A note on minimal partial clones

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1. Introduction

Recent developments of universal algebra show the importance of clones as a tool of analysis and classification. A clone is a superposition closed set of finitary operations on a fixed finite set A containing all projections. Maximal clones of total (= everywhere defined) operations are very important for primality (completeness) and are fully known [Ros 65, Ros 70].

Equally minimal clones play a significant role for the following reversed completeness criterion: Given a set \( R \) of relations under which conditions is the clone consisting of all operations preserving all relations in \( R \) reduced to the set of all projections? As the lattice \( \mathcal{L}_A \) of all clones is atomic, such a criterion might be based on the knowledge of all operations generating a minimal clone. At the present stage it seems that the determination of minimal clones even for small \( |A| \) is a very complex task [M 83, Pail 84, Pail 86, Ros 83].

Clones of partial operations also play an important role in the theory of partial algebras and in computer science (cf. e.g. [Bur 86]). Nevertheless they are much less known. Recently maximal partial clones were completely described by combinatorial properties [Had-R]. In the present paper we address the following problem: What are the atoms of the lattice \( \mathcal{L}_A \) of all partial clones? We show that they are either the atoms of \( \mathcal{L}_A \) or are generated by partial projections defined on a totally reflexive and totally symmetric domain (Theorem 2.5).

* The goal of this note is to present the results of [Bör-H-P].

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2. Preliminaries and the main result

2.1. Let \( A \) be a non-empty set. For a positive integer \( n \), an \( n \)-ary partial operation on \( A \) is a map \( f : \text{dom } f \to A \) where \( \text{dom } f \) is an \( n \)-ary relation on \( A \) called the domain of \( f \). Let \( P_A^{(n)} \) denote the set of all \( n \)-ary partial operations on \( A \) and let \( P_A := \bigcup_{n \geq 1} P_A^{(n)} \), moreover set

\[
O_A^{(n)} := \{ f \in P_A^{(n)} : \text{dom } f = A^n \}
\]

and

\[
O_A := \bigcup_{n \geq 1} O_A^{(n)} \quad (= \text{set of total operations}).
\]

Furthermore for \( n \geq 1 \) we denote by \( \text{Rel}_A^{(n)} := \{ R : R \subseteq A^n \} \) the set of all \( n \)-ary relations on \( A \) and let \( \text{Rel}_A := \bigcup_{n \geq 1} \text{Rel}_A^{(n)} \). In the sequel we shall distinguish between empty relations of different arities and consequently we shall also distinguish between nowhere defined operations (i.e. operations with empty domains) of different arities (cf. 4.7).

For \( C \subseteq P_A \) and \( n \geq 1 \), put \( C^{(n)} := C \cap P_A^{(n)} \) and let

\[
\mathcal{D}(C) := \{ \text{dom } f : f \in C \}.
\]

Also for \( R \subseteq \text{Rel}_A \) let \( R^{(n)} := R \cap \text{Rel}_A^{(n)} \). For \( n, m \geq 1 \), \( f \in P_A^{(n)} \) and \( g_1, \ldots, g_n \in P_A^{(m)} \), we define
the superposition of $f$ and $g_1, \ldots, g_n$, denoted by $f[ g_1, \ldots, g_n ]; \in P_A^{(n)}$, by setting
\[ \text{dom}( f[ g_1, \ldots, g_n ] ) := \{ (a_1, \ldots, a_n) \in A^n : (a_i, \ldots, a_m) \in \bigcap_{i=1}^m \text{dom } g_i \}
\]
and
\[ f[ g_1, \ldots, g_n ] (a_1, \ldots, a_n) := f( g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m) ) \]
for all $(a_1, \ldots, a_n) \in \text{dom } f[ g_1, \ldots, g_n ]$.

For every positive integer $n$, every $n$-ary relation $D$ and each $1 \leq i \leq n$ let $e_{i,D}$ denote the $n$-ary $i$-th partial projection defined by $e_{i,D}(z_1, \ldots, z_n) = z_i$ for all $(z_1, \ldots, z_n) \in D$. For notational ease we shall write $e_i$ for $e_{i,1}$. Furthermore for $\mathcal{R} \subseteq \mathcal{R}_{\lambda A}$ put
\[ \mathcal{P}(\mathcal{R}) := \{ e_{i,D} : 1 \leq i \leq n < \omega, \ D \in \mathcal{R}^{(n)} \} \]
hence $\mathcal{P}(\mathcal{R})$ denotes the set of all partial projections with domains in $\mathcal{R}$. Furthermore let
\[ J_A := \{ e_i : 1 \leq i \leq n < \omega \} \]
be the set of all total projections.

