1. INTRODUCTION

Multiple-valued Post logics originated in Post (1921) have been extensively investigated in recent years. The algebraization of the logics is provided by means of Post algebras. They play a role of semantical structures similar to the role of Boolean algebras for the classical logic. Post algebras have been introduced in Rosenbloom (1942) and investigated by many authors, for example Epstein (1960), Rousseau (1970), Rasiowa (1973), Traczyk (1967).

In Orlowska (1987, 1988, 1990a) a relationship has been established between several nonclassical logics and relation algebras. It has been shown that formulas of the logics can be treated as some relations and propositional operations as relational operations. That observation becomes an inspiration for the development of nonclassical relation and cylindric algebras (Orlowska 1973), Traczyk (1967).

Relational semantics for Post logics leads to a Post-Kripke frame which satisfies the conditions below:

(P1) W is a nonempty set (of states), R is a nonempty binary relation in W (accessibility relation), each di : W → W is a function in W

(P2) R is reflexive

(P3) R is transitive

(P4) di dj x = dj x

(P5) (x, d1 x) ∈ R

(P6) (di x, x) ∈ R for any i

(P7) (x, y) ∈ R implies (di x, di y) ∈ R for any i

(P8) (di x, x) ∈ R iff not (x, di x) ∈ R

(P9) For any x ∈ W there is i such that (di x, x) ∈ R.

Models of the logic are systems M = (K, m) where K is a Post-Kripke frame and m : VARPROP → P(W) is a meaning function which assigns sets of states to propositional variables and satisfies the condition:

(m) If (s, t) ∈ R and s ∈ m (p), then t ∈ m (p) for p ∈ VARPROP.

As usually, we define satisfiability of formulas by states in a model:

\[ M, s \text{ sat } p \text{ iff } s \in m (p) \text{ for } p \in \text{VARPROP} \]

\[ M, s \text{ sat } A \lor B \text{ iff } M, s \text{ sat } A \text{ or } M, s \text{ sat } B \]

\[ M, s \text{ sat } A \land B \text{ iff } M, s \text{ sat } A \text{ and } M, s \text{ sat } B \]

\[ M, s \text{ sat } A \rightarrow B \text{ iff for all } t \in W \text{ if } (s, t) \in R \text{ and } M, t \text{ sat } A, \text{ then } M, t \text{ sat } B \]

\[ M, s \text{ sat } A \text{ iff for all } t \in W \text{ if } (s, t) \in R, \text{ then not } M, t \text{ sat } A \]

\[ M, s \text{ sat } D_1 A \text{ iff } M, s \text{ sat } A \]

\[ M, s \text{ sat } E_j \text{ iff } (d_j s, s) \in R, 0 < j < \omega \]

\[ M, s \text{ sat } E_0, \text{ not } M, s \text{ sat } E_0 \text{ for any } s \in W. \]

A formula A is true in a model M if M, s sat A for all s ∈ W. A formula is valid if it is true in all models.

Axiomatization of the logic has been given in Rasiowa (1973). The axioms and rules are listed below:

A0 Axioms of the intuitionistic propositional calculus

A1 D1 (A ∨ B) ↔ D1 A ∨ D1 B

A2 D1 (A ∧ B) ↔ D1 A ∧ D1 B

A3 D1 (A → B) ↔ (D1 A → D1 B) ∧ ...

A4 D1 (¬ A) ↔ ¬ D1 A

A5 D1 D1 A ↔ D1 A
A6 \( D_i E_j \) for \( i \leq j \), \( \neg D_i E_j \) for \( i > j \)
A7 \( D_i + A \rightarrow D_i A \)
A8 \( E_\omega \)
A9 \( D_1 A \vee \neg D_1 A \)
A10 \( D_1 A \wedge E_1 \rightarrow A \)

The rules of inference are modus ponens and
\begin{align}
(1) & A \quad \text{for any } i & (2) & D_i A \quad \text{for every } i
\end{align}

Completeness of the given axiomatization with respect to an algebraic semantics has been proven in Rasiowa (1973), and completeness with respect to the class of all models based on Post-Kripke frames has been given in Maximova and Vakarelov (1974).

Post logics of order \( m \) include finitely many operations \( D_i \) for \( 1 \leq i \leq m-1 \), and finitely many constants \( E_j \) for \( 0 \leq j \leq m-1 \). Axiomatization of the logic can be obtained by replacing axioms A8 and A10 by the following:

A8' \( E_{m-1} \)
A10' \( A \leftrightarrow E_1 \wedge D_1 A \vee \ldots \vee E_{m-1} \wedge D_{m-1} A \).

