Axiomatizing Net Computations and Processes

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Abstract

Descriptions of concurrent behaviors in terms of partial orderings (called *nonsequential processes* or simply *processes* in Petri net theory) have been recognized as superior when information about distribution in space, about causal dependency or about fairness must be provided. However, at least in the general case of Place/Transition (P/T) nets, the proposed models lack a suitable, general notion of *sequential composition*.

In this paper, a new algebraic axiomatization is proposed, where, given a net N, a term algebra $\mathcal{P}[N]$ with two operations of parallel and sequential composition is defined. The congruence classes generated by a few simple axioms are proved isomorphic to a slight refinement of classical processes.

Actually, $\mathcal{P}[N]$ is a symmetric monoidal category,[1] parallel composition is the monoidal operation on morphisms and sequential composition is morphism composition. Besides $\mathcal{P}[N]$, we introduce a category $\mathcal{S}[N]$ containing the classical occurrence and step sequences. The term algebras of $\mathcal{P}[N]$ and of $\mathcal{S}[N]$ are in general incomparable, and thus we introduce two more categories $\mathcal{K}[N]$ and $\mathcal{T}[N]$ providing a most concrete and a most abstract extremum respectively. A simple axiom expressing the functoriality of parallel composition allows us to map $\mathcal{K}[N]$ to $\mathcal{P}[N]$ and $\mathcal{S}[N]$ to $\mathcal{T}[N]$, while commutativity of parallel composition maps $\mathcal{K}[N]$ to $\mathcal{S}[N]$ and $\mathcal{P}[N]$ to $\mathcal{T}[N]$ (see Figure 4).

Morphisms of $\mathcal{K}[N]$ constitute a new notion of concrete net computation, while the strictly symmetric monoidal category $\mathcal{T}[N]$ was introduced previously by two of the authors as a new algebraic foundation for P/T nets [7]. In this paper, the morphisms of $\mathcal{T}[N]$ are proved isomorphic to the processes recently defined in terms of the "swap" transformation by Best and Devillers [1]. Thus the diamond of the four category gives a full account in algebraic terms of the relations between interleaving and partial ordering observations of P/T net computations.

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[1] We refer to Mac Lane's book [5] for a precise definition of this notion. However, the basic idea is simple. There is a binary operation, defined both on the objects and on the morphisms, that is functorial and satisfies the axioms of a monoid up to a natural isomorphism. If the monoid operation is commutative (again, up to a natural isomorphism), the monoidal category is called *symmetric*. For example, the cartesian product of sets is a symmetric monoidal operation. If the natural isomorphisms are identities, then we get a *strict* version of the notion. The idea will be further illustrated by the construction of "term models" such as $\mathcal{P}[N]$ and other related categories.

1 Introduction

Many models have been proposed to specify systems the sub-parts of which can progress in parallel and synchronize. The main differences among these models stem from the way in which the occurrence of concurrent events is described. The most common ways are often referred to as:

1. interleavings of events;
2. interleavings of multisets of events; and
3. partial (or causal) orderings of events.

To make the discussion more precise, let us focus on the well-known and easy to understand model of *Place/Transition Petri nets*. In its classical description [13], the model consists of two sets S and T of places and transitions respectively. To every transition t, two (usually finite[2]) multisets of places t and t called *preset* and *postset* are associated. Global states consist

[2] In this paper we restrict ourselves to the case where - even in the presence of an infinite number of places and transitions - presets, postsets and markings are finite.

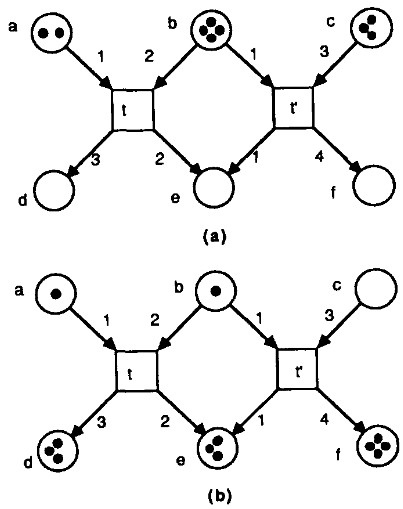

Figure 1: A net before and after the simultaneous firing of t and t'.

of *markings*, i. e. of multisets of places. Given a marking u, a transition t is enabled by it if ${}^\bullet t \subseteq u$, and in this case a step $u[[t > v$ may take place, with $v = (u - {}^\bullet t) \cup t^\bullet$. In general, a *step* may consist of the firing of a multiset G of transitions, each disjointly enabled.

Let us consider the net in Figure 1. It has a set of places $S = \{a, b, c, d, e, f\}$ and a set of transitions $T = \{t, t'\}$. Incoming and outgoing arrows of a transition and the associated numbers specify presets and postsets. For instance, ${}^\bullet t = \{a, 2b\}$ and $t^\bullet = \{3d, 2e\}$.

Figure 1(a) shows a marking $u = \{2a, 4b, 3c\}$, namely a marking where there are two *tokens* on place a, four on b and three on c. Figure 1(b) describes the state $v = \{a, b, 3d, 3e, 4f\}$ reached after the simultaneous firings of t and t', namely after the step $u[[t, t' > v$.

Let us go back to the problem of specifying concurrent systems. Given a net N an initial marking u and a final marking v, we can characterize its behaviour as a set[3] of abstract computations from u to v. Following our previous discussion, we have three options:

1. *occurrence sequences*: sequences of steps, each consisting of the firing of a single transition;
2. *step sequences*: sequences of steps, each with at least one firing transition;
3. *processes*: unfoldings of N from u to v.

[3] Using a *set* of abstract computations to specify a concurrent system is a common but simplistic view. To fully specify what a concurrent system both may and must do, it is usually necessary to consider tree-like structures, like synchronization trees or event structures. In this paper we will not be concerned with this aspect of the problem.

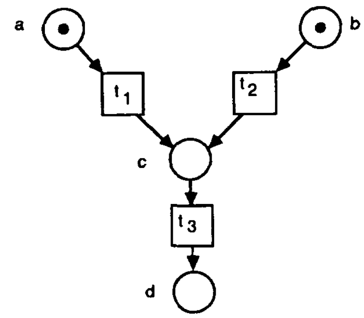

Figure 2: A net discussed by Best and Devillers.

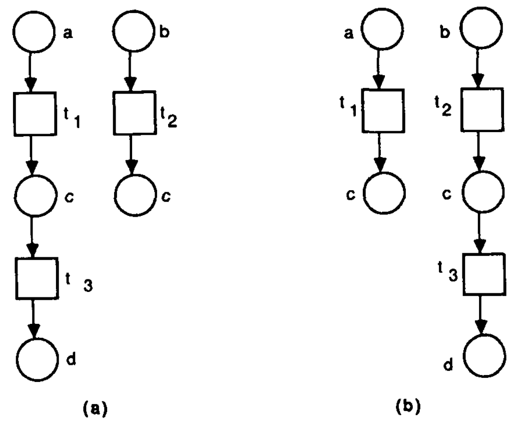

Figure 3: The processes of a net discussed by Best and Devillers.

