A DECADE OF SPECTRAL TECHNIQUES

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Abstract

This paper summarizes some of the significant achievements of the last ten years in the area of "Spectral Techniques". We have chosen to give as many references as possible to help the reader interested in particular aspects rather than going through the technicalities of the proof of some theorems. From the several known fields of Applications we have selected to give a review on Pattern Analysis with emphasis in the study of self-similarity of Patterns.

Foreword

Ten years ago, at the 11th. International Symposium on Multiple-valued Logic, Mark Karpovsky gave a first Main Lecture on Spectral Methods for decomposition, design and testing of multiple-valued digital systems [KARP 81]. The efforts to relate Spectral Methods and Multiple-valued Switching Theory, may however be traced back to the very beginning of the ISMVL-Series: at the first Meeting (1971), still under the name "Symposium on the Theory and Applications of Multiple-valued Logic Design", Tadahiro Kitahashi presented a paper on the orthogonal expansion of multi-valued functions and their threshold-realizability [KITA 71]. Increasing research efforts directed to both theoretical and applications oriented aspects of Spectral Methods (within the English publishing community) may be noticed in the second half of the seventies (See references dated 1977-1980); particularly after the appearance of the pioneering paper of R. Lechner [LECH 71] and the books of N. Ahmed and K.R. Rao [AHRA 75] and of M.G. Karpovsky [KARP 76].

The present paper intends to point out some of the most important achievements of the last decade and summarize some of the contributions of the author to this process. A possibly complementary review may be found in [MORA 85].

Introduction

The name "Spectral Techniques" has been accepted in the scientific literature to denote developments in Abstract Harmonic Analysis oriented to possible Applications in e.g. Switching Theory and Logic Design [HUMM 85], [MORA 88a], [LUMO 89]; Fault Detection [KARP 85], [HUMM 85], [MILL 82, 87]; Coding Theory [BLAH 83], [FUMY 86], [XIMO 88], [XIAO 90] and Pattern Analysis [SEMO 82], [MOSE 83], [MOSE 85], [MOSE 86], [MORA 89], [LUMO 90].

Abstract Harmonic Analysis emerged as a generalization of Fourier Analysis [FOUR 22] by replacing complex-valued functions on the real group, by complex-valued functions on an arbitrary (locally compact Abelian) group. The co-domain of the functions has been mostly the complex field. It did take quite long until the concept of derivative found a proper expression in Abstract Harmonic Analysis. This was due to the work of J.E. Gibbs [GIBB 67]. (An up-to-date Bibliography of Gibbs derivatives may be found in [BUST 90] pp. XIV-XXIV)

In the area of "Spectral Techniques" related to multiple-valued systems we usually deal with real-valued functions defined on a finite locally compact Abelian group and with co-domain the same group. Some of the most important transforms that take the place of the Fourier transform in this case are the Chrestenson [CHRE 55], which is a generalization of the Walsh transform [WALS 23], the Watari [WATA 56], which is a generalization of the Haar transform [HAAR 10] and the
Zhang-Hartley [ZHAN 84], [MORA 87a, 87b], [MOPO 90a, 90b], which is a real valued transform related to the Chrestenson, in an analogous way as the Hartley [HART 42] is related to the Fourier transform. Another perhaps less known real valued transform (including the Hartley as a special case) is the W-Transform [WANG 84, 89], originally introduced in China. A different approach is taken by using Number Theoretic Transforms (NTT), which are generalizations of the discrete Fourier Transform over residue class rings of rational integers [AGBU 75], [KIBL 77], [CREU 88], by using polynomial transforms [ZHMO 88], [KSHZ 90], generalized Reed-Muller Transforms [CHWU 83], [ORLL 90] or by using discrete Trigonometric transforms [AHNR 74], [ZHPA 90]. Finally, from the theoretical point of view, there have been some important developments, which should be reported here. The F-Transform [CRTA 85], which has been introduced to work with signals defined on a finite abelian group with values in a commutative ring with unity and includes the discrete Fourier and Walsh transforms as well as several polynomial and number theoretic transforms as special cases; the Algebraic Discrete Fourier Transform [BEFM 83], [BETH 84], which possesses a fast symbolic DFT-Algorithm; fast Fourier Transforms on finite non-abelian groups [KARP 77], [BETH 84] and metabelian groups [CLAU 89]; Hilbert Transforms on both abelian and non-abelian finite groups [MORA 87b], [MOSA 86, 88], [STMO 90] as well as further contributions to the theory of Gibbs derivatives [STMO 89], [ZHMO 89], [BUST 90].