Rules of inference are modus ponens and (r1).

3. Post relation algebras

Let a Post-Kripke frame \( K = (W, R, [d_i] 1 \leq i < \omega) \) be given. Following a method developed in Orlowska (1990b) we define a class of relation algebras corresponding to the Post logic. The elements of each of these algebras are binary relations of the form \( X \times W \) for \( X \subseteq W \), that is they are sets (or equivalently unary relations) which are `dummy embedded' into \( W \times W \). The operations in the algebras are counterparts of propositional operations in the Post logic. For the sake of simplicity they are denoted in the same way as the respective logical operators. Let:

- \( A, B \in \{ X \times W : X \subseteq W \} \), then we define:
  - \( (p \cup) A \cup B = \{(x, y) : (x, y) \in A \text{ or } (x, y) \in B\} \)
  - \( (p \cap) A \cap B = \{(x, y) : (x, y) \in A \text{ and } (x, y) \in B\} \)
  - \( (p \rightarrow) A \rightarrow B = \{(x, y) : \text{for all } z \text{ if } (x, z) \in R \text{ and } (z, y) \in A \text{ then } (z, y) \in B\} \)
  - \( (p \rightarrow) A \rightarrow B = \{(x, y) : \text{for all } z \text{ if } (x, z) \in R \text{ then } (z, y) \in A \} \)
  - \( (p \mathrm{D}) D_i A = \{(x, y) : (d_i x, y) \in A\} \)
  - \( (p \mathrm{E}) E_j = \{(x, y) : (d_j x, y) \in R\} \) for \( 0 < j < \omega \), \( E_\omega = \emptyset, E_0 = W \times W \).

Clearly, \( \cup, \cap, \neg \) are Boolean operations. \( E_j \) are constants or, equivalently, 0-ary relational operations. We also have:

- \( (d) \) \( D_i A = D_i - A \)

Proposition 3.1

(P10) If \( i \leq j \), then \( (d_j x, d_i x) \in R \)
(P11) \( (d_j x, y) \in R \) implies \( (x, d_1, y) \in R \)
(P12) \( (d_j x, y) \in R \) and \( (d_i x, y) \in A \) imply \( (x, y) \in A \)
(P13) \( (d_j x, y) \in R \) and \( (x, y) \in A \) imply \( (d_i x, y) \in A \)

Proof: (P10) follows easily from (P3) and (P6).

Proof of (P11): Let \( (d_i x, y) \in R \). By (P5) \( (y, d_1 y) \in R \). By (P3) \( (d_j x, d_1 y) \in R \). By (P7) and (P4) we have \( (d_j x, d_1 x, y) \in R \). By (P5) \( (x, d_1 x, y) \in R \). By (P3) \( (d_j x, d_1 y) \in R \).

Proof of (P12) is by induction with respect to the complexity of A. (i) If \( A \in \text{CON} \), then the condition holds by (m2).

(ii) If \( A = E_j \), then \( (d_j x, y) \in E_j \) iff \( (d_j d_1 x, x) \in R \). By (P4) \( (d_j x, x) \in R \). Hence \( (x, y) \in E_j \).

(iii) If \( A = D_1 B \), then \( (d_j x, y) \in D_1 B \) iff \( (d_j x, y) \in B \). Suppose that \( (x, y) \in D_1 B \). It follows that \( (d_j x, y) \in B \), a contradiction.

(iv) Let \( A = A \). We have \( (d_j x, y) \in A \) iff \( (d_j x, z) \in R \) for \( z \neq 0 \). By (P3) \( (d_j x, u, u) \in R \). By (a) \( (u, u) \in B \), a contradiction.

(vi) Consider relation of the form \( A \rightarrow B \). We have \( (d_j x, y) \in B \) iff \( (a) \) for all \( z \neq 0 \). \( (d_j x, z) \in R \) implies \( (z, y) \in B \). Suppose that \( (x, y) \not\in B \). Hence there is \( u \) such that \( (x, u) \in R \), \( (u, y) \in A \), and \( (y, y) \not\in B \). By (P3) \( (d_j x, u) \in R \). By (a) \( (u, u) \in B \), a contradiction.

In the remaining cases the proof of (P12) is similar.

Proof of (P13): Suppose that \( (d_i x, y) \in -A \). Hence \( (d_j d_i s, y) \in D_j A \). By (P4) \( (d_j d_i s, y) \in A \). Since \( (x, y) \in A \), we have \( (d_j x, y) \in D_i A \), a contradiction.

Proposition 3.2

For any Post-Kripke frame set \( \{X \times W : X \subseteq W\} \) is closed with respect to the relational operations defined above.

