{Z})$. Then at the wavelet decomposition

$$a_L = D^i(x * \gamma_L), \quad d_l = D^i(x * \gamma_l), 1 \leq l < L$$

at the reconstruction


\[ x = U^l(a_l) \ast g_L + \sum_{i=1}^{l} U^i(d_i) \ast f_i \]  

(6)

We can prove the NRWT theory in \(l^2(\mathbb{Z}_c)\) by Lemma 3.

**Theorem 1** Suppose \(x, g_L, f_{1stL} \in l^2(\mathbb{Z}_c)\). Then at the wavelet decomposition,

\[ a_L = D^L_0(x \ast \bar{g}_L), \quad d_L = D^0_0(x \ast \bar{f}_L), 1 \leq l \leq L; \]

at the reconstruction,

\[ x = K^N_0(U^1(a_L) \ast g_L) + \sum_{i=1}^{N} K^N_i(U^i(d_i) \ast f_i). \]

**Proof** Suppose \(E_u = [0, N-1], E_v = [0, \lambda - 1]. \) Extend \(x, g_L, f_{1stL}\) to two ends by infinite zeros and get \(ex, e g_L, e f_{1stL}, \) then \(ex, e g_L, e f_{1stL} \in l^2(\mathbb{Z}). \) By Lemma 3,

\[ a'_i = D(ex \ast e g_L), \quad d'_i = D(ex \ast e f_i), 1 \leq l \leq L \]

\[ ex = U^i(a'_i) \ast e g_L + \sum_{i=1}^{L} U^i(d'_i) \ast e f_i \]  

(7)

By the translation-invariant property of convolutions, \(x \ast \bar{g}_L\) is equal to \(ex \ast e \bar{g}_L\) on the region \(E_x \ast \bar{g}_L. \) By the definitions of \(D_0\) and \(D(\text{see Appendix})\), if we extend the definition domain of \(D_0\) to \(\mathbb{Z}\), \(D_0\) and \(D\) is equivalent. Hence,

\[ a_L = K_{E_{L}}(a'_L) \]

(8)

Similarly,

\[ d_L = K_{E_{L}}(d'_L), 1 \leq l \leq L \]

(9)

Let \(r = U^1(a_L) \ast g_L + \sum_{i=1}^{L} U^i(d_i) \ast f_i. \) By (7), (8), (9) and the definitions of \(U^i\) and \(U^1\), \(r = K_{E_x}(ex). \) Since \(E_x \subset E_{xL}, E_r \subset E_{xL}\) and \(x = K_{E_x}(ex), \) to prove \(x = K_{E_x}(r) = K^N_E(r)\) we need to prove \(E_x \subset E_r. \)

First, we prove \(E_x \subset E_{U^1(a_L) \ast g_L}. \) Let \(w = U^1(a_L) \). Since \(a_L = D^L_1(x \ast \bar{g}_L), \)

\[ E_w = [2^{L}E^1_{x \ast \bar{g}_L} / 2^{L}, 2^{L}[(E^1_{x \ast \bar{g}_L} / 2^{L})] \]  

(10)

By Lemma 2, \(E_{xL} = [0, \lambda - 1]. \) Hence

\[ E_{xL} = [-((\lambda - 1), 1) / 2] \]

(11)

 Applied to (10) yields \(E_w = [-2^{L}((\lambda - 1) / 2^{L}), 2^{L}[(N - 1) / 2^{L}] \)

Hence, \(E_{wL} = [-2^{L}((\lambda - 1) / 2^{L}), 2^{L}[(N - 1) / 2^{L}] \) \( \lambda - 1). \) Obviously, \(E_x \subset E_{wL}. \) Similarly, for \(1 \leq l \leq L, E_x \subset E_{U^i(d_i) \ast f_i}. \) Hence \(E_x \subset E_r. \)

### 3.2. NRWT in \(l^2(\mathbb{Z}_c^+)\)

It’s not convenient to implement the NRWT in \(l^2(\mathbb{Z}_c)\), because we have to deal with the noncausal filtering. Therefore, we study the NRWT in \(l^2(\mathbb{Z}_c^+)\). Let \(D^l(\cdot)\) denote the downsampling operator used in the \(l\)th stage wavelet transform. We first discuss the value of \(k_l.\)