Work in the area of Spectral Techniques has been highlighted by a series of international Workshops. The first one was chaired by Mark Karpovsky at the Boston University in 1983 followed by a Workshop chaired by Michael Corinnaus at the École Polytechnique de Montréal in 1986. Germany hosted the 1988 Workshop at the University of Dortmund, followed by another Workshop organized by C. Moraga and Zhang Gongli within the International Conference on Signal Processing in Beijing, P.R. China in 1990. Germany may probably host the next international Workshop on Spectral Techniques in 1992. These activities have supported the following publications: [KARP 85], [MORA 88b], [ICSP 90] (Vol. II, pp 1178-1232) and [MOCR 91].

Pattern Analysis

Notation and Definitions.

Under Pattern Analysis we understand here the study of structural properties of patterns by means of two-sided Spectral Transforms. In a way, this is a natural development of former studies on symmetries and co-symmetries of binary truth-vectors [HURS 77], [HUMM 85], [MORA 88a] and multiple-valued vectors [MILL 81], [KHOD 89].

In the context of this paper, a pattern is just a digitized picture i.e., it is an array of pixels. Pixels are elementary picture atoms and carry a single color. It becomes apparent that the size of a pixel depends on the chromatic and geometric resolution chosen to digitize the original picture. At the same time we represent patterns as real-valued matrices and we define a one to one mapping between the set of values and the set of colors required to represent patterns and to operate upon patterns. Both are unique representations of a same pictorial object. Operations will be carried out using the pattern representation for which they have the lowest complexity; but the resulting pattern will "exist" both with a pixel-representation and a numerical one.

Definition 1.

Let \( \Pi(m,n) \) be the set of all matrices with \( p^m \) rows and \( p^n \) columns for some \( p \in \mathbb{N}, \ p \geq 2 \) and \( m,n \in \mathbb{N}_0 \). If \( A \in \Pi(m,n) \) we may simply write \( A(\ldots) \) and if \( m=n \) we just write \( A(m) \).

Definition 2.

The set of Chrestenson functions \([CHRE 55]\) forms a complete orthogonal system. These functions are defined as follows:

\[
\chi_w : (\mathbb{Z}_p)^n \to \mathbb{C}, \ \text{with } w \in (\mathbb{Z}_p)^n
\]

\[
\chi_w(z) = \exp \left( iw^\top z (2\pi/p)^{1/2} \right)
\]

where \( \forall \ w, z \in (\mathbb{Z}_p)^n \). \( \langle w, z \rangle \) denotes the inner product of \( w \) and \( z \).
The Chrestenson kernel will be denoted by \((p) \mathbf{X}_{(n)}\). If no confusion arises, the parameter \(p\) will be omitted. For a given \(p\) the kernel may be recursively defined as shown below:

\[
\begin{align*}
\mathbf{X}_{(0)} &= [1] \\
\mathbf{X}_{(1)} &= [x_0(c)] \quad 0 \leq q \leq p-1 \\
\mathbf{X}_{(n)} &= \mathbf{X}_{(1)} \otimes X_{(n-1)} = \mathbf{X}_{(n-1)} \otimes \mathbf{X}_{(1)}
\end{align*}
\]

where \(\otimes\) denotes the Kronecker product.

The chrestenson Kernel is obviously orthogonal:

\[
\mathbf{X}_{(n)} \mathbf{X}^*_{(n)} = \mathbf{X}^*_{(n)} \mathbf{X}_{(n)} = p^n \mathbf{I}_{(n)}
\]

where \(\mathbf{X}^*_{(n)}\) denotes the complex conjugate of \(\mathbf{X}_{(n)}\).