The proof is by an easy verification.

By a full Post relation algebra determined by frame \( K \) we mean an algebra

\( \text{fullPoRA} (K) = \{ [X \times W : X \subseteq W], \cup, \cap, \rightarrow, \neg, [D_j] 1 \leq i \leq \omega, [E_j] 0 \leq j \leq \omega \} \)

Consider the class

\( \text{fullPoRA} = \{ \text{fullPoRA} (K) : K \text{ is a Post-Kripke frame} \}

Then the class of Post relation algebras is defined as

\( \text{PoRA} = \{ \text{PoRA} : \text{SP (fullPoRA)} \}

where \( S \) and \( P \) are the operations of taking isomorphic copies of subalgebras and direct products, respectively.
Consider an algebra of binary relations over a set W:

\[ \text{fullIRA}(W) = (P \times W) \cup \cup \cup \cup, \succ, \sim, 1, 0, -1, 1 \]

where \( \cup, \cap, \prec, \sim, - \) are Boolean operations, \( 1 = W \times W, 0 \) is the relational composition, \( -1 \) is the converse operation, and \( 1 \) is the identity relation. In a standard way (Nemeti 1990) we define the classes of set relation algebras:

\[ \text{fullRA} = (\text{fullIRA} (W) : W \text{ is a set}) \]

\[ \text{RA} = \text{SP (fullIRA).} \]

Post relation algebras are closely related to the \( \omega^+ \)-valued Post logic. Let \( \tau : \text{VARPROP} \rightarrow \text{CON} \) be a one-to-one mapping of the set of propositional variables into the set of relational constants. We define a translation function \( t \) from formulas of the Post logic into relational terms:

\[ t(p) = \tau(p) \text{ for } p \in \text{VARPROP}, \quad t(E_j) = E_j, \]

\[ t(\neg A) = \neg t(A), \quad t(D_i \ A) = D_i \ t(A), \]

\[ t(A \cup B) = t(A) \cup t(B), \quad t(A \land B) = t(A) \land t(B), \]

\[ t(A) \rightarrow t(B). \]

The respective definitions lead to the following theorem:

**Proposition 3.3**

The following conditions are equivalent:

(a) A formula \( F \) is valid in \( \omega^+ \)-valued logic

(b) \( t(F) = E \) holds in every algebra from PoRA.

We show that operations \( \rightarrow, \sim, D_i \) for \( 1 \leq i \leq \omega \), and \( E_j \) for \( 0 \leq j \leq \omega \) as well as conditions (P2), \ldots , (P8) are definable by means of relational terms over RA. Let \( D_i \subseteq W \times W \), for \( 1 < i < \infty \) be relational terms in \( W \) such that for every \( i \) we have \( D_i \cap D_i^{-1} \subseteq 1 \), that is they all are functions. They are intended to be a relational counterpart of functions \( D_i \) in frame \( K \), and for the sake of simplicity we denote them by the same symbols.

**Proposition 3.4**

(a) \( A \rightarrow (R (A \cap B)) \)

(b) \( \neg A = \neg (R A) \)

(c) \( D_i A = D_i \ o \ A \)

(d) \( E_j = (d (\cap (d_j \ o \ R)) \ o \ 1 \text{ for } 0 < j < \omega, E_0 = -1, \quad E_0 = 1 \).

**Proposition 3.5**

(a) Condition (P2) holds in a frame \( K \) iff \( -1 \cap R = 1 \) holds in fullIRA over set \( W \) of states of \( K \)

(b) Condition (P3) holds iff \( - (R_0 R) \cap R = 1 \)

(c) Condition (P4) holds iff \( -1 \cup d_0 \ o \ d_1 \ o \ 1 \ o \ d_1^{-1} = 1 \)

(d) Condition (P5) holds iff \( -1 \cup d_0 \ o \ R^{-1} = 1 \)

(e) Condition (P6) holds iff \( -1 \cup d_{i+1} \ o \ R_0 d_i^{-1} = 1 \)

(f) Condition (P7) holds iff \( -R \cup d_i \ o \ R_0 d_i^{-1} = 1 \)

(g) Condition (P8) holds iff \( -1 \cup R \ o (d_{i+1} \ o \ R^{-1} = 1) \)

\[ - (d_{i+1} \ o \ R^{-1}) \cup (d_i \ o \ R) \cap (d_{i+1} \ o \ R^{-1}) = 1 \]

It seems that condition (P9) is not expressible in a form of a relational equation. However, if we confine ourselves to algebras with finitely many operations \( D_i \), then the respective class turns out to be a generalized reduct of the class of set relation algebras.