The orthogonal wavelet transforms require uniform sampling\(^{[984]}\), i.e., the beginning indexes of downsampling should be coherent for different stages. According to downsampling step, downsampling can be divided into 3 cases. (1). \(k_L\) can be equal to the minimal sampling step in each stage, i.e., \(k_L = 2; (2). k_L\) can be equal to the sampling step in each stage, i.e., \(k_L = 2; (3). \) Map the convolution result \(x_L\) in \(l^2(\mathbb{Z}_c)\) to \(x_L^\prime\) in \(l^2(\mathbb{Z}_c^+)\), to downsample \(x_L\) from 0 equals to downsample \(x_L^\prime\) from

\[ 0 \ast i = \lambda - 2[(\lambda - 1) / 2] \]

(12)

which means \(k_L\) can be equal to \(0. \) In addition, obviously, the beginning index can also be equal to \(k_L - 1\) in the last 3 cases. Theorem 2 proves that signals can be reconstructed perfectly if we make NRWT using \(D^l(\cdot)\) when \(k_L = 1, 2^l - 1, 2^l, 0, 1, \) or \(0.\)

**Theorem 2** Suppose \(x, g_L, f_{1stL} \in l^2(\mathbb{Z}_c^+)\). Then for \(k_L = 1, 2^l - 1, 2^l, 0, 1, \) or \(0, \) at the wavelet decomposition,

\[ a_L = D^L_0(x \ast \bar{g}_L), \quad d_L = D^0_0(x \ast \bar{f}_L), 1 \leq l \leq L; \]

at the reconstruction,

\[ x = K^N_0(U^1(a_L) \ast g_L) + \sum_{i=1}^{N} K^N_i(U^i(d_i) \ast f_i). \]

**Proof** By Proposition 2, \(d_0 = D_0^L(x \ast \bar{f}_0) = D_0 \left(R_{-k}(x \ast \bar{f}_0)\right) \) since \(k = 1, 2^l - 1, 2^l, 0, 1, \) or \(0. \) and \(l \geq 1. \) By \(z[n] = R_{\lambda + 1}(z[n]) \) (see Appendix), \(d_0 = D_0 \left(x \ast R_{-k}(\bar{f}_0)\right). \) And we also can prove that for any vector \(z \in l^2(\mathbb{Z}_c), R_{\lambda + 1}(z[n]) = (R_{\lambda}(z)^{\prime})^\prime. \)

Hence \(d_0 = D_0 \left(x \ast (R_{-\lambda}(\bar{f}_0))\right)^{\prime}. \) Similarly,

\[ a_L = D^L_0 \left(x \ast (R_{-\lambda}(\bar{f}_0)\right)^{\prime} \sin^{(1)}(\lambda - 1)\), \]

(13)

From Theorem 2, we can show that the conjugate reflection operator “*” does not need to be used, noncausal filtering cannot happen in the NRWT in \(l^2(\mathbb{Z}_c^+)\), which offers us a convenient algorithm to implement the NRWT.

**Table 1. The maps between \(k_L\) and \(C_k\)**

<table>
<thead>
<tr>
<th>(k_L)</th>
<th>(C_{-1})</th>
<th>(C_{-2})</th>
<th>(C_0)</th>
<th>(C_k)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2^l - 1)</td>
<td>(2^l)</td>
<td>(0)</td>
<td>(0)</td>
<td>(k_L - 1)</td>
</tr>
</tbody>
</table>

Now, we have 6 different downsampling in the NRWT in \(l^2(\mathbb{Z}_c^+)\). For convenience, we use some notations to denote them: \(C_k\) denote the result of NRWT under different downsampling mode, where the meanings of \(k\) is listed in Table 1; \(C_k^l\) denote the result of NRWT in \(l\)th stage; \(W_k\) denote the wavelet transform to get \(C_k\); \(W_k^l\) denote the \(l\)th stage wavelet transform to get \(C_k^l.\)
Several new downsampling modes, \( k_l = 3, 4 \ldots \) or \( \lambda \), have been added into Table 1. \( k_1 \) is limited to be less than or equal to \( \lambda \), because \( \lambda_l - k_1 + 1 \geq 1 \) in the reconstruction stage. Obviously, \( C_{3s \leq k \leq \lambda} \) can not reconstruct input signal in general but \( C_{3s \leq k \leq \lambda} \) can in some special cases, which will be discussed in section 4.2. Now we have \( \lambda + 4 \) different downsampling in the NRWT in \( \ell^2(\mathbb{Z}^+_\lambda) \). Which mode is optimal? What about the relations among them? These problems will be discussed in section 4.

4. Comparisons of \( W_{-3s \leq k \leq \lambda} \)

In order to measure \( W_{-3s \leq k \leq \lambda} \), we introduce the conception of redundancy degree.