2.2. Definitions. A partial clone on $A$ is a superposition closed subset of $P_A$ containing $J_A$ (for an equivalent definition see [Had-R]). If a partial clone $C$ contains an $n$-ary operation $f$ with $\text{dom } f \neq A^n$, then it is called a proper partial clone. If otherwise $C \subseteq O_A$ then it is called a total clone. Moreover a partial projection clone is a clone containing only partial projections. For $C \subseteq P_A$, let clone$(C)$ denote the partial clone generated by $C$, i.e. the least clone containing $C$.

2.3. Example. Let $0 \in A$ and let
\[ \text{Pol}(0) := \bigcup_{n \geq 1} \{ f \in P_A^{(n)} : (0, \ldots, 0) \in \text{dom } f \Rightarrow f(0, \ldots, 0) = 0 \}, \]
then Pol$(0)$ is a proper partial clone on $A$.

The partial clones (respectively the total clones) on $A$, ordered by inclusion form an algebraic lattice $\mathcal{L}_{P_A}$ [Had-R] (respectively $\mathcal{L}_{O_A}$) in which every meet is the set-theoretical intersection. For $F \subseteq P_A$, the partial clone generated by $F$ is the intersection of all partial clones containing the set $F$ (or equivalently is the set of term operations of the partial algebra $\langle A, F \rangle$). A minimal partial clone is an atom of $\mathcal{L}_{P_A}$, i.e. a partial clone covering the set $J_A$. The problem of determining all the atoms of $\mathcal{L}_{O_A}$ for $A$ finite was raised in [Pöss-K 79], it has been studied by several authors [Csád 83, Pál 84, Pál 86, Ros 83, Sze 86], and remains unsolved at the present time. However, for $A$ finite, it is known that $\mathcal{L}_{O_A}$ is an atomic lattice (i.e. every total clone on $A$ distinct from $J_A$ contains a minimal total clone) with a finite number of atoms. The five types of minimal total clones are discussed in [Ros 83, Sze 86].

2.4. Definitions. Let $n \geq 1$, $R$ be an $n$-ary relation on $A$ and $S_n$ be the group of permutations on $\{1, \ldots, n\}$.

The relation $R$ is said to be
\begin{enumerate}
  \item totally symmetric if for all $\pi \in S_n$ and $(a_1, \ldots, a_n) \in A^n$,
  \[ (a_1, \ldots, a_n) \in R \Leftrightarrow (a_{\pi(1)}, \ldots, a_{\pi(n)}) \in R, \]
  \item totally reflexive if for every $(a_1, \ldots, a_n) \in A^n$ and all $1 \leq i < j \leq n$, the equality $a_i = a_j$ implies that $(a_1, \ldots, a_n) \in R$,
  \item nontrivial if $R \neq A^n$.
\end{enumerate}

Note that any subset of $A$ (including the empty set $\emptyset$) is considered as a totally reflexive and totally symmetric relation.

In the sequel we shall consider a finite set $A$ with $|A| > 1$.

The following lemma shows that a minimal proper partial clone is in fact a partial projection clone.

2.5. Lemma. Let $C$ be a partial clone. Then the set $C' := \mathcal{P}(\mathcal{D}(C))$ is a partial projection clone contained in $C$ and such that $\mathcal{D}(C') = \mathcal{D}(C)$.

From Lemma 2.6 we deduce that any partial clone $C$ contains a partial projection clone $C'$. If $C$ is a total clone, then $C' = J_A$ but if $C$ is a proper partial clone, then $C'$ contains a not everywhere defined partial projection.
Hence we have

2.6. Corollary. Every minimal clone on $A$ is either a total minimal clone (i.e. an atom in both $C_{o_A}$ and $C_{p_A}$) or is a proper partial projection clone.

Clearly a projection clone $C$ is uniquely determined by the set $D(C)$ of all the domains of its operations. This leads us to investigate some special sets of relations on $A$ which we call weak systems of relations.

3. Weak systems of relations

We want to characterize the subsets of $\text{Rel}_A$ of the form $D(C)$ where $C$ is a partial clone on $A$. For this we have to introduce the following concepts:

3.1. Definition. For an integer $n \geq 1$ we denote $\omega_n := \{1, \ldots, n\}$. Let $n, m \geq 1$ be integers and let $s : \omega_n \rightarrow \omega_m$ be a map. Define the mapping $t_s : \text{Rel}_{A^n} \rightarrow \text{Rel}_{A^m}$ by setting

$$t_s(R) := \{ (a_{s(1)}, \ldots, a_{s(n)}) \in A^m : (a_{s(1)}, \ldots, a_{s(n)}) \in R \}$$

for all $R \in \text{Rel}_{A^n}$.

3.2. Examples. Let $s : \omega_n \rightarrow \omega_m$ be a map.

1) Let $n = m$ and let $s$ be a permutation on $\omega_n$. Then $t_s$ performs a re-arrangement of the arguments of any $R \in \text{Rel}_{A^n}$. In particular an $n$-ary relation $R$ is totally symmetric if and only if $t_s(R) = R$ for all $s \in S_n$. Also $R$ is totally reflexive if and only if $t_s(R) = A^n$ for every non-injective $r : \omega_n \rightarrow \omega_n$.