Let a Post-Kripke frame of order \( m \) be given. By a full Post relation algebra of order \( m \) over \( K \) we mean an algebra of the form

\[ \text{fullPostRA}(K) = (X \times W : X \subseteq W, \cup, \cap, \sim, \rightarrow, \neg, D_1, \ldots, D_{m-1}, E_0, \ldots, E_{m-1}) \]

where the operations \( \cup, \cap, \sim, \rightarrow \) are Boolean operations, \( \neg \) and \( \rightarrow \) are defined by \( (p \rightarrow) \) and \( (p \neg) \), respectively, \( D_i \) for \( i = 1, \ldots, m - 1 \) and \( E_j \) for \( j = 0, m - 1 \) are defined by \( (pD) \) and \( (pE) \) respectively, with \( E \) replaced by \( E_{m-1} \).

In a similar way we define classes of algebras:

\[ \text{fullPostRA} = (\text{fullPostRA}(K) : K \text{ is a Post-Kripke frame of order } m) \]

\[ \text{PostRA} = \text{SP (fullPostRA)} \]

Let \( (P9') \) be the condition obtained from \( (P9) \) by assuming that \( i = 1, \ldots, m - 1 \).

**Proposition 3.6**

Condition \( (P9') \) holds in a Post-Kripke frame of order \( m \)

\[ i f f \quad -1 \cup (d_1 \ldots \cup d_{m-1}) \ o \ R = 1 \text{ holds in the algebra fullIRA (W) where W is a set of states of K.} \]

In view of propositions 3.2, 3.3, and 3.4 we conclude that any Post relation algebra of order \( m \) is a generalized reduct of a set relation algebra, namely we have:

**Proposition 3.7**

\[ \text{PostRA} = S(RdRA). \]

It is easy to see that Post relation algebras of order \( m \) correspond to the \( m \)-valued Post logic, namely we have:

**Proposition 3.8**

The following conditions are equivalent:

(a) A formula \( F \) is valid in \( m \)-valued Post logic

(b) \( t(F) = E_{m-1} \) holds in every algebra from PoRA.

In a natural way we can obtain a set of equations defining the class of Post relation algebras of order \( m \). Let \( \text{EQPoRA} \) be the following set of equations derived from axioms of the \( m \)-valued Post logic.

\[ (c1) A \cap (A \rightarrow B) = A \cap B \]

\[ (c2) (A \rightarrow B) \cap B = B \]

\[ (c3) (A \rightarrow B) \cap (A \rightarrow C) = (A \rightarrow (B \cap C) \]

\[ (c4) (A \rightarrow A) \cap B = B \]

\[ (c5) \neg (A \rightarrow A) \cup B = B \]

\[ (c6) A \rightarrow \neg (A \rightarrow A) = \neg A \]

\[ (c7) D_1 (A \cup B) = D_1 A \cup D_1 B \]

\[ (c8) D_1 (A \cap B) = D_1 A \cap D_1 B \]

\[ (c9) D_1 (A \rightarrow B) = (D_1 A \rightarrow D_1 B) \cap \ldots \]

\[ \cap (D_1 A \rightarrow D_1 B) \]

\[ (c10) D_1 (\neg A) = \neg D_1 A \]

\[ (c11) D_1 D_1 A = D_1 A \]

\[ (c12) D_1 E_j = E_{m-1} \text{ for } i < j, D_1 E_j = E_0 \text{ for } i > j \]

\[ (c13) D_1 A \cap \neg D_1 A = E_{m-1} \]

\[ (c14) A = (E_1 \cap D_1 A) \cup \ldots \cup (E_{m-1} \cap D_{m-1} A) \]

\[ 300 \]
Proposition 3.9
Class \( P_{\text{om}} \) RA is definable by \( E \) \( P_{\text{om}} \).

It would be interesting to investigate Post relation algebras within the relation-algebraic framework (Henkin, Monk and Tarski 1985, Nemeti 1990). In particular the algebras introduced in the present paper should be treated as abstract algebras, not necessarily set algebras, and a representation theory for them could be developed.

4. A relational logic for Post relation algebras

In the present section we define a logic \( L \) \( P \) intended to provide a formal tool to verify equations in Post relation algebras. Expressions of the language of \( L \) are constructed with symbols from the following pairwise disjoint sets:

- \( \text{VAROB} \) a set of object variables
- \( \text{CON} \cup \{ \text{Ej} \} \) \( 0 \leq j \leq \omega \cup \{ \text{R} \} \) a set of relational constants
- \( \{ \cup, \cap, \rightarrow \} \) the set of binary relational operations
- \( \{ \rightarrow, \text{D}_1, \text{D}_2, \ldots \} \) the set of unary relational operations.