**Definition 4** Define \( rd(W_k) = (A_{c_v} - A_x) / A_x \). \( rd(W_k) \) is called the redundancy degree of \( W_k \).

4.1. The Analysis of Perfect Reconstruction and Redundancy Degree

By Theorem 2, we know that only \( C_{-3s \leq k \leq \lambda} \) can reconstruct the input signal perfectly. By the definition of the redundancy degree, the larger \( A_{c_v} \), the higher \( rd(W_k) \). Therefore, to measure the performance of \( W_{-3s \leq k \leq \lambda} \) is equivalent to compare the lengths of \( C_{-3s \leq k \leq \lambda} \).

Let \( \alpha = (L + 1)(\lambda - 1) \), \( \beta = N - \lambda + 2 \), \( \gamma_l = [(l + 1) / (L + 1)] + 1 \). By Theorem 2 and Proposition 1,

\[
A_{c_{v_{1,2}}} = \alpha + \sum_{l=1}^{L} \gamma_l \left(\frac{\beta - k}{2^l}\right)
\]

\[
A_{c_{v_{2,3}}} = \alpha + \sum_{l=1}^{L} \gamma_l \left(\frac{\beta - 2^l - k - 2}{2^l}\right)
\]

\[
A_{c_{v_{4,6}}} = \alpha + \sum_{l=1}^{L} \gamma_l \left(\frac{N - k}{2^l} - \frac{\lambda - 1}{2^l}\right)
\]

By (13)–(15), we can get the following redundancy degree theory.

**Theorem 3** Suppose \( x, y, f_{1 \leq l \leq L} \) in \( \ell^2(\mathbb{Z}^+_\lambda) \). Then

i. \( rd(W_{-3}) \leq rd(W_0) \leq rd(W_{-1}) \leq rd(W_1) \), \( k = -3, -1, 2 \)

ii. If \( N = 2^m, m \in \mathbb{Z} \) and \( m \geq L \), \( rd(W_{-3}) = rd(W_0) \).

**Proof**

(1) Compute the values (13), (14) and (15) in (c).

(2) For \( N = 2^m, m \in \mathbb{Z} \) and \( m \geq L \), by equation (14),

\[
A_{c_{v_{2,3}}} = \alpha + \sum_{l=1}^{L} \gamma_l \left[ (N - \lambda + 2 - 2^l) / 2^l \right]
\]

\[
= \alpha + \sum_{l=1}^{L} \gamma_l \left[ N / 2^l - [(\lambda - 2^l - 1)] - 1 \right]
\]

\[
= \alpha + \sum_{l=1}^{L} \gamma_l \left[ N / 2^l - ((\lambda - 2^l) / 2^l) + 1 \right]
\]

By (15),

\[
A_{c_{v_{4,6}}} = \alpha + \sum_{l=1}^{L} \gamma_l \left[ N / 2^l - [(\lambda - 1) / 2^l] \right]
\]

Hence, to prove \( A_{c_{v_{4,6}}} = A_{c_{v_{2,3}}} \), we need to prove

\[
\frac{N - 2^l - 1}{2^l} = \frac{N - 2^l}{2^l} + 1
\]

Suppose \( \frac{N - 2^l}{2^l} = \xi + \frac{\mu}{2^l}, \xi, \mu \in \mathbb{Z} \) and \( 0 \leq \mu < 2^l \), \( \frac{\mu}{2^l} = 0 \), \( \frac{\mu + 1}{2^l} = 1 \), then \( \frac{N - 2^l}{2^l} + 1 = \frac{N - 2^l + 1}{2^l} \).

\[
\xi + \frac{N - 2^l + 1}{2^l} = \xi + \frac{N - 2^l}{2^l} + 1 = \xi + \frac{N - 2^l}{2^l} + 1
\]

1. Hence, \( \frac{N - 2^l}{2^l} + 1 = \frac{N - 2^l + 1}{2^l} \). It is proved.

**Theorem 3** proves that, among \( W_{-3 \leq k \leq \lambda} \), \( W_0 \) is optimal, \( W_0 \) is suboptimal, and \( rd(W_0) = rd(W_{-2}) \) if \( N = 2^m \).

4.2. The Analysis of Relative Error and Redundancy Degree

In this section, we compare \( W_{-3 \leq k \leq \lambda} \) from reconstruction error and redundancy degree. The Relative Error [3] between the vector \( x \) and \( w \) is defined as \( ||x - w|| / ||x|| \), where \( ||x|| \) denote the norm operator.