2) Let $n < m$ and assume $s(k) = k$ for all $k = 1, \ldots, n$. Then $t_s$ adds fictive arguments to any $R \in \text{Rel}_{A^n}$.

3) If $s(i) = s(j)$ for some $i \neq j \in \omega_n$, then $t_s$ performs the identification of the $i$-th and $j$-th argument of any $R \in \text{Rel}_{A^n}$.

The following results are easy to check:

3.3. Fact. Let $n, m, k \geq 1$ be integers, $s : \omega_n \rightarrow \omega_m$ and $s' : \omega_m \rightarrow \omega_k$ be maps and let $R, Q \in \text{Rel}_{A^n}$. Then

$$t_s(A^n) = A^m,$$

$$t_s(R \cap Q) = t_s(R) \cap t_s(Q),$$

$$t_{s'}(t_s(R)) = t_{s \circ s'}(R)$$

where $s$ denotes the composition of maps: $(s' \circ s)(i) := s'(s(i))$.

3.4. Definition. A set $R \subseteq \text{Rel}_A$ is said to be a weak system of relations on $A$ if the following conditions hold for all $n, m \geq 1$:

(i) $A^n \in R$,

(ii) for all $R, Q \in R^{(n)}$ we have $R \cap Q \in R^{(n)}$ (i.e. $R$ is closed under finite intersections),

(iii) for all $s : \omega_n \rightarrow \omega_m$ and all $R \in R^{(n)}$ we have $t_s(R) \in R^{(m)}$ (i.e. $R$ is closed under all operations $t_s$).

The following result gives the relationship between partial clones and weak systems of relations:

3.5. Proposition. Let $C$ be a partial clone on $A$. Then $D(C)$ is a weak system of relations. Conversely if $R$ is a weak system of relations on $A$ then $P(R)$ is a partial (projection) clone.

3.6. Denote by $\mathcal{W}_A$ the set of all weak systems on $A$. $\mathcal{W}_A$ can be considered as the set of all subalgebras of a many-sorted (heterogeneous) algebra with carrier set $(\text{Rel}_{A^n})_{n \geq 1}$ and the operations $A^n$ (constants), $\cap$ and $t_s$ (cf. 3.4(i)-(iii)). Therefore $<\mathcal{W}_A, \subseteq>$ forms a complete algebraic lattice. The smallest element in this lattice is clearly the weak system

$$T_A := \{ A^n \mid n \geq 1 \}.$$

Moreover, given a set $R \subseteq \text{Rel}_A$, the weak system generated by $R$, which we shall denote by $< R >_{\mathcal{W}_A}$, is the least weak system containing $R$. Hence every relation in $< R >_{\mathcal{W}_A}$ can be obtained from $R$ by applying finitely many times the operations 3.4(i) – (iii) to the relations in
4.1. Lemma. Let \( R \) be a partial clone. If \( R \) is a proper projection clone whose domain is a nontrivial totally reflexive and totally symmetric relation, then there exists a nontrivial totally reflexive and totally symmetric relation \( Q \) such that \( R \) is incomparable to \( Q \).

4.2. Lemma. Let \( n \geq 1 \) and \( R \in \text{Rel}_A^{(n)} \) be nontrivial, totally reflexive and totally symmetric. Then \( <R>_{w^*} \) is a minimal weak system of relations on \( A \), i.e. \( <R>_{w^*} \) is an atom of \( \mathcal{W}_A; \).

4.3. Remark. Let \( n, m \geq 1 \), \( R \in \text{Rel}_A^{(n)} \), \( Q \in \text{Rel}_A^{(m)} \) be two nontrivial totally reflexive and totally symmetric relations. Then the equality \( <R>_{w^*} = <Q>_{w^*} \) implies that \( n = m \) and \( R = Q \). Indeed from the proof above we get that \( R = t_u(Q) \) and \( Q = t_v(R) \) for some maps \( u : m \to n \) and \( v : n \to m \). As both \( R \) and \( Q \) are nontrivial and totally reflexive we have that both \( u \) and \( v \) are injective (cf. 3.2(1)). Hence \( n = m \) and \( u, v \in \text{S}_n \). Moreover from \( R \) and \( Q \) we conclude that \( R = Q \).

We collect the above results (4.1, 4.2, 3.7(d)) to obtain

4.4. Corollary. Let \( A \) be a finite set and \( C \) be a proper partial clone on \( A \). Then \( C \) is a minimal proper partial clone if and only if \( P(C) \) is a minimal weak system of relations whence if and only if \( C \) is generated by a proper partial projection whose domain is a nontrivial totally reflexive and totally symmetric relation.