Set \( \text{EREL} \) of relational expressions is the smallest set including set \( \text{C} \cup \{ \text{R} \} \) of constants and closed with respect to all the relational operations.

Set \( \text{FOR} \) of formulas is the smallest set satisfying the conditions:

- If \( x, y \in \text{VAROB} \) and \( A \in \text{EREL} \), then
  \[ x A y \in \text{FOR} \]
- \( x R y, x - R y \in \text{FOR} \).

Formulas built with relational expressions from set \( \text{EREL} \) are said to be nondegenerate. Now we define models of the language. By a model of the language of logical \( L \) we mean any system of the form:

\[ M = (\text{OB}, R, \{ \text{di} \}; 1 \leq i \leq \omega, m) \]

where \( \text{OB} \neq \emptyset, R \subseteq \text{OB} \times \text{OB} \) is a reflexive and transitive relation intended to provide interpretation of relational constant \( R \), for every \( i \) \( \text{di} \) is a function in \( \text{OB} \) satisfying conditions (P4), \ldots (P9), and \( m \) is a meaning function which assigns binary relations in \( \text{OB} \) to relational expressions and satisfies the following conditions:

- If \( P \in \text{CON} \), then \( m(P) \) is of the form \( x \times \text{OB} \) for a certain \( x \in \text{OB} \)
- If \( (x, y) \in R \) and \( (x, z) \in m(p) \), then \( (y, z) \in m(p) \)
- For \( 1 \leq j \leq \omega, m(\text{Ej}) = \emptyset, m(\text{E}0) = \text{OB} \times \text{OB} \), \( m(R) = R \)
- \( m(\neg A) = \neg m(A), m(\rightarrow A) = \rightarrow m(A), m(\text{D}_i A) = \text{D}_i m(A) \)
- \( m(A \cup B) = m(A) \cup m(B), m(A \cap B) = m(A) \cap m(B), m(A \rightarrow B) = m(A) \rightarrow m(B) \).

By a valuation over \( M \) we mean a function \( v : \text{VAROB} \rightarrow \text{OB} \) which assigns objects to object variables. We say that valuation \( v \) satisfies formula \( x A y \) in model \( M \) whenever \( (v(x), v(y)) \in m(A) \). A formula is true in model \( M \) if it is satisfied by all valuations over \( M \). A formula is valid if it is true in all models.

Observe that every model of logic \( L \) determines an algebra from the class \( P \) RA, namely the algebra generated by \( \{ m(P) : P \in \text{C} - \{ \text{R} \} \} \).

The following theorems provide relationship between Post logics and the relational logic.

Proposition 4.1
For every model of the \( \omega^+ \)-valued Post logic there is a model \( M' \) of relational logic \( L \) such that for any formula \( F \) we have \( F \) is true in \( M \) if and only if \( F \) is true in \( M' \).

Proof: Let a model \( M = (K, m) \) be given, where \( K = (w, R, \{ dj \}; 1 \leq i \leq \omega) \). We define model \( M' = (w, R, \{ dj \}; 1 \leq i \leq \omega, m') \) whose set of objects coincides with \( K \), relation \( R \) and functions \( dj \) are the same as in \( M \), and meaning function \( m' \) is defined as:

\[ m'(P) = m(p) \times w, \quad \forall p \in \text{CON} \]

where \( p \) is a propositional variable whose relational translation is \( P \), that is \( t(p) = P \); for compound relational expressions \( m' \) satisfies conditions (m3), (m4), (m5); \( m'(R) = R \), \( m'(\text{Ej}) = \{(x, y) : (dj x, x) \in E \} \) for \( 0 \leq j < \omega, E_0 = \emptyset, E = \text{W} \times \text{W} \).

Clearly, condition (m2) is satisfied by \( m' \) because \( m \) satisfies the corresponding condition (m).

We show that for any \( s \in \text{W} \) and for any formula \( F \) of the Post logic we have

(i) \( M,s \) sat \( F \) iff \( (s, y) \in m'(t(F)) \)

The proof is by induction with respect to the complexity of \( F \). If \( F \) is a propositional variable, then (i) holds by definition of \( m' \). Let \( F = \text{Ej} \) for \( 0 \leq j < \omega \). We have \( M,s \) sat \( \text{Ej} \) iff \( (dj s, s) \in R \) iff \( (s, y) \in (x, z) : (dj x, x) \in E \) for \( 0 \leq j < \omega, E_0 = \emptyset, E = \text{W} \times \text{W} \).