Definition 3.

The (two sided) Chrestenson Spectrum of a pattern \(A(m, n)\) is given by:

\[
T_A = p^{-(m+n)} \mathbf{X}_{(m)} A(m, n) \mathbf{X}^*_{(n)}
\]

It is interesting to point out that in this context, the matrix representation of the Gibbs differential operator for \(p\)-valued functions [MORA 83], up to a scaling factor \(p^n\), may be considered as the spectrum of a diagonal pattern whose elements on the main diagonal correspond to those of \(Z_{p^k}\).

Definition 4.

The Zhang-Hartley kernel [ZHAN 84], [MORA 87] is defined as follows:

\[
\mathbf{ZH}_{(n)} = Re \mathbf{X}_{(n)} + Im \mathbf{X}_{(n)}
\]

It has been shown [ZHAN 84] that this kernel is orthogonal and symmetric, i.e., \(\mathbf{ZH}_{(n)} \mathbf{ZH}_{(n)} = p^n \mathbf{I}_{(n)}\).

Definition 5.

The (two sided) Zhang-Hartley Spectrum of a Pattern \(A(m, n)\) is given by:

\[
T'_A = p^{-(m+n)} \mathbf{ZH}_{(m)} A(m, n) \mathbf{ZH}^*_{(n)}
\]

Definition 6.

Let \(Q_{(1)} = \{ q_{jk} \} \), \(0 \leq j, k < p\), with \(q_{jk} = 1\) if \(j+k \equiv 0 \mod p\) and \(q_{jk} = 0\) otherwise. It becomes apparent that \(Q\) is a permutation matrix with the following properties, which are simple to prove:

\[
\begin{align*}
Q_{(1)}Q_{(1)} &= I_{(1)} \\
Q_{(n)} &= Q_{(1)} \otimes Q_{(n-1)} = Q_{(n-1)} \otimes Q_{(1)} \\
X_{(n)}Q_{(n)} &= Q_{(n)}X_{(n)} = X^*_{(n)} \\
\mathbf{ZH}_{(n)}Q_{(n)} &= Q_{(n)}\mathbf{ZH}_{(n)}
\end{align*}
\]

Definition 7.

\[
\begin{align*}
A(m, n) \text{ is even iff } & \quad A(m, n) = Q_{(m)} A(m, n) Q_{(n)} \\
A(m, n) \text{ is odd iff } & \quad A(m, n) = -Q_{(m)} A(m, n) Q_{(n)} \\
A(m, n) \text{ is strong-even iff } & \quad A(m, n) = Q_{(m)} A(m, n) = A(m, n) Q_{(n)} = Q_{(m)} A(m, n) Q_{(n)}
\end{align*}
\]

Theorem 2. [MOW 90a]

A pattern is even iff its Chrestenson spectrum is real, and is odd iff its Chrestenson spectrum is imaginary.

Corollary 1.1 [MORA 89], [MOPO 90a]

A pattern is even iff its Chrestenson and its Zhang-Hartley spectra are identical. A pattern is odd iff its Chrestenson spectrum equals its Zhang-Hartley spectrum multiplied by \((-1)^Q\).

Definition 8.

A non-zero matrix with entries from \(\{0, 1\}\) will be called binary. Meanwhile a non-zero matrix with entries from \(\{0, k\}, k \in \mathbb{R}\) will be called two-valued.

Definition 9.

A Chrestenson self-similarity matrix is a two-valued matrix which has a two-valued Chrestenson spectrum.

Theorem 2.

If a matrix is Chrestenson self-similar (CSS) then it is even and its spectrum is also CSS.
Definition 10.
A Zhang-Hartley self-similarity matrix is a two-valued strong even matrix which has a (strong even) two-valued Zhang-Hartley spectrum.

It becomes apparent that if a matrix is Zhang-Hartley self-similar (ZHSS) then the ZH-spectrum of such a matrix is also ZHSS. Moreover both the matrix and its ZH-spectrum are CSS.

Definition 11.
A pattern $A$ is in a Chrestenson or Zhang-Hartley self-similarity relation