We are now in a position to state the following result:

4.5 Theorem. Let \( A \) be a finite set with \( |A| > 1 \). The lattice \( L_{P_A} \) of all partial clones on \( A \) is atomic and contains a finite number of atoms. Moreover \( C \) is a minimal partial clone if and only if \( C \) is a minimal total clone or is generated by a single partial projection with a nontrivial totally reflexive and totally symmetric domain.

We first show that the lattice \( L_{P_A} \) is atomic. Let \( C \) be a partial clone. If \( C \) is a total clone then it contains a minimal total clone since \( L_{O_A} \) is atomic (cf. e.g. [Pöps-K 79]; 3.1.5). Otherwise \( C \) is a proper partial clone. Thus \( C \) contains \( P(D(C)) \) (by 2.6), and \( D(C) \) contains a minimal weak system \( R \) by 4.1 and 4.2. Therefore \( C \) contains the minimal partial clone \( P(R) \) (cf. 3.7(d)). Hence \( L_{P_A} \) is atomic.
As there are only finitely many minimal total clones (cf. e.g. [Sze 86; 1.14], [Pös-K 79], [Ros 83]) and finitely many non-trivial totally reflexive and totally symmetric relations on $A$ (their arities are bounded by $|A|$), we deduce (cf. 2.7 and 4.4) that the number of minimal partial clones on $A$ is finite. The remaining part of Theorem 2.5 is Corollary 4.4.

Let $|A| = k > 1$ and denote by $r(k)$ the number of minimal total clones on $A$ (i.e. $r(k)$ is the number of atoms of $L_0$, and this number is unknown for $k > 3$).

Using 2.5, 4.4 and 4.3 we get

4.6. Corollary. The number $m(k)$ of minimal partial clones on $A$ with $|A| = k \geq 2$ is

$$m(k) = r(k) + \sum_{\ell=1}^{k-1}(2^{\ell} - 1).$$

In particular we have $r(2) = 7$, $m(2) = 11$, $r(3) = 84$, $m(3) = 99$.

Proof: By 4.3 and 4.4, we have to count all different totally reflexive and totally symmetric relations on $A$. Let $1 \leq \ell \leq k$ and $\rho \in \text{Rel}^R_0$ be totally reflexive and totally symmetric. Thus $\rho$ contains the relation

$$R_0 := \{(a_1, \ldots, a_\ell) \in A^\ell : |(a_1, \ldots, a_\ell)| < \ell\}

and moreover $(a_1, \ldots, a_\ell) \in \rho$ implies $(a_{i(1)}, \ldots, a_{i(\ell)}) \in \rho$ for all $i \in S_\ell$. Consequently either $\rho = R_0$ or

$$\rho = R_0 \cup \bigcup_{i=1}^{\ell-1} \{(a_{i(1)}, \ldots, a_{i(\ell)}) : x \in S_\ell\}

for some $\ell$-element subsets $A_i = \{a_{i(1)}, \ldots, a_{i(\ell)}\} \subseteq A$, $i \in \{1, \ldots, \ell\}$, where $1 \leq s \leq \binom{\ell}{s}$. There are $\binom{\ell}{s}$ such subsets and hence we have $2^{\binom{\ell}{s}} - 1$ different choices for $\rho$ (the $-1$ comes from the fact that $R = A^\ell$ is excluded, $R = R_0$ corresponds to the choice of no subset $A_i$). This establishes the formula for $m(k)$. Now it is well known that $r(2) = 7$, $r(3) = 84$ ([Pös 41], [Cás 83]).

4.7. Remarks. 1) For every $n \geq 1$ denote by $\theta_n$ the $n$-ary partial operation on $A$ with empty domain; i.e. $\text{dom} \theta_n = \theta_n$ (where $\theta_n \in \text{Rel}^A_n$) denotes the empty set considered as the empty $n$-ary relation). Then every $\theta_n$ generates the weak system

$$<\theta_n>_{\text{w}} = <\theta_n>_{\text{r}} = T_n \cup \{\theta_n : n \geq 1\}

(since $t_{\text{w}}(\theta_n) = \theta_n$ for $x : n \rightarrow m$) which is minimal since $\theta_n \in \text{Rel}^A_n$ is totally reflexive and totally symmetric. The corresponding minimal partial clone is

$$P(<\theta_n>_{\text{r}}) = I_n \cup \{\theta_n : n \geq 1\}.$$

This clone actually corresponds to $\ell = 1$ and $R = \theta_1$ in the counting above.

2) In [Had-R-S] partial clones are defined differently and one can show that our definition and the definition given in [Had-R-S] are not equivalent (every partial clone in the sense of 2.2 is a clone in the sense of [Had-R-S] but not conversely). The problem of describing all minimal partial clones according to the definition in [Had-R-S] is still open.

References