Clearly, condition (m2) is satisfied by \( m' \) because \( m \) satisfies the corresponding condition (m).

From (i) the theorem follows easily.

Proposition 4.2
For every model \( M \) of the relational logic \( L \) there is a model \( M' \) of the \( \omega^+ \)-valued Post logic such that for any formula \( F \) we have \( F \) is true in \( M \) if and only if \( F \) is true in \( M' \).

Proof: Let a model \( M = (\text{OB}, R, \{ dj \}; 1 \leq i \leq \omega, m) \) be given satisfying conditions (m1), \ldots (m5). We define model \( M' = (\text{OB}, R, \{ dj \}; 1 \leq i \leq \omega, m') \) whose set of states coincides with \( \text{OB} \), relation \( R \) and functions \( dj \) are the same as in \( M \), and meaning function \( m' \) is defined as:

\[ m'(p) = X, \quad \text{where } X \text{ is a subset of } \text{OB} \text{ such that } m(p) = X \times \text{OB} \text{ for } p(t(p)). \]
Clearly, \( m' \) satisfies conditions (P1), \ldots, (P9). Due to (m2) it also satisfies (m). The rest of the proof is similar to the proof of 4.1.

Propositions 4.1 and 4.2 lead to the following theorem.

**Proposition 4.3**
The following conditions are equivalent:
(a) A formula \( F \) of \( \omega^* \)-valued Post logic is valid
(b) Formula \( x t(F) y \) is valid in the relational logic.

5. Deduction rules

In the present section we define deduction rules for the relational logic. We define two kinds of rules: Decomposition rules, which enable us to decompose relational formulas into some simpler formulas, depending on symbols of relational operations occurring in the formulas; specific rules, which correspond to semantical postulates assumed in the models of the relational logic. The rules apply to finite sequences of formulas. As a result of application of a rule we obtain a family of new sequences. Let \( G, H \) denote finite (possibly empty) sequences of relational formulas. We admit the following rules:

**Decomposition rules**

\[
\begin{align*}
\frac{G, x - A y, H}{G, x - A y, H} & \quad (\rightarrow) \\
\frac{G, x A y, x B y, H}{G, x A y, x B y, H} & \quad (\wedge) \\
\frac{G, x - (A \cup B) y, H}{G, x - A y, H, G, x - B y, H} & \quad (\rightarrow) \\
\frac{G, x (A \cap B) y, H}{G, x A y, H, G, x B y, H} & \quad (\cap) \\
\frac{G, x - (A \cap B) y, H}{G, x A y, H, G, x B y, H} & \quad (\neg) \\
\frac{G, x (A \rightarrow B) y, H}{G, x - R z, z - A y, z B y, H} & \quad (\neg) \\
\frac{G, x - (A \rightarrow B) y, H}{G, x - (A \rightarrow B) y, H} & \quad (\neg)
\end{align*}
\]

where \( z \) is a variable

\( H_1 = G, x R z, H, x - (A \rightarrow B) y \)
\( H_2 = G, z A y, H, x - (A \rightarrow B) y \)
\( H_3 = G, z - B y, H, x - (A \rightarrow B) y \)

\[
\begin{align*}
\frac{G, x - A y, H}{G, x - A y, H} & \quad (\rightarrow) \\
\frac{G, x - R z, z - A y, H}{z \text{ is a new variable}} & \quad (\neg)
\end{align*}
\]

\[
\begin{align*}
H_1 = G, x R z, H, x - A y \\
H_2 = z A y, H, x - A y & \quad (\neg)
\end{align*}
\]

\[
\begin{align*}
\frac{G, x D_1 (A \cup B) y, H}{G, x D_1 A y, x D_1 B y, H} & \quad (D) \\
\frac{G, x D_1 - (A \cup B) y, H}{G, x D_1 - A y, x D_1 - B y, H} & \quad (D - \cup)
\end{align*}
\]

\[
\begin{align*}
\frac{G, x D_i - (A \rightarrow B) y, H}{G, x D_i A y, H, G, x D_i B y, H} & \quad (D \cap) \\
\frac{G, x D_i - A y, x D_i - B y, H}{G, x D_i - A y, x D_i - B y, H} & \quad (D - \cap) \\
\frac{G, x D_i - (A \rightarrow B) y, H}{G, x (D_j A \rightarrow D_j B) y, H} & \quad (D \rightarrow) \\
\frac{G, x D_i - (A \rightarrow B) y, H}{G, x (D_j A \rightarrow D_j B) y, H} & \quad (D - \rightarrow) \\
\frac{G, x D_1 - A y, H}{G, x - R z, z - D_1 A y, H} & \quad (D) \\
\frac{G, x D_1 - A y, H}{G, x - D_1 A y, H} & \quad (D -)
\end{align*}
\]