![Figure 3. Comparison of relative errors and redundancy degrees of \( W_k \) using Db6 when \( N = 128, L = 8, K \leq k \leq \lambda \)](image)

Figure 3 shows how the relative errors and redundancy degrees vary with \( k \) when we make \( W_k \) on Doppler and Blocks respectively using Db6. First, we analyze the change of relative errors. As seen in Figure 3(a), the frequency of Doppler varies quickly on the region \([1, 20]\), which leads to the relative error climbs up quickly when \( k \) becomes large, as seen in Figure 3(b). However, since the value of Doppler is very small in the region \([0, 4]\), the relative error varies very small for \( k = 3, 4 \). Figure 3(b') shows the change of relative errors on Blocks in Figure 3(a'). As seen in Figure 3(a'), the values of Blocks on the region \([1, 10]\) are unchanged and keep zeros. Therefore, when \( k \) become large and \( k \leq 10 \), the relative error change little and \( C_k \) can reconstruct the input signal perfectly.

Now analyze the change of redundancy degrees. As seen in Figure 3(c) and Figure 3(c'), the redundancy degrees of \( W_k \) are identical although the input signals are
different, which proves again that the redundancy degree is only related with the length of the input signal and decrease as $1 \leq k \leq \lambda$, i.e., $rd(W_{k+1}) \leq rd(W_k)$.

To sum up, we can get the following conclusion:

**Conclusion 1** Suppose $x \in \ell^2(\mathbb{Z}^+), \lambda \geq 4$. If the values of $x$ on the region $[1, \lambda]$ change little and close to zeros, $C_{2, \lambda,k, \lambda}$ also can reconstruct $x$ perfectly and $rd(W_{k+1}) \leq rd(W_k)$.

5. Conclusion

We extend the NRWT theory in $\ell^2(\mathbb{Z})$ from 2 aspects. First, we limit the transform space into $\ell^2(\mathbb{Z}^+)$. Second, we extend the downsampling modes to 6 species. These extendings not only adapt NRWT to different wavelet transforms, but also make NRWT compatible with the exiting RWT. At last, we compare the redundancy degree of $W_k$ from perfect reconstruction and relative errors and point out that $rd(W_{-2})$ is minimal and $rd(W_0) = rd(W_{-2})$ if $N = 2^m$. We can choose an appropriate wavelet transform mode according to the actual situation.

NRWT theory in this article can be extended to the general frame\cite{[11]}, such as the biorhogonal wavelet transform\cite{[12]}, binary wavelet transform\cite{[6][13]} etc. However, all of the extendings are limited to the one dimension space. For two-dimension wavelet transform\cite{[6]}, we have to consider the transpose operator. For example, the 1\textsuperscript{st} -stage approximation coefficient in 2-dimension wavelet transform is $a_1 = D \left( (D(x * u))^T * u \right)$, where $T$ denotes transpose. Obviously, we cannot implement the NRWT in the 2-dimension case through computing the convolution of $u$ and $U_1^T(u)$ at first. Then, how can we make NRWT in the 2-dimension case? We will study this problem in future.

6. Appendix: Notation

- $\mathbb{Z}^+$: Positive integers
- $\mathbb{N}$: Nonnegative integers
- $z[i, j]$:
  - $z[i, j] = \{z[i], z[i + 1], \ldots, z[j]\}$
- $v_{n=1}^{|z(n)|}$:
  - $v_{n=1}^{|z(n)|} = z(1) * z(2) * \ldots * z(N)$
- $R_k(z)$: Shift operator\cite{[3]}:
  - $R_k(z)[n] = z[n - k]$
- $z \equiv w$: Integers $n \leq z$
- $[z]$: Largest integer $n \leq z$
- $[z]$: Smallest integer $n \geq z$
- $U(z)$: Upsampling\cite{[3]}:
  - $z \in \ell^2(\mathbb{Z})$, $U(z)[n] = \begin{cases} z(n/2) & 2 \mid n \\ 0 & 2 \nmid n \end{cases}$
- $D(z)$: Downsampling\cite{[3]}:
  - $D(z)[n] = z[2n]$
- $\bar{z}$: Conjugate reverse:
  - $z \in \ell^2(\mathbb{Z}), \bar{z}[n] = \bar{z}[-n]$

7. Acknowledgement

We would like to thank particularly Zhang Yuan and Huang Wei, who corrected my English grammar. I am also grateful to Mao Feiqiao for her revising the abstract and conclusion of this paper.

8. References