A Small Universal Model for System Executions

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Abstract

Lamport has introduced a language for concurrency which uses two relations, strong (temporal) precedence, and weak (causal) precedence. Systems of concurrent activities under this scheme are called System Executions. We show that every consistent set of atomic relations has a unified model of size roughly $O(n^2)$. This model can be used to give a simplified proof of completeness of some axioms. We give several complexity results for deciding the theory of several classes of axiom sets, for both partial models and global-time models, showing many such variations to have the same complexity as transitive closure or matrix multiplication. Finally, we show that deciding disjunctive axioms is NP-complete, for both global-time and the standard model.

1 Introduction

In this paper we give a small model for Lamport’s System Executions [3,4]. This small model will allow us to prove results about the complexity of deciding various theories of system executions. We also characterize the relationship between some of these theories of the standard model and theories of the global-time model of Ben-David.

A System Execution is a triple

$$(E, \rightarrow, \rightsquigarrow),$$

where $E$ is a set of event executions and $\rightarrow$ and $\rightsquigarrow$ are relations on $E$. In addition, $\rightarrow$ is an irreflexive partial order. Together, they obey the following axioms:

A1. $A \rightarrow B \supset (B \rightarrow A)$.
A2. $A \rightarrow B \land B \rightarrow C \supset A \rightarrow C$.
A3. $\neg(A \rightarrow A)$.
A4. $A \rightarrow B \supset A \rightarrow B \land B \not\rightarrow A$.
A5. $(A \rightarrow B \rightsquigarrow C \lor A \rightsquigarrow B \rightarrow C) \supset A \rightarrow C$.
A6. $A \rightarrow B \rightarrow C \rightarrow D \supset A \rightarrow D$.
A7. $A \rightarrow A$.
A8. $A \rightarrow B \rightarrow C \rightarrow D \supset A \rightarrow D$.

Lamport gave a model of system executions as motivation for axioms A1 through A6. In this model, which we shall refer to as the standard model, one begins with a poset $P$. Event executions correspond to non-empty subsets of $P$. The relations $\rightarrow$ and $\rightsquigarrow$ are defined in terms of the partial...
order. Weak or causal precedence is defined as

$$ A \rightarrow B \equiv \exists a \in A. \exists b \in B. a \leq b $$

and strong or temporal precedence as

$$ A \rightarrow B \equiv \forall a \in A. \forall b \in B. a < b. $$

It is known that axioms A1-A8 are complete for this model, due to Anger and independently Abraham and Ben-David, both in an as-yet unpublished work cited in [5]. In [5], Ben-David goes on to examine the consequences of adding an additional axiom, namely, the global-time axiom, which is

$$ A9. \ A \rightarrow B \lor B \rightarrow A. $$

Ben-David shows the completeness of axioms A1-A9 for a class of models known as MRE-models. An MRE-model is a collection of Global time models, each of which is a system execution. The event executions of a Global-time model are intervals on the real line, under the standard ordering. Hence $A \rightarrow B$ iff every point in $A$ is smaller than every point of $B$, and $A \rightarrow \rightarrow B$ if there is some point of $A$ which is less than or equal to some point of $B$. A Global-time model can be seen to be a standard model in which the partial order is in fact total. We say that $A \rightarrow B$ (resp. $A \rightarrow \rightarrow B$) in an MRE-model $M$ iff $A \rightarrow B$ ($A \rightarrow \rightarrow B$) holds in every global-time model of $M$.

An atomic assertion is one of the form either $A \rightarrow B$ or else $B \rightarrow \rightarrow A$. A set of atomic assertions $\Gamma$ is also described as atomic. The atomic theory of some set of assertions $\Gamma$ (written $AT(\Gamma)$) is the set of atomic assertions which are logical consequences of $\Gamma$.

A literal assertion is either an atomic assertion or else the negation of an atomic assertion. Literal sets and literal theories are defined analogously.

A disjunctive assertion is either an atomic assertion or else the disjunction of two disjunctive assertions. Disjunctive sets contain only disjunctive assertions, and the disjunctive theory of a set of (disjunctive) axioms $\Gamma$ is the set of disjunctive assertions which are logical consequences of $\Gamma$.

A useful notion is that of a universal model of $\Gamma$. A universal model of $\Gamma$ is a model in which only the logical consequences of $\Gamma$ hold and no others. Such models are quite useful in deciding the theory of $\Gamma$.

## 2 The small model

Universal MRE-models are in general quite large. For example, suppose we have the relation $A \rightarrow B$, but not $B \rightarrow \rightarrow A$, nor $A \rightarrow B$. Any single global-time model which satisfies $A \rightarrow B$ will either satisfy $B \rightarrow \rightarrow A$ or $A \rightarrow B$, hence two global-time models must be included in the MRE model, one with $\neg(B \rightarrow \rightarrow A)$ and the other with $\neg(A \rightarrow B)$. Straightforward application of Ben-David's techniques produces at least two such global-time models for every such pair $A, B$; each global model has size proportional to the number of variables. It is not clear in [5] whether these global models can be constructed in time linear in their size, but even if they can, a universal model of some set of axioms $\Gamma$ will be of size roughly $O(n^3)$, where $n$ is the number of variables appearing in $\Gamma$.

We shall build a model of size $O(n^2)$. Let $TC(n)$ be the amount of time needed to compute the transitive closure of a relation with $n$ elements in the worst case. Then this model can be built in time $O(TC(n))$. 
2.1 The construction

We let $\mathbb{2}$ denote the familiar poset on $\{0,1\}$ with the usual ordering $0 < 1$.

Let $\Gamma$ be a set of atomic propositions. We construct $\mathcal{U}(\Gamma)$ as follows. Let $V$ be the set of variables appearing in $\Gamma$. Let $W$ and $S$ be subsets of $(V \times 2) \times (V \times 2)$ as follows. Let

$$(A,0) W (B,1)$$

hold if and only if $A = B$ or $A \rightarrow B$ is an element of $\Gamma$. Let

$$(A,1) S (B,0)$$

hold if and only if $A \rightarrow B$ is a proposition of $\Gamma$.

Observe that $W^+ = W$ and $S^+ = S$, by construction. We shall let our model be the poset

$$((V \times 2), (W \cup S)^*)$$

we define the subset corresponding to each variable $A$ to be the set $\{(A,0), (A,1)\}$. Now $(W \cup S)^*$ is reflexive and transitive by construction. Eventually, we will show that it is also anti-symmetric if and only if $\Gamma$ is consistent. First we prove two lemmas.

**Lemma 1** Let $\Gamma$ be a set of atomic propositions. For all variables $A$ and $B$,

$$(A,0) (W \cup S)^* (B,1)$$

if and only if

$$\Gamma \models A \rightarrow B.$$  

*Proof.* First let us suppose that

$$(A,0) (W \cup S)^* (B,1).$$

If $A = B$, then

$$\Gamma \models A \rightarrow A$$

by axiom A7. If $(A,0) (W \cup S) (B,1)$, then we must have $A \rightarrow B \in \Gamma$ by construction. Otherwise, there are variables $C, D$ such that

$$(A,0) W (C,1) S (D,0) (W \cup S)^* (B,1).$$

But then we have $\Gamma \models D \rightarrow B$ by induction. Also $A \rightarrow C$ and $C \rightarrow D$ must be members of $\Gamma$ by the definition of $W$ and $S$. But then axiom A8 gives us $\Gamma \models A \rightarrow B$.

On the other hand, let us suppose that

$$\Gamma \models A \rightarrow B.$$  

Then we must have

$$(A,i) (W \cup S)^* (B,k)$$

for some $i, k$ between 0 and 1. Since

$$(X,0) (W \cup S)^* (X,1)$$

for all variables $X$, it follows that

$$(A,0) (W \cup S)^* (B,1).$$

Q.E.D.

**Lemma 2** Let $\Gamma$ be a set of atomic propositions. For all variables $A$ and $B$,

$$(A,1) (W \cup S)^* (B,0)$$

if and only if

$$\Gamma \models A \rightarrow B.$$  

*Proof.* First, suppose that

$$(A,1) (W \cup S)^* (B,0).$$

If $A \rightarrow B \in \Gamma$, then we are done. Otherwise, we must have $C$ and $D$ such that

$$(A,1) S (C,0) W (D,1) (W \cup S)^* (B,0).$$
But then we have $\Gamma \models D \rightarrow B$ by induction. Also, we see that $A \rightarrow C$ and $C \rightarrow D$ are members of $\Gamma$ by construction. Hence $\Gamma \models A \rightarrow B$ by axiom A6.

Conversely, suppose that $\Gamma \models A \rightarrow B$. Then it follows that

$$(A, i) \ (W \cup S)^* \ (B, k)$$

for all $i, k$ between 0 and 1. So then it certainly follows that

$$(A, 1) \ (W \cup S)^* \ (B, 0).$$

Q.E.D.

Now we may prove that if a set of axioms is satisfied any model at all, then it is satisfied by our universal model.