\( z \) is a new variable

\[
\frac{G, x - (A \rightarrow B) y, H}{G, x D_1 - A y, H, x D_1 - A y, H} & \quad (D - \neg)
\]

\( z \) is a new variable

\( (D -) \setminus \{G, x D_1 - A y, H\} \cup \{G, x D_1 - A y, H\} \cup \{G, x D_1 - A y, H\} \cup \{G, x D_1 - A y, H\} \cup \{G, x D_1 - A y, H\} \cup \{G, x D_1 - A y, H\} \cup \{G, x D_1 - A y, H\} \)

\( z \) is a variable
Specific rules

(p1) \[\frac{G, x A y, H}{G, x A z, H}\]
z is a variable, \(A \in \text{CON}\)

(p2) \[\frac{G, z A y, H}{G, z R x, H, G, z A y, H}\]
z is a variable, \(A \in \text{CON}\)

(p3) \[\frac{G, x D_j A y, H, x D_j A y}{G, x D_j A y, H}\]
for \(j \geq i\)

(p4) \[\frac{G, x A y, H}{G, x D_j A y, H} \quad 1 \leq i < \omega\]

(p5) \[\frac{G, x E_j y, H}{G, x E_j y, H, x E_j y}\]
for \(j \leq i\)

(R tran) \[\frac{G, x R y, H}{G, x R z, x R y, H} \quad G, z R y, x R y, H\]
z is a variable

Fundamental sequences play the role of axioms in the deduction system.

A sequence \(H\) of formulas is valid iff for every model \(M\) and for every valuation \(v\) there is a formula \(F\) in \(H\) such that \(v\) satisfies \(F\) in \(M\). A rule of the form \(H/\{H_i\}\)

\(1 \leq i < \omega\) is admissible whenever sequence \(H\) is valid iff for all \(i\) sequences \(H_i\) are valid.

**Proposition 5.1**

(a) Decomposition rules are admissible
(b) Specific rules are admissible
(c) Fundamental sequences are valid.

**Proof:** We show that rule \((D \rightarrow)\) is admissible. Assume that sequence above the line is valid. Let \((s, t) \in D_1 (A \rightarrow B)\). It follows that \((d_j s, t) \in A \rightarrow B\), and hence (i) for all \(z (d_j s, z) \in R\) implies that if \((z, t) \in A\) then \((z, t) \in B\). Suppose that there is \(k \leq i\) such that \((s, t) \in D_k A \rightarrow D_k B\). This means that there is \(u\) such that \((s, u) \in R\) and \((u, t) \in D_k A\) but \((u, t) \in D_k B\). Hence \((d_k u, t) \in A\) and \((d_k u, t) \in B\).

\(D_1 (d_j s, d_j u) \in R\). By \((P7)\) \((d_j s, d_j u) \in R\). By \((P10)\) \((d_j u, u) \in R\). By \((P3)\) \((d_j s, d_j u) \in R\). But due to (i) if \((d_k u, t) \in A\), then \((d_k u, t) \in B\), a contradiction. We conclude that if a valuation satisfies formula \(D_1 (A \rightarrow B)\), then that valuation satisfies every one of \(D_k A \rightarrow D_k B\) for \(k \leq i\).

If a valuation satisfies a formula from \(G\) or \(H\), then clearly that valuation satisfies the same formula in all the sequences under the line. Hence all the sequences under the line are valid.

Now assume that all the sequences under the line of rule \((D \rightarrow)\) are valid. Let for all \(j \leq i (s, t) \in D_j A \rightarrow D_j B\).

In particular for \(j = 1\) we have (ii) for all \(z (s, z) \in R\) and \((z, t) \in D_1 A\) imply \((z, t) \in D_1 B\). Suppose that \((s, t) \in D_1 (A \rightarrow B)\). Hence \((d_1 s, t) \in A \rightarrow B\).

It follows that there is \(u\) such that \((d_1 s, u) \in R\) and \((u, t) \in A\), and \((u, t) \in B\). But from (ii) applied to \(z = d_1 u\) we obtain a contradiction. In all the remaining cases the respective valuation satisfies some formula above the line. We conclude that sequence above the line is valid, which completes the proof of admissibility of rule \((D \rightarrow)\). Now we prove that rule \((D \leftarrow)\) is admissible. If \((s, t) \in D_1 \rightarrow A\), then \((d_1 s, t) \in \rightarrow A\), and hence (iii) for all \(z (d_1 s, z) \in R\) then \((z, t) \in A\). Suppose that there is \(u\) such that (iv) \((s, u) \in R\) and (v) \((d_1 u, t) \in D_1 A\).