Processes [4] are defined as acyclic nets with single multiplicity of places, called *occurrence nets*, equipped with mappings to the original net.

For instance, in Figure 2 we see a net discussed by Best and Devillers [1]). If we let $u = \{a, b\}$ and $v = \{c, d\}$, the set of occurrence sequences consists of

$$\{a,b\}[[t_1 > \{b,c\}[[t_2 > \{2c\}[[t_3 > \{c,d\}$$
$$\{a,b\}[[t_2 > \{a,c\}[[t_1 > \{2c\}[[t_3 > \{c,d\}$$
$$\{a,b\}[[t_1 > \{b,c\}[[t_3 > \{b,d\}[[t_2 > \{c,d\}$$
$$\{a,b\}[[t_2 > \{a,c\}[[t_3 > \{a,d\}[[t_1 > \{c,d\},$$

while the set of step sequences includes the four sequences above and also

$$\{a,b\}[[t_1,t_2 > \{2c\}[[t_3 > \{c,d\}$$
$$\{a,b\}[[t_1 > \{b,c\}[[t_2,t_3 > \{c,d\}$$
$$\{a,b\}[[t_2 > \{a,c\}[[t_1,t_3 > \{c,d\}.$$

Finally, in Figure 3(a) and (b) we see the two processes describing the desired behaviour in the third way.

Descriptions of process behaviors in terms of partial orderings like item 3 above have been recognized as superior when information about distribution in space, about causal dependency (or independency) or about fairness must be provided. Besides processes [4], many partial ordering-based structures (POS) have been proposed in the literature, like traces [6], event structures [10], pomsets [12], Winkowsky processes [14] and concurrent histories [2,3]. In all these cases, two drawbacks can be observed:

- no operation of sequential composition has been defined between POS, except for the trivial case where all the events of p' precede all the events of p'' in the resulting partial ordering $p'; p''$. A partial exception are Petri net processes and Winkowski processes, where however the *1-safe condition* (i.e. no concurrent places with the same label) is required. For concurrent histories, a nondeterministic operation, called in infix form _ *before* _ *gives* _, has been defined. Again, the operation becomes deterministic when the 1-safe condition holds;
- no simple algebraic way of defining and handling POS has been devised. As a consequence, the less descriptive but more elegant approach based on interleaving is often preferred for defining the semantics of *process description languages* like CCS [9].

The aim of the present paper is to solve both problems. In fact, an algebraic approach is proposed, where a term algebra on two operations (parallel and sequential composition) is defined. The congruence classes generated by a few simple axioms are proved isomorphic to a slight refinement of classical processes.

Our first starting point is the previous work by two of the authors [7,8], where a new algebraic definition for Place/Transition Petri nets has been proposed. Some account of it is given in Section 2. In short, a Petri net is seen as a graph where the nodes form a free commutative monoid. A closure operation $\mathcal{T}[\_]$, easily definable in categorical terms, enables us to freely generate from a net N a strictly symmetric monoidal category $\mathcal{T}[N]$, whose morphisms describe (congruence classes of) net computations.

The second starting point was a paper by Best and Devillers [1], where the three notions of abstract computation mentioned above are discussed in detail. They observe that, while one might expect processes to be more abstract than occurrence sequences and thus many occurrence sequences to correspond to the same process, the two notions are in fact incomparable. For instance, for the net in Figure 2 the occurrence sequences

$$\{a,b\}[[t_1 > \{b,c\}[[t_2 > \{2c\}[[t_3 > \{c,d\}$$
$$\{a,b\}[[t_2 > \{a,c\}[[t_1 > \{2c\}[[t_3 > \{c,d\}$$
$$\{a,b\}[[t_1 > \{b,c\}[[t_3 > \{b,d\}[[t_2 > \{c,d\}$$

correspond to the process in Figure 3(a), and the sequences

$$\{a,b\}[[t_1 > \{b,c\}[[t_2 > \{2c\}[[t_3 > \{c,d\}$$
$$\{a,b\}[[t_2 > \{a,c\}[[t_1 > \{2c\}[[t_3 > \{c,d\}$$
$$\{a,b\}[[t_2 > \{a,c\}[[t_3 > \{a,d\}[[t_1 > \{c,d\}$$

correspond to the process in Figure 3(b), where the sequences

$$\{a,b\}[[t_1 > \{b,c\}[[t_2 > \{2c\}[[t_3 > \{c,d\}$$
$$\{a,b\}[[t_2 > \{a,c\}[[t_1 > \{2c\}[[t_3 > \{c,d\}$$

correspond to both processes. A similar observation has been made by two of the authors in [3], where a new notion of abstract computation has been proposed, from which both occurrence sequences and processes can be abstracted out and which is *initial* among all other such abstractions (and therefore is the most concrete–i.e., making the fewest identifications).

Best and Devillers looked for a similar notion of computation, more abstract than both occurrence sequences and processes. In a somewhat *ad hoc* manner, they defined a *swapping* operation on processes: when two concurrent instances of the same place can be found (as the two places labelled c in Figure 3(a)), their causal consequences can be *swapped* (obtaining for our example the process in Figure 3(b)). Equivalence classes with respect to swapping, which we may call *commutative processes*, are recognized as the least abstract model which is more abstract than both occurrence sequences and processes. They are suggested as the correct observation level for nets. In our example in Figure 2, the two processes in Figure 3(a) and (b) and the four occurrence sequences

$$\{a,b\}[[t_1 > \{b,c\}[[t_2 > \{2c\}[[t_3 > \{c,d\}$$
$$\{a,b\}[[t_2 > \{a,c\}[[t_1 > \{2c\}[[t_3 > \{c,d\}$$
$$\{a,b\}[[t_1 > \{b,c\}[[t_3 > \{b,d\}[[t_2 > \{c,d\}$$
$$\{a,b\}[[t_2 > \{a,c\}[[t_3 > \{a,d\}[[t_1 > \{c,d\}$$

all correspond to a single commutative process.

Our results are as follows. In the spirit of the categorical approach to Petri nets proposed in [7,8], given a net N we define four categories $\mathcal{K}[N]$, $\mathcal{S}[N]$, $\mathcal{P}[N]$ and $\mathcal{T}[N]$, each of them freely generated by N in a different equationally axiomatized class of categories. These categories are naturally related to each other by a diamond of quotient homomorphisms (i.e., structure-preserving functors) in such a way that $\mathcal{P}[N]$ and $\mathcal{S}[N]$ are in general incomparable, while $\mathcal{K}[N]$ provides the most concrete (i.e., making fewest identifications) and $\mathcal{T}[N]$ the most abstract (i.e., making most identifications) model among the four.

The category $\mathcal{K}[N]$ represents a notion of net computation more concrete than both occurrence sequences and processes. It assumes only associativity of parallel composition, but contains a subcategory of *symmetries* expressing the fact that in a marking the tokens on the

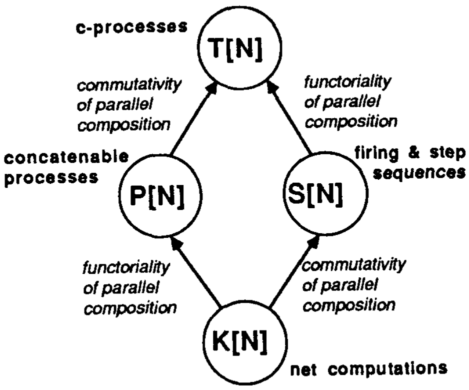

Figure 4: The categories $\mathcal{K}[N]$, $\mathcal{S}[N]$, $\mathcal{P}[N]$ and $\mathcal{T}[N]$ and their semantic relationships.

same place can be permuted. Furthermore, a *coherence axiom* holds, which equates any parallel composition α of processes with another parallel composition α' of the same processes, where the different order between α and α' is compensated by composing suitable symmetries in sequence before α and after α'.

The category $\mathcal{S}[N]$ is obtained as a quotient of $\mathcal{K}[N]$ by identifying all symmetries with the identities. The coherence axiom thus forces commutativity of parallel composition. The morphisms of $\mathcal{S}[N]$ can be seen as generalizations of occurrence sequences and of step sequences. Sequences of generators (i.e. sequences of transitions of N) can be proved to correspond to occurrence sequences, and sequences of parallel compositions of generators correspond to step sequences, but more complicate terms also exist in the algebra.

The category $\mathcal{P}[N]$ is obtained as a quotient of $\mathcal{K}[N]$ simply by adding a distributivity axiom which specifies the *functoriality* of parallel composition. Thus, $\mathcal{P}[N]$ is a monoidal category, and it is also symmetric. We show that the morphisms of $\mathcal{P}[N]$ are just a slight refinement of classical processes which we call *concatenable processes*. The refinement consists of imposing a total ordering among those minimal places (or "heads") of a process that are instances of the same place, and a similar ordering for the maximal places (or "tails"). This allows us to define a new general notion of *sequential composition* of processes.

Taking the union of the axioms of $\mathcal{S}[N]$ and of $\mathcal{P}[N]$ actually produces the strictly symmetric monoidal category $\mathcal{T}[N]$. This allows us to prove our last result, i.e. that the morphisms of $\mathcal{T}[N]$, as defined in [7,8] by two of the authors, coincide with the commutative processes defined by Best and Devillers [1]. Figure 4 gives a schematic summary of our results.

2 Petri Nets Are Monoids

Here we recall the definition of Petri (Place/Transition) nets proposed in [7,8] and the $\mathcal{T}[N]$ construction. The symbol $\oplus$ used in [7] for parallel composition is replaced here by the symbol $\otimes$, since, although commutative in $\mathcal{T}[N]$, $\otimes$ is not always commutative in this paper.

Definition 1 *(Petri Nets)*
A Place/Transition Petri Net (a net, in short) is a graph where the nodes form a free commutative monoid. Namely, a net is a quadruple $N = (S^\oplus, T, \partial_0, \partial_1)$, where $S^\oplus$ is a free commutative monoid of nodes[4] *over a set of* places S; T *is a set of* transitions; *and* $\partial_0, \partial_1 : T \longrightarrow S^\oplus$ *are functions associating to every transition a* source *and a* target *node respectively.* ∇

Notice that, in the finite case, multisets over a set S coincide with the elements of the free commutative monoid having S as set of generators.

A *Petri net morphism* h from N to N' is a pair of functions $< f, g >$, $f : T \longrightarrow T'$ and $g : S^\oplus \longrightarrow S'^\oplus$, such that $g \circ \partial_0 = \partial'_0 \circ f$ and $g \circ \partial_1 = \partial'_1 \circ f$, where g is a monoid morphism (i.e. leaving 0 fixed and respecting the monoid operation $\oplus$).

With this definition of morphism, nets form a category equipped with products and coproducts. In this paper we will not emphasize the categorical structure "in the large" of our models.

Definition 2 *(From nets to categories)*
Given a net $N = (S^\oplus, T, \partial_0, \partial_1)$, the category $\mathcal{T}[N]$ is the strictly symmetric strict[5] *monoidal category freely generated by N*[6]*. The objects of $\mathcal{T}[N]$ are the nodes of N, i.e. $S^\oplus$; the arrows of $\mathcal{T}[N]$ are obtained from the transitions T of N by adding for every object u an identity morphism (also denoted by u) and by closing freely with respect to the operations of* parallel composition $\_ \otimes \_$ *and of* sequential composition $\_ ; \_$. *The resulting category is made into a strictly symmetric (strict) monoidal category by imposing functoriality and strict commutativity axioms on $\otimes$. In detail, the category $\mathcal{T}[N]$ is defined by the following rules of inference*

[4]The elements of $S^\oplus$, called the *markings* of N, will be represented as formal sums $n_1 a_1 \oplus \ldots \oplus n_k a_k$ with the order of the summands being immaterial, with the a_i in S, the n_i in the set of natural numbers $\mathcal{N}$, addition defined by $(\bigoplus_i n_i a_i) \oplus (\bigoplus_i m_i a_i) = (\bigoplus_i (n_i + m_i) a_i)$ and 0 as the neutral element.

[5]A monoidal category is *strict* when the restriction of the binary operation to the *objects* is an ordinary monoid. All the categories considered in this paper are strict and have as its monoid of objects the free commutative monoid $S^\oplus$. However, not all of them are *strictly symmetric*, i.e., in general the monoidal operation need not be commutative on the morphisms, and therefore explicit natural isomorphism must be used to permute the factors.

[6]Therefore, $\mathcal{T}[N]$ can be abstractly characterized by a suitable universal property [7].

$$\frac{t : u \to v \text{ in } N}{t : u \to v \text{ in } \mathcal{T}[N]} \quad \frac{u \text{ in } S^{\oplus}}{u : u \to u \text{ in } \mathcal{T}[N]}$$

$$\frac{\alpha : u \to v, \beta : v \to w \text{ in } \mathcal{T}[N]}{\alpha;\beta : u \to w \text{ in } \mathcal{T}[N]}$$

$$\frac{\alpha : u \to v, \alpha' : u' \to v' \text{ in } \mathcal{T}[N]}{\alpha \otimes \alpha' : u \oplus u' \to v \oplus v' \text{ in } \mathcal{T}[N]}$$

and axioms, expressing the fact that the arrows form a commutative monoid:

$$(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma) \quad \alpha \otimes \beta = \beta \otimes \alpha \quad \alpha \otimes 0 = \alpha$$

the fact that $\mathcal{T}[N]$ *is a category:*

$$\alpha; \partial_1(\alpha) = \partial_0(\alpha); \alpha = \alpha \quad (\alpha;\beta);\gamma = \alpha;(\beta;\gamma)$$

and the functoriality of $\otimes$*:*

$$(\alpha \otimes \alpha');(\beta \otimes \beta') = (\alpha;\beta) \otimes (\alpha';\beta') \quad u \otimes v = u \oplus v. \ \nabla$$

The intuitive meaning of the monoidal operation $\_ \otimes \_$ is parallel composition of arrows, and of course the meaning of the categorical operation $\_;\_$ is sequential composition of arrows. Since the generators of the arrows of $\mathcal{T}[N]$ are the transitions of N, the inference rules above amount to closing the transitions of N with respect to parallel and sequential composition, giving origin to a generalized notion of net computation. Furthermore, the axioms above define arrows of $\mathcal{T}[N]$ as equivalence classes of such computations. Understanding the actual meaning of these equivalence classes and their relevance to net theory is the main aim of this paper.

We end this section by reconsidering in this setting the two examples discussed in Section 1.

Example 3 *(Nets as monoids)*

The net in Figure 1 can be represented in our algebraic setting by the two generators

$$t : a \oplus 2b \to 3d \oplus 2e$$
$$t' : b \oplus 3c \to e \oplus 4f.$$

The step consisting of the concurrent firings of t and t' is represented by the arrow

$$t \otimes t' : a \oplus 3b \oplus 3c \to 3d \oplus 3e \oplus 4f.$$

∇

Example 4 *(From Best and Devillers [1])*

In the net in Figure 2, the term

$$t_1 \otimes t_2; t_3 \otimes c$$

where $\otimes$ *takes precedence with respect to ;, corresponds to starting with a token in place a and a token in place b and to executing transitions* t_1 *and* t_2 *simultaneously. One of the tokens is then left in c, while the other is used for executing transition* t_3*. Similarly, the four terms*

$$t_1 \otimes b; t_2 \otimes c; t_3 \otimes c \quad t_2 \otimes a; t_1 \otimes c; t_3 \otimes c$$

$$t_1 \otimes b; t_3 \otimes b; t_2 \otimes d \quad t_2 \otimes a; t_3 \otimes a; t_1 \otimes d$$

correspond to different sequential executions of the transitions, in the order indicated. It is easy to see that the five terms above are equivalent, and thus define a single arrow of $\mathcal{T}[N]$. ∇

3 Computations

In this section we want to define a new category $\mathcal{K}[N]$ whose morphisms represent a notion of net computation that is more concrete than both occurring sequences and processes. For this purpose, we relax the equational constraints of $\mathcal{T}[N]$ by dropping the axioms $\alpha \otimes \beta = \beta \otimes \alpha$ and $(\alpha \otimes \alpha');(\beta \otimes \beta') = (\alpha;\beta) \otimes (\alpha';\beta')$, expressing the commutativity and the functoriality of $\otimes$ respectively. Furthermore, we introduce a new set of constant transitions called *symmetries* and axiomatize them. Finally, we impose axioms expressing that the sequential composition of a basic transition of N with a symmetry does not change the transition, and that the factors can be exchanged in any parallel composition of transitions, provided that suitable symmetries are composed sequentially before and after.

Notions of substitution similar to symmetries appear naturally in the study of monoidal categories and have been studied by many authors, in particular by Pfender [11]. To define symmetries, let us consider a finite set I labelled by a function $l : I \longrightarrow S$ which associates to every element x a label $l(x)$. When defined up to isomorphism (i.e. up to label-preserving bijections), the set I corresponds to an element $u = n_1 a_1 \oplus \ldots \oplus n_k a_k$ of $S^{\oplus}$, where $n_i = |\{x \mid l(x) = a_i\}|$, $i = 1, \ldots k$. A *symmetry* p of the labelled set I is a bijective endofunction $p : I \longrightarrow I$ that is label-preserving, i.e., such that $l = l \circ p$. We can associate it to u and write $p : u \to u$. It is clear that, by choosing a linear order for each of the sets $\{x \mid l(x) = a_i\}$, $i = 1, \ldots k$, p can be expressed as a vector of permutations. Also, operations of parallel and sequential composition can be defined on symmetries.

Definition 5 *(Defining symmetries)*

Given $u = n_1 a_1 \oplus \ldots \oplus n_k a_k$ *in* $S^{\oplus}$, *a* symmetry $p : u \to u$ *on u is a vector of permutations:*

$$< \sigma_{a_1}, \ldots, \sigma_{a_k} > \text{ with } \sigma_{a_i} \in \Pi(n_i).$$

We then define the parallel composition $p \otimes p' : u \oplus v \rightarrow u \oplus v$ *of two symmetries* $p : u \rightarrow u$ *and* $p' : v \rightarrow v$ *by:*

$$p \otimes p' = < \sigma_{a_1} \otimes \sigma'_{a_1}, \ldots, \sigma_{a_k} \otimes \sigma'_{a_k} >$$

where

$$\begin{aligned} \sigma \otimes \sigma'(x) &= \sigma(x) & \textit{if } 0 < x \leq |\sigma| \\ &= \sigma'(x - |\sigma|) + |\sigma| & \textit{if } |\sigma| < x \leq |\sigma| + |\sigma'|. \end{aligned}$$

Finally, we define the sequential composition $p; p' : u \rightarrow u$ *of* $p : u \rightarrow u$ *and* $p' : u \rightarrow u$ *by:*

$$p; p' = < \sigma_{a_1}; \sigma'_{a_1}, \ldots, \sigma_{a_k}; \sigma'_{a_k} >$$

where

$$\sigma; \sigma'(x) = \sigma'(\sigma(x)).$$

∇

Notice that both ; and $\otimes$ are associative, but not commutative. It follows from the definition that for each u in $S^{\oplus}$ the symmetry $u = < id_{a_1}, \ldots, id_{a_n} >: u \rightarrow u$ consisting of identity permutations is an identity for sequential composition. Furthermore, $\otimes$ and ; satisfy the equation:

$$(p \otimes p'); (q \otimes q') = (p; q) \otimes (p'; q').$$

Thus, we have defined a monoidal category of symmetries.

Definition 6 *(Defining* $Syms_S$*)*
Given a set S*, let* $Syms_S$ *be the monoidal category whose objects are the elements of the free commutative monoid* $S^{\oplus}$ *and whose arrows are symmetries with the above defined operations of* $\otimes$ *and* ; *. Then,* $Syms_S$ *is a (strict) monoidal category.* ∇

Symmetries can be suggestively represented in graphical form. For instance in Fig. 5 (a) and (b) we see examples of parallel and sequential composition of symmetries. In both cases the first operand is symmetry $p = < \sigma_a, \sigma_b >: 3a \oplus 2b \rightarrow 3a \oplus 2b$, where $\sigma_a = \{1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 1\}$ and $\sigma_b = \{1 \rightarrow 2, 2 \rightarrow 1\}$.

Before defining our notion of abstract computation, we need a third operation on symmetries, called *interchange*. The idea can be easily grasped from Fig. 5 (c). Given elements $u_1 = 2a \oplus b$ and $u_2 = 3a \oplus 2b$ of $S^{\oplus}$, their permutation $\pi = \{1 \rightarrow 2, 2 \rightarrow 1\}$ obviously generates a symmetry $\pi(u_1, u_2) = < \sigma_a, \sigma_b >$, where $\sigma_a = \{1 \rightarrow 4, 2 \rightarrow 5, 3 \rightarrow 1, 4 \rightarrow 2, 5 \rightarrow 3\}$ and $\sigma_b = \{1 \rightarrow 3, 2 \rightarrow 1, 3 \rightarrow 2\}$. Notice that $\pi(u_1, u_2) : u_1 \oplus u_2 \rightarrow u_1 \oplus u_2$.

Definition 7 *(Defining the interchange operation on symmetries)*

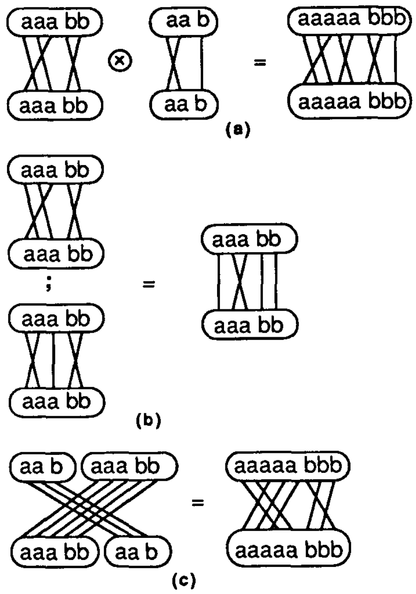

Figure 5: Three operations on symmetries.

Given π *in* $\Pi(m)$ *and* m *elements* $u_i = n_{i1}a_1 \oplus \ldots \oplus n_{ik}a_k$, $i = 1, \ldots, m$, *in* $S^{\oplus}$*, we define the* interchange symmetry $\pi(u_1, \ldots, u_m) = < \sigma_{a_1}, \ldots, \sigma_{a_k} >$, *with* $\pi(u_1, \ldots, u_m) : u_1 \oplus \ldots \oplus u_m \rightarrow u_1 \oplus \ldots \oplus u_m$*, as follows:*

$$\sigma_{a_j}(x) = x - \sum_{i=1}^{h-1} n_{ij} + \sum_{\pi(i) < \pi(h)} n_{ij} \quad \textit{if} \quad \sum_{i=1}^{h-1} n_{ij} \leq x < \sum_{i=1}^{h} n_{ij}, \quad j = 1, \ldots, k.$$

∇

In particular, the permutation $\gamma = 1 \rightarrow 2, 2 \rightarrow 1$ provides a "coherence isomorphism" $\gamma(u, v) : u \oplus v \rightarrow u \oplus v$ for which it is easy to check that, for any symmetries $p : u \rightarrow u$ and $p' : v \rightarrow v$ the identity

$$\gamma(u, v); (p' \otimes p) = (p \otimes p'); \gamma(u, v).$$

Therefore $Syms_S$ is a *symmetric* monoidal category [5].

Definition 8 *(Defining* $\mathcal{K}[N]$*)*
Given a net $N = (S^{\oplus}, T, \partial_0, \partial_1)$*, the category* $\mathcal{K}[N]$ *includes* $Syms_S$ *as a subcategory, and has as additional arrows those defined by the following rules of inference:*

$$\frac{t : u \rightarrow v \textit{ in } N}{t : u \rightarrow v \textit{ in } \mathcal{K}[N]}$$

$$\frac{\alpha : u \rightarrow v, \beta : v \rightarrow w \textit{ in } \mathcal{K}[N]}{\alpha; \beta : u \rightarrow w \textit{ in } \mathcal{K}[N]}$$

$$\frac{\alpha : u \rightarrow v, \alpha' : u' \rightarrow v' \textit{ in } \mathcal{K}[N]}{\alpha \otimes \alpha' : u \oplus u' \rightarrow v \oplus v' \textit{ in } \mathcal{K}[N]}$$

plus axioms, expressing the fact that the arrows form a monoid:

$$(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma) \quad 0 \otimes \alpha = \alpha \quad \alpha \otimes 0 = \alpha$$

and the fact that $\mathcal{K}[N]$ is a category[7]

$$\alpha ; \partial_1(\alpha) = \partial_0(\alpha); \alpha = \alpha \quad (\alpha;\beta);\gamma = \alpha;(\beta;\gamma).$$

In addition, there are axioms involving symmetries. The first pair states that generators (i.e. the basic transitions of N) are symmetrical:

$t;s = t$ *where* $t : u \to v \in T, s : u \to u$ *in* Sym_S
$s;t = t$ *where* $t : u \to v \in T, s : u \to u$ *in* Sym_S.

Finally, the last axiom expresses coherence, *i.e. the fact that the factors can be exchanged in any parallel composition of transitions, provided that suitable symmetries are composed sequentially before and after.*

$\pi(u_1, \ldots, u_m); \alpha_{\pi(1)} \otimes \ldots \otimes \alpha_{\pi(m)} =$
$\alpha_1 \otimes \ldots \otimes \alpha_m; \pi(v_1, \ldots, v_m)$
where $\alpha_i : u_i \to v_i \in \mathcal{K}[N]$,
$i = 1, \ldots, m$, *π is any permutation of m elements and $\pi(u_1, \ldots, u_m)$ and $\pi(v_1, \ldots, v_m)$ are interchange symmetries as defined above.*

∇

Although $\otimes$ is not functorial and therefore $\mathcal{K}[N]$ is *not* a monoidal category, thanks to the coherence axiom, $\mathcal{K}[N]$ enjoys some of the properties typical of symmetric monoidal categories. The category $\mathcal{K}[N]$ is the free category generated by the net N in the class of categories with a multiplication $\otimes$ that satisfy the axioms above, and therefore $\mathcal{K}[N]$ can be abstractly characterized by a universal property (details in the full paper).

In order to give an example of application of the coherence axiom, let us consider again Example 4 and Figure 2. It is easy to see that in $\mathcal{K}[N]$ we have

$$t_1 \otimes t_2; t_3 \otimes c = t_2 \otimes t_1; p; t_3 \otimes c = t_2 \otimes t_1; c \otimes t_3$$

where $p = < \sigma_a, \sigma_b, \sigma_c >$, $|p| = 2c$, with $\sigma_a = \sigma_b = \emptyset$, $\sigma_c = \{1 \to 2, 2 \to 1\}$, while

$$t_1 \otimes t_2; t_3 \otimes c \neq t_2 \otimes t_1; t_3 \otimes c$$

i.e. commutativity of $\otimes$ does not hold. In fact, there are in all twelve arrows from $a \oplus b$ to $c \oplus d$ in $\mathcal{K}[N]$, which can be represented by the following terms:

$\alpha_1 = t_1 \otimes b; t_2 \otimes c; t_3 \otimes c \qquad \alpha_2 = t_1 \otimes b; c \otimes t_2; t_3 \otimes c$
$\alpha_3 = t_2 \otimes a; t_1 \otimes c; t_3 \otimes c \qquad \alpha_4 = t_2 \otimes a; c \otimes t_1; t_3 \otimes c$
$\alpha_5 = t_1 \otimes b; t_3 \otimes b; t_2 \otimes d \qquad \alpha_6 = t_2 \otimes a; t_3 \otimes b; t_1 \otimes d$
$\alpha_7 = (t_1; t_3) \otimes t_2 \qquad \alpha_8 = (t_2; t_3) \otimes t_1$
$\alpha_9 = t_1 \otimes t_2; t_3 \otimes c \qquad \alpha_{10} = t_2 \otimes t_1; t_3 \otimes c$
$\alpha_{11} = t_1 \otimes b; t_2 \otimes t_3 \qquad \alpha_{12} = t_2 \otimes a; t_1 \otimes t_3$

Notice that while in this example it is possible to find a representative without symmetries for each equivalence class, this is not true in general.

4 Occurrence and Step Sequences

In this section we define a new category $\mathcal{S}[N]$ by identifying all symmetries in $\mathcal{K}[N]$. Its morphisms include the classical occurrence and step sequences.

Definition 9 *(Defining $\mathcal{S}[N]$)*
Given a net $N = (S^{\oplus}, T, \partial_0, \partial_1)$, the category $\mathcal{S}[N]$ is obtained from the category $\mathcal{K}[N]$ presented in Definition 8 by the quotient map specified by the following additional axiom:

$p = u$ *where* $p : u \to u$ *is any symmetry.*

∇

The coherence axiom of $\mathcal{K}[N]$ then yields:

Proposition 10 *($\otimes$ is commutative)*
The operation $\otimes$ is commutative in $\mathcal{S}[N]$.∇

The following theorem expresses our first result.

Theorem 11 *(Defining occurrence and step sequences algebraically)*
The following expressions are canonical in $\mathcal{S}[N]$ (up to associativity of ; and commutativity of $\otimes$):

$w_1 \otimes t_1; \ldots ; w_n \otimes t_n \; : \; w_1 \oplus u_1 \to w_n \oplus v_n$
where $t_i : u_i \to v_i \in T$

$w_1 \otimes \bigotimes_j t_{1j}; \ldots ; w_n \otimes \bigotimes_j t_{nj}$:
$w_1 \oplus \bigoplus_j u_{1j} \to w_n \oplus \bigoplus_j v_{nj}$
where $t_{ij} : u_{ij} \to v_{ij} \in T$.

Furthermore, they are in bijection with the occurrence and step sequences[8] *of N respectively:*

$w_1 \oplus u_1 [[t_1 > w_1 \oplus v_1 \ldots w_n \oplus u_n [[t_n > w_n \oplus v_n$
where ${}^{\cdot}t_i = u_i$ *and* $t_i^{\cdot} = v_i$

[7] Notice that there is no inference rule for generating identities, since they are already included in Sym_S. Recall that we represent with the same symbol both the objects and their associated identities.

[8] As mentioned previously, we consider only sequences with finite markings and firings, and we assume the obvious identification between finite markings on S and elements of $S^{\oplus}$.

$w_1 \oplus \bigoplus_j u_{1j}[[t_{11}, \ldots, t_{1m_1} >$
$w_1 \oplus \bigoplus_j v_{1j} \ldots w_n \oplus \bigoplus_j u_{nj}[[t_{n1}, \ldots, t_{nm_n} >$
$w_n \oplus \bigoplus_j v_{nj}$
where $^{\cdot}t_{ij} = u_{ij}$ *and* $t_{ij}^{\cdot} = v_{ij}$.

In the following, we will thus refer to the canonical forms above as occurrence *and* step sequences. ▽

The first property above is an obvious consequence of the identification of symmetries in the presence of the coherence axiom. It is easy to see that an equivalent presentation of $\mathcal{S}[N]$ would consist of omitting coherence axiom and symmetries altogether (except unities) and of asserting commutativity of $\otimes$ instead.

Considering our running example, commutativity of $\otimes$ generates the following nine equivalence classes, which thus are the arrows from $a \oplus b$ to $c \oplus d$ in $\mathcal{S}[N]$:

$$\{\alpha_1, \alpha_2\}\ \{\alpha_3, \alpha_4\}\ \{\alpha_5\}\ \{\alpha_6\}$$
$$\{\alpha_7\}\ \{\alpha_8\}$$
$$\{\alpha_9, \alpha_{10}\}\ \{\alpha_{11}\}\ \{\alpha_{12}\}$$

The first four classes and the last three classes together are the occurrence sequences and the step sequences as listed in the same order in Section 1. The remaining two classes

$$\{\alpha_7\} = (t_1; t_3) \otimes t_2 \quad \{\alpha_8\} = (t_2; t_3) \otimes t_1$$

represent more general computations obtained through parallel composition of sequences.

5 Processes

In this section we define a new symmetric monoidal category $\mathcal{P}[N]$ from a net N by adding to $\mathcal{K}[N]$ the axiom expressing the functoriality of $\oplus$. We also construct a category $\mathcal{CP}[N]$ whose morphisms are a small refinement of classical processes called *concatenable processes*. Concatenable processes are obtained from processes by imposing a total ordering both on the minimal places (or "heads") that are instances of the same place and, similarly, on the maximal places (or "tails"). In this way, we endow processes with a new operation of sequential composition. We then prove that $\mathcal{P}[N]$ and $\mathcal{CP}[N]$ are isomorphic and therefore we obtain a new, entirely algebraic, axiomatization of processes.

Definition 12 *(Defining $\mathcal{P}[N]$)*
Given a net $N = (S^{\oplus}, T, \partial_0, \partial_1)$, the monoidal category $\mathcal{P}[N]$ is obtained from the category $\mathcal{K}[N]$ presented in Definition 8 by the quotient map specified by the following additional axiom:

For $\alpha : u \to v$, $\alpha' : u' \to v'$, $\beta : v \to w$, $\beta' : v' \to w'$:
$\alpha \otimes \alpha' \; ; \; \beta \otimes \beta' \; = \; (\alpha; \beta) \otimes (\alpha'; \beta')$.

▽

The category $\mathcal{P}[N]$ thus defined is the free[9] symmetric monoidal category generated by the Petri net N.

Example 13 *(Exchanging disjointly enabled transitions)*
To see how the new axiom works, it is convenient to prove that $\alpha_2 = \alpha_3$ for our running example. This shows that two disjointly enabled transitions can be executed in any order. Applying the axiom and cancelling identities we get

$$\alpha_2 = t_1 \otimes b; c \otimes t_2; t_3 \otimes c = (t_1; c) \otimes (b; t_2); t_3 \otimes c = t_1 \otimes t_2; t_3 \otimes c = \alpha_9$$

i.e. two disjointly enabled transitions can be executed in parallel. Applying the argument symmetrically and terminating with the coherence axiom, we get

$$\alpha_9 = t_1 \otimes t_2; t_3 \otimes c = (a; t_1) \otimes (t_2; c); t_3 \otimes c = a \otimes t_2; t_1 \otimes c; t_3 \otimes c = t_2 \otimes a; t_1 \otimes c; t_3 \otimes c = \alpha_3.$$

▽

In all, for our running example there are two arrows from $a \oplus b$ to $c \oplus d$ in $\mathcal{P}[N]$, corresponding to the following equivalence classes:

$$\alpha_2 = \alpha_3 = \alpha_5 = \alpha_7 = \alpha_9 = \alpha_{11}$$
$$\alpha_1 = \alpha_4 = \alpha_6 = \alpha_8 = \alpha_{10} = \alpha_{12}$$

It is easy to see that the two classes above represent processes depicted in Figure 3 (a) and (b) respectively.

In general, processes do not contain enough information to represent morphisms of $\mathcal{P}[N]$. For example, the terms $t_1 \otimes t_2$ and $t_2 \otimes t_1$ are clearly not equivalent, but both of them would correspond to the same process, depicted in the intersection of (a) and (b) in Figure 3. To achieve a one-to-one correspondence with morphisms, we introduce a slightly different notion of process. The first definition introduces the notion of *label-indexed ordering function* and the second defines *concatenable processes* as finite processes with ordered labels on both minimal and maximal places.

Definition 14 *(Label-indexed ordering functions)*
Given a set S with a labeling function $l : S \longrightarrow S'$, a label-indexed ordering function *is a family $\beta = \{\beta_a\}$, $a \in S'$ of bijections, where $\beta_a : [a] \longrightarrow \{1, \ldots, |[a]|\}$, with $[a] = \{b \in S | l(b) = a\}$.* ▽

Definition 15 *(Concatenable processes)*
A concatenable process for a net N is a triple $C = <\pi, \beta, \gamma>$, where:

[9] Therefore, it can be abstractly characterized by a suitable universal property.

- *$\pi =< f, g >$ is a net morphism from a finite occurrence net*[10] *P to N. The function g must map places of P to places of N. The places of P which are minimal in the partial ordering associated to it are called* heads, *the maximal places are called* tails*;*
- *β is a label-indexed ordering function on the heads of P;*
- *γ is a label-indexed ordering function on the tails of P.*

Isomorphic[11] *concatenable processes are identified.* ∇

The function g, when restricted to the heads and tails, defines two multisets and therefore two associated elements of $S^{\oplus}$ that we shall denote by u and v. Therefore, we can picture a concatenable process C of N as an arrow $C : u \to v$. It is then easy to define operations on concatenable processes. The constants (i.e. the generators) and the operations $\otimes$ and ; are defined in a straightforward way. For example, Figure 6 (a) shows the evaluation of the term:

$t_2 \otimes t_1; p; t_3 \otimes c$ with $p =< \emptyset, \emptyset, \{1 \to 2, 2 \to 1\} >$.

Label-indexed ordering functions defined on heads and tails are represented as superscripts and subscripts of the main labels $a, b, \ldots$ respectively. Indices are omitted when $[a]$ is a singleton. Figure 6 (b) shows the result of the evaluation, which is isomorphic to the concatenable process in Figure 3 (a).

Since concatenable processes contain the symmetries as a subcategory (represented as concatenable processes with only places and no transitions) and satisfy all the inference rules and axioms of $\mathcal{P}[N]$, and since, as already mentioned, $\mathcal{P}[N]$ is the free symmetric monoidal category generated by the net N, the following properties are easy consequences.

Proposition 16 *(Defining $\mathcal{CP}[N]$)*
Concatenable processes on a net N form a symmetric, strict monoidal category $\mathcal{CP}[N]$. ∇

Proposition 17 *(from $\mathcal{P}[N]$ to $\mathcal{CP}[N]$)*
Given a net N, there is a unique homomorphism H from $\mathcal{P}[N]$ to $\mathcal{CP}[N]$ preserving all the operations and leaving N fixed when viewed as a subnet of $\mathcal{P}[N]$ and of $\mathcal{CP}[N]$ via the obvious inclusions.∇

[10]Remember that an occurence net is a net where the preset $^{\cdot}t$ and the postset $t^{\cdot}$ of any transition t are sets, $F^{\bullet}$ is a partial ordering and the preset $^{\cdot}a$ and the postset $a^{\cdot}$ of any place a contain at most one element, with xFy iff $x \in \partial_0(y)$ or $y \in \partial_1(x)$, $^{\cdot}x = \{y | yFx\}$ and $x^{\cdot} = \{y | xFy\}$.

[11]Two concatenable processes C and C' are isomorphic if there is a net isomorphism $< f'', g'' >$ from P to P' preserving all labels, namely with $f(t) = f'(f''(t))$, $g(b) = g'(g''(b))$, $\beta(b) = \beta'(g''(b))$ and $\gamma(b) = \gamma'(g''(b))$.

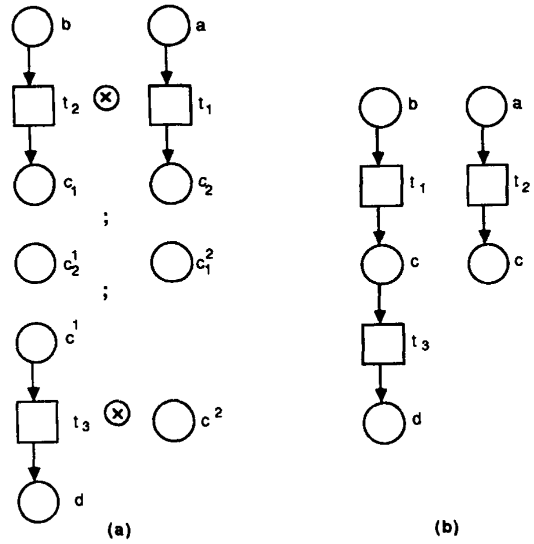

Figure 6: The evaluation of a term in $\mathcal{CP}[N]$, in (a), and the corresponding result, in (b).

The following lemmas and theorem prove the main result of this section, i.e. that H is an isomorphism.

Lemma 18 *(Canonical terms with maximal parallelism)*
Given a concatenable process $C = < \pi, \beta, \gamma >$, $\pi = < f, g >: P \longrightarrow N$, and a term α which evaluates to it, there is a term α' with maximal parallelism *of the form*

$$p_1; u_1 \otimes \bigotimes_j t_{1j}; p_2; \ldots; p_n; u_n \otimes \bigotimes_j t_{nj}; p_{n+1}$$

which is equivalent to α in $\mathcal{P}[N]$ and thus evaluates to C in $\mathcal{CP}[N]$ as well, satisfying the following conditions:

1. *the instances t_{ij}, $j = 1, \ldots, m_i$ of transitions of N corresponds exactly to the transitions of P at depth*[12] *i;*
2. *the identity $u_i = \bigoplus_j a_{ij}$ is the set containing those places a_{ij} of P which satisfy both conditions: they are either minima or in the postset of a transition at depth smaller than i; and they are either maxima or in the preset of a transition at depth larger than i;*
3. *the ordering of transitions t_{ij} in the noncommutative product $\bigotimes_j t_{ij}$ is canonical, in the sense that it respects some predetermined, fixed ordering of the transitions of N. In particular, all the instances of the same transition are adjacent.*

[12]An element t is *at depth* n in a partial ordering P if n is the length of the longest chain of elements of P which are predecessors of, or equal to, t.

Note that two such terms α'_1 and α'_2 for the same concatenable process C may differ only in the symmetries p_i.