**Theorem 3** Let $\Gamma$ be a set of atomic formulas and let $(W \cup S)^*$ be constructed from it as above. The relation $(W \cup S)^*$ is a partial order if and only if $\Gamma$ is consistent.

*Proof.* Suppose that $(W \cup S)^*$ is a partial order. Then $U(\Gamma)$ as described above is a standard model of $\Gamma$, hence $\Gamma$ is consistent.

Conversely, suppose $(W \cup S)^*$ is not a partial order. It is transitive and reflexive by construction, so it must be that there are distinct points $(A, i)$ and $(B, j)$ such that both

$$(A, i) \ (W \cup S)^* \ (B, j)$$

and

$$(B, j) \ (W \cup S)^* \ (A, i).$$

By the construction of $(W \cup S)^*$, we can find such points that satisfy $i \neq j$. Without loss of generality, assume that $i = 0$ and $j = 1$. But then we have $\Gamma \models A \dashv \vdash B$ by Lemma 1, and $\Gamma \models B \rightarrow A$ by Lemma 2, from which we get $\neg(A \dashv \vdash B)$ by axiom A4.

Q.E.D.

Lemmas 1 and 2 give us the result we were looking for, namely, the universality of $U(\Gamma)$.

**Theorem 4** Let $\sigma$ be an atomic sentence. Then $\Gamma \models \sigma$ if and only if $U(\Gamma) \models \sigma$.

One of the theorems in [5] stated that every system execution was isomorphic to a finitary system execution, i.e., one in which the event execution corresponding to each variable contained only a finite number of points. Our model can strengthen this if we define a binary system execution to be one in which each event execution contains exactly two elements.

**Theorem 5** Every system execution is isomorphic to a binary system execution.

*Proof.* If $S$ is a system execution, let $\Gamma_S$ be the set of atomic sentences which hold in $S$. Then Theorem 4 gives us that $U(\Gamma_S)$ is isomorphic to $S$.

Q.E.D.

We might have proved this result directly by first adding, for each event execution $A$, two points $A_0$ and $A_1$. We would make the augmented partial order have $A_0 < a$ and $a < A_1$ for all $a \in A$. Furthermore for all $b$ in the universe other than those in $A$, the augmented order would have $b < A_1$ if $b$ was an $a \in A$ with $b < a$ and, dually, $b > A_0$ if $b$ was an $a \in A$ with $b < a$. Likewise, for each $b$ strictlypreceding every element of $A$, make $b < A_0$ and, dually, for each $b$ strictly preceded by every element of $A$ make $b > A_1$. Carrying out this construction for each $A$ in turn will result in an isomorphic system execution. Then dropping all points but the “endpoints” $A$, for each variable $A$ will give us a binary system execution isomorphic to $S$.

### 3 Complexity Results

It should be clear that, for a finite $\Gamma$ which contains references to $v$ variables, $U(\Gamma)$ con-
tains exactly \(2v\) points. In order to represent the partial order of \(U(\Gamma)\), we will need in general \(4v^2\) space. Calculating the partial order \((W \cup S)^*\) from \(W\) and \(S\) will take as long as it takes to compute the transitive closure of a relation on \(2v\) points, which is known to be equivalent to multiplying matrices of size \(O(2v)\). Letting \(M(n)\) be the time required to multiply \(n \times n\) matrices, we have the following theorem.

**Theorem 6** If \(\Gamma\) is a set of atomic formulas of size \(n\), then \(\Gamma \models p\) for an atomic formula \(p\) can be decided in time \(O(M(n))\).

Warshall’s algorithm gives an \(O(n^3)\) algorithm. We note that the transitive closure need only be computed once for each \(\Gamma\), giving us the following result.

**Corollary 7** For fixed \(\Gamma\) and atomic \(p\) and \(\Gamma\), \(\Gamma \models p\) can be decided in \(O(1)\).

This is in contrast to Ben-David’s global time method, which contains \(O(n^2)\) global time models, each of which must be checked to verify \(\Gamma \models p\).

These results can be extended to entailment under the Global-Time Axiom. We write \(\Gamma \models_G p\) to denote that \(p\) is a logical consequence of axioms A1-A8 plus the global-time axiom. The following fact is a consequence of a theorem of Ben-David’s [5]. We give an independent proof for expository purposes.

**Lemma 8** If \(\Gamma\) and \(p\) are atomic, then \(\Gamma \models p\) if and only if \(\Gamma \models_G p\).

**Proof.** Suppose \(\Gamma \models p\). Any global-time model is a partial order model, so that \(\Gamma \models_G p\). Now suppose that \(\Gamma \not\models p\). Then \(U(\Gamma) \not\models p\). Since \(p\) is atomic, this amounts to saying that there are two points \(x\) and \(y\) in \(U(\Gamma)\) such that \(x \not\leq y\). (For \(p = A \rightarrow B\), these would be \((A, 0)\) and \((B, 1)\), for \(p = A \rightarrow B\), \((A, 1)\) and \((B, 0)\).)

A global-time model of \(\Gamma\) may be formed from \(U(\Gamma)\) by completing the underlying partial order to a total order. This can always be done to make \(x > y\). But this will mean that \(p\) does not hold in the resulting global-time model, hence \(\Gamma \not\models_G p\).

**Q.E.D.**

This means that we can compute atomic consequences of the axioms in matrix time.

**Corollary 9** Given atomic \(\Gamma\) and \(p\), \(\Gamma \models_G p\) may be decided in time \(O(M(|\Gamma|))\).

Moving on from atomic formulas, we examine literal formulas. First we note that literal formulas are no more difficult to decide than atomic formulas. Let \(\Gamma^+\) denote the atomic propositions in \(\Gamma\) and let \(\Gamma^-\) denote the negated propositions in \(\Gamma\).

**Theorem 10** Given literal \(\Gamma\), satisfiability of \(\Gamma\) may be checked in time \(O(M(|\Gamma|))\).

**Proof.** The algorithm is as follows. First construct \(U(\Gamma^+)\) and check all of the propositions of \(\Gamma^-\) for consistency with \(U(\Gamma^+)\). For instance, for a proposition \(\neg(A \rightarrow B)\) we check that \(A \rightarrow B\) does not hold in \(U(\Gamma)\).

If \(U(\Gamma^+)\) is consistent, and if each negative proposition is satisfied in \(U(\Gamma^+)\), then \(\Gamma\) has a model, namely \(U(\Gamma^+)\).

Conversely, if \(U(\Gamma^+)\) is not a system execution, we know from Lemma 3 that \(\Gamma^+\) and hence \(\Gamma\) are not satisfiable. If some negated proposition \(\neg\sigma\) is inconsistent with some fact \(\sigma\) such that

\[
U(\Gamma^+) \models \sigma
\]

we have that

\[
\Gamma^+ \models \sigma
\]
by Theorem 4. By completeness, we have
\[ \Gamma^+ \vdash \sigma, \]
hence
\[ \Gamma \vdash \sigma. \]
However, since \( \neg \sigma \in \Gamma \), we also have
\[ \Gamma \vdash \neg \sigma, \]
hence
\[ \Gamma \models \neg \sigma. \]
Thus \( \Gamma \) is not satisfiable.

Q.E.D.

Since \( \Gamma \models \sigma \) if and only if \( \Gamma \cup \{ \neg \sigma \} \) is unsatisfiable, we have the following corollary.

**Corollary 11** If \( \Gamma \) and \( \sigma \) are literal, then whether \( \Gamma \models \sigma \) can be decided in \( O(M(|\Gamma|)) \) time.

Surprisingly, we can use the partial order model to decide entailment in the global-time model. This is because negations do not add to the expressive power of the language under the global time assumption.

**Lemma 12** Let \( \Gamma \) be literal. Then there is a global-time equivalent atomic \( \Gamma' \).

**Proof.** The global-time axiom, together with axiom A4 give us the following two equivalences for all \( A, B \) in any global time model.

\[
\begin{align*}
A \rightarrow B & \iff \neg(B \rightarrow A) \quad (1) \\
A \rightarrow B & \iff \neg(B \rightarrow A) \quad (2)
\end{align*}
\]

This gives us a method for constructing \( \Gamma' \). For each atomic formula \( \sigma \in \Gamma \) let \( \sigma \in \Gamma' \). For each formula \( \neg(A \rightarrow B) \in \Gamma \), let \( B \rightarrow A \in \Gamma' \), and for each formula \( \neg(A \rightarrow B) \in \Gamma \), let \( B \rightarrow A \) belong to \( \Gamma' \). Equivalence of \( \Gamma \) and \( \Gamma' \) follows from the equations above, and the completeness of the global-time axioms.

Q.E.D.

This lemma gives us a method for deciding global-time satisfiability, and hence entailment for global time models. Furthermore, in the proof of the last theorem it turns out that \( |\Gamma| = |\Gamma'| \). This gives the following result.

**Corollary 13** For literal \( \Gamma \) and \( \sigma \), whether \( \Gamma \models_\sigma \) is decidable in time \( O(M(|\Gamma|)) \).

We note the fact that the theory of literal formulas is not in general the same for global time models as the standard model. In particular, every global-time model which satisfies \( \neg(A \rightarrow B) \) will also satisfy \( B \rightarrow A \). However, there are standard models in which neither \( A \rightarrow B \) or \( B \rightarrow A \).

4 **NP-completeness**

It is clear that in any theory which includes all the boolean connectives, satisfiability must be NP-complete. However, in this section we will show that merely adding the disjunction operation, without adding negation, is enough to make satisfiability NP-complete.

**Theorem 14** Let \( \Gamma \) be a set of disjunctive axioms. Then global-time satisfiability of \( \Gamma \) is NP-complete.

**Proof.** Global-time satisfiability of \( \Gamma \) is in NP since we may guess a global time model and verify it.

To show that global-time satisfiability is NP-hard, we reduce it to SAT, the original NP-complete problem. An instance of SAT
consists of a set of variables $V$ and a collection $C$ of clauses $c_i$ each of which is a set of literals over $V$.

To construct an instance of global-time satisfiability from an instance of SAT, for each variable $v \in V$ we create two variables $A_v$ and $B_v$. Next we define for each literal $x$ over $V$ the atomic formula $\alpha(x)$. We take $\alpha(v) = A_v \rightarrow B_v$ for unnegated variables $v$, and $\alpha(\neg v) = B_v \rightarrow A_v$ for negated variables. Next, extend $\alpha$ to clauses by setting $\alpha(c_i)$ to be the disjunction, over all literals $x \in c_i$, of $\alpha(x)$. Then we define

$$\Gamma_C = \{\alpha(c_i) : c_i \in C\}.$$

Now suppose that $\Gamma_C$ has a global-time model. Then for each $v \in V$, we will have either $A_v \rightarrow B_v$ or else $B_v \rightarrow A_v$. We use this to define a truth assignment $\tau$. In the former case, let $\tau$ assign the variable $v$ the value $\text{true}$, and in the latter, let $\tau$ assign the variable $v$ the value $\text{false}$. We claim that such a truth assignment will satisfy all the clauses of $C$. This is because $\alpha(c_i)$ is satisfied if and only if $\tau$ satisfies $c_i$.

Conversely, suppose that $\tau$ is a satisfying truth assignment for $C$. Then construct a standard model for $\Gamma_C$ by letting $A_v \rightarrow B_v$ for each variable $v$ such that $\tau(v)$ is $\text{true}$, and letting $B_v \rightarrow A_v$ for each variable $v$ such that $\tau(v)$ is $\text{false}$. Since for $v \neq w$ there is no relation between $A_v$ and $B_w$, this collection of atomic formulas must have a standard model, and hence a global-time model. Furthermore, each clause $\alpha(c_i)$ is satisfied since $\tau$ satisfies $c_i$.

Therefore $\Gamma_C$ is satisfiable if and only if the clauses $C$ are satisfiable. This gives us NP-hardness.

The theorem follows then as before. $Q.E.D.$

### 5 Conclusion

We have shown that use of the standard, partial-order model can lead to models which contain only one object rather than many objects such as the MRE-models. We find this more intuitively satisfying and do not understand the claim of some authors that total order semantics are simpler and more intuitive. While everyone’s intuition may differ, it seems clear that partial-order semantics are no more complex than global-time semantics.

One of the things that makes global-time semantics attractive is that there is a canonical total order, namely the real line, in which the global-time models can be embedded. This represents a substantial savings in the space needed to store a global-time model, since one may merely record the
value in the real line to which each point of the model is mapped (or the beginning and ending points of an interval), and the total order on the canonical order is easily computed in constant time. Thus the need for computing the transitive closure is bypassed, since we already know the ordering relation on the reals.

If we could find some analogous canonical partial ordering, it would lead to improved running time of the algorithms presented. What we have in mind is a partial order in which we could embed all of the universal models, or at least all of the finite ones, and in which the ordering relation can be computed very quickly. This is an area for further research.

Another direction for further research is to add an equivalence relation on the event executions and study the resulting logic. What we have in mind is an algebraic kind of model, where not all event executions are unique. Equivalent executions would then represent distinct calls to the same procedure, such as the semaphore $P$ and $V$ operations, or distinct execution instances of some block, such as a critical section.

We find this direction of study particularly interesting since it is only when such identifications are made that partial-order semantics becomes strictly more expressive than total-order semantics [2,1]. On the other hand, we prefer partial-order semantics useful even when they are of the same expressive power as total-order semantics, because they allow one to ignore incidental interleavings and concentrate on the temporal relationships which are of concern.

**References**