By (iv) and (v) we conclude that \((d_1 s, d_1 u) \in R\). By \((P10)\) \((d_1 u, d_1 u) \in R\). By \((P3)\) \((d_1 s, d_1 u) \in R\). By (v) \((d_1 s, t) \in A\). Hence we have \((d_1 s, d_1 u) \in R\) and \((d_1 u, t) \in A\) which is in conflict with (iii).

Now assume that for all \(z (s, z) \in R\) or \((z, t) \in D_1 A\). Thus we have (vi) for all \((s, z) \in R\) implies \((z, t) \in D_1 A\). Suppose that \((s, t) \in D_1 \rightarrow A\). Hence \((d_1 s, t) \in A\).

It follows that there is \(u\) such that \((d_1 s, u) \in R\) and \((u, t) \in A\). By \((P11)\) we have (vii) \((s, d_1 u) \in R\). Since \((u, t) \in A\), we have (viii) \((d_1 u, t) \in D_1 A\). Applying (vi) to \(z = d_1 u\) we have...
(s d1 u) ∈ R implies (d1 u, t) ∉ D1 A, which contradicts (vii) and (viii). In all the remaining cases the respective formulas come from sequences G or H. We conclude that rule (D→) is admissible. Admissibility of the remaining decomposition rules can be obtained in a similar way. For example, rule (DD) corresponds to condition (P4), rule (→ D) is a counterpart of (d).

Admissibility of rules (p1) and (p2) is obtained from (m1) and (m2), respectively.

Now we prove admissibility of rule (p3). If the sequence above the line of the rule is valid, then clearly the sequence under the line is valid. Assume that (s, t) ∈ D1 A. Hence (d1 s, t) ∈ A. Suppose that (s, t) ∈ D1 A, that is (d1 s, t) ∈ A. Since j ≥ i, by (P10) we have (dj s, dj s) ∈ R. By (P13) (dj s, t) ∈ A, a contradiction.

Now we prove admissibility of the infinitary rule (p4). Let (s, t) ∈ A, and suppose that there is i such that (dj s, t) ∈ A. Hence (d1 s, t) ∈ D1 A. By (P4) and (d) we have (d1 s, t) ∈ A. Since (s, t) ∈ A, we have (d1 s, t) ∈ D1 A, a contradiction. Now assume that for all i we have (s, t) ∈ D1 A, that is (d1 s, t) ∈ D1 A, and also by (P4) (d1 s, t) ∈ D1 A. Suppose that (s, t) ∈ A. It follows that (d1 s, t) ∈ D1 A = D1 A, a contradiction. All the remaining valuations satisfy a formula which occurs either in G or in H. Admissibility of (p5) can be easily obtained from (P3), (P10), and the respective definitions. Admissibility of (R tran) is a consequence of (P3).

The proof of condition (c) is by an easy verification.

Given a nondegenerate formula of the form x A y, where A is a compound relational expression, we can decompose it by successive application of the decomposition rules, or we can apply a specific rule. In the process of application of the rules we form a tree whose vertices consist of finite sequences of relational formulas. If rule (p4) is applied to a formula in a vertex of the tree, then the vertex has infinitely many immediate successors. All the other rules produce a finite number of successor vertices. We stop applying formulas in a vertex after obtaining a fundamental sequence. Trees obtained in this way are called decomposition trees.

A branch of decomposition tree is said to be fundamental if it contains a vertex with a fundamental sequence. A decomposition tree is fundamental if all its branches are fundamental. Observe that all the branches in a fundamental decomposition tree are finite.

In general there is no unique decomposition tree for a given formula. The trees are determined by the choice of formulas in a vertex to which the rules are to be applied, and by the choice of variables which are introduced by rules (→), (→→), (→), (D→), (D→), (p1), (p2), (R tran).

By the method similar to that developed in Orlowska (1987, 1988) we can prove the following completeness theorem for the relational logic.

Proposition 5.2
The following conditions are equivalent:
(a) A nondegenerate formula x A y of the relational logic is valid.
(b) There is a fundamental decomposition tree of x A y.

Example 5.1
We show that relational formula F = x (D1 (A ∪ B) → (D1 (A ∪ D1 B))) y is valid.

Example 5.2
We prove that formula F = x (D1 (A ∩ E1) → A) y is valid.
node. Again the sequences obtained in this way are fundamental. We conclude that formula F is valid.

References