Proof: *Outline. It is always possible to reduce α to a sequentialization of parallel compositions by applying functoriality of $\otimes$. It is easy to see that, since they must evaluate to C, all such terms are step sequences, with suitable symmetries and orderings of transitions, compatible with the partial ordering of P. Furthermore, applying the technique shown in Example 13 it is possible to exchange transitions, eventually deriving a term corresponding to a step sequence satisfying the first two conditions above. Finally, the transitions in the same parallel composition can be permuted as to satisfy the last condition by using the coherence axiom.* ∇

Lemma 19 *(Handling isomorphisms of concatenable processes)*

Given a concatenable process C and a term α for it with maximal parallelism, let p be a symmetry such that $\alpha;p$ evaluates in $\mathcal{CP}[N]$ to C too. Then in $\mathcal{P}[N]$ we have

$$\alpha;p = \alpha$$

Proof: *Outline. If α and $\alpha;p$ evaluate to the same C, then there is an isomorphism $\mathcal{H}$ of C which "induces" p on the tails of C and which does not modify its heads. Formally, let $C =< \pi,\beta,\gamma >$, $\pi =< f,g >: P \longrightarrow N$ and $\mathcal{H} =< f',g' >$, where g' is an identity for the heads of P. Then $p =< \sigma_{a_1},\ldots,\sigma_{a_k} >$ is the symmetry on $S^{\oplus}$ defined as*

$$\sigma_{a_i} = \{\gamma_{a_i}(b) \rightarrow \gamma_{a_i}(g'(b)) \mid b \text{ is a tail of } P \text{ with } g(b) = a_i\}.$$

Starting from $\alpha;p$ and using the coherence axiom, we can "push back" p through all the subterms $u_i \otimes \bigotimes_j t_{ij};p_{i+1}$, which are left invariant by the permutations induced by the homomorphism. Eventually, the extra symmetry corresponding to the heads will be the identity and thus α will be obtained. ∇

Theorem 20 *($\mathcal{P}[N] = \mathcal{CP}[N]$)*

Given a net N, the unique homomorphism H from $\mathcal{P}[N]$ to $\mathcal{CP}[N]$ is an isomorphism.

Proof: *To prove H injective, thanks to Lemma 18, we have to show that two terms with maximal parallelism for the same concatenable process C:*

$$\alpha = p_1;u_1 \otimes \bigotimes_j t_{1j};p_2;\ldots;p_n;u_n \otimes \bigotimes_j t_{nj};p_{n+1}$$

$$\alpha' = p'_1;u_1 \otimes \bigotimes_j t_{1j};p'_2;\ldots;p'_n;u_n \otimes \bigotimes_j t_{nj};p'_{n+1}$$

can be proved equivalent in $\mathcal{P}[N]$. If $n = 0$, the thesis is trivial: $p_1 = p'_1$ since they represent the same C. Otherwise, let us assume inductively $p_i = p'_i$, $i = 1,\ldots,n$, and let us prove that no matter what p_{n+1} and p'_{n+1} are, if α and α' represent the same C, then $\alpha = \alpha'$. In fact, let $p = p_{n+1}^{-1};p'_{n+1}$, i.e. such that $p'_{n+1} = p_{n+1};p$. Then Lemma 19 applies. Finally, H is easily proved surjective, by constructing a term for every concatenable process. ∇

6 Commutative Processes

Adding to the axioms for $\mathcal{K}[N]$:

- the extra axiom for $\mathcal{S}[N]$

  $p = u$ where $p : u \rightarrow u$ is any symmetry.

- the extra axiom for $\mathcal{P}[N]$

  For $\alpha : u \rightarrow v$, $\alpha' : u' \rightarrow v'$, $\beta : v \rightarrow w$, $\beta' : v' \rightarrow w'$:
  $\alpha \otimes \alpha' ; \beta \otimes \beta' = (\alpha;\beta) \otimes (\alpha';\beta')$.

we get a category isomorphic to $\mathcal{T}[N]$, and therefore we obtain the diagram of quotients in Figure 4. We call *c-processes* the arrows of $\mathcal{T}[N]$. Since the axioms for $\mathcal{T}[N]$ are the union of the axioms for $\mathcal{S}[N]$ and for $\mathcal{P}[N]$, a property holds similar to that claimed in [1], last paragraph of page 129, i.e. that commutative processes are the least abstract model which is more abstract than both occurrence sequences and processes.

Theorem 21 *(C-processes are as concrete as possible, Part I)*

Given a net N, the quotient from $\mathcal{K}[N]$ to $\mathcal{T}[N]$ is the join of the quotients from $\mathcal{K}[N]$ to $\mathcal{S}[N]$ and from $\mathcal{K}[N]$ to $\mathcal{P}[N]$. In other words, the diamond of Figure 4 is a pushout diagram. ∇

We can also give an alternative characterization of c-processes: they are the most concrete abstraction of classical processes (without assuming any refinement of the notion, such as the one provided by concatenable processes) that possesses both parallel and sequential composition.

Theorem 22 *(C-processes are as concrete as possible, Part II)*

Given a net N, let us consider the category $\mathcal{P}[N]$ and let us add the following rule:

For α and α' evaluating to $C =< \pi,\beta,\gamma >$ and $C' =< \pi,\beta',\gamma' >$ in $\mathcal{CP}[N]$ respectively, let $\alpha = \alpha'$.

The resulting quotient coincides with $\mathcal{T}[N]$.

Proof: *When applied to a pair of symmetries p and p' with $|p| = |p'| = u$, the axiom identifies them.* ∇

It is easy to relate commutative processes by Best and Devillers with our c-processes. We first express in our terms their Definition 7.2. in [1].

Definition 23 *(the relation $\equiv_0$ on occurrence sequences)*
Let $\alpha \equiv_0 \alpha'$ whenever

$$\alpha = \alpha_1; u_2 \otimes u \otimes t_1; v_1 \otimes u \otimes t_2; \alpha_2 \text{ and}$$
$$\alpha' = \alpha_1; u_1 \otimes u \otimes t_2; v_2 \otimes u \otimes t_1; \alpha_2$$
$$\text{with } t_1 : u_1 \rightarrow v_1 \text{ and } t_2 : u_2 \rightarrow v_2$$

∇

Theorem 24 *(C-processes and commutative processes coincide)*
Two occurrence sequences α and α' are identified in the quotient from $\mathcal{S}[N]$ to $\mathcal{T}[N]$ iff $\alpha \equiv_0^* \alpha'$. Furthermore, every class of the quotient contains at least one occurrence sequence.

Proof: *Outline. The effect of the functoriality axiom on occurrence sequences is that of exchanging two independently enabled transitions by equating both sequences to the parallel composition of the two transitions. The same result is achieved directly by the axiom $\alpha \equiv_0 \alpha'$.*
∇

7 Conclusions

In this paper, we have defined a diagram of four categories which gives a full account in algebraic terms of the relationships between interleaving and partial ordering observations of Place/Transition net computations. Our method has been entirely axiomatic, i.e., we have given equational axioms characterizing each of the notions as elements of a free algebra modulo the corresponding equations. As a byproduct of our approach, processes have been endowed with a new natural operation of sequential composition. The key to providing a common algebraic framework for both approaches has been the introduction of an algebra of symmetries and of a coherence axiom. Symmetries essentially take into account the fact that in the partial ordering approach tokens in the same place are distinguishable and that one must keep track of how they are permuted. However, this result has been obtained, rather surprisingly, *without* explicitly introducing any notion of individual tokens. Therefore, our models, although fully expressing the notion of process, are abstract enough to adhere to the standard concept of Place/Transition net based on the notion of marking.

8 Acknowledgements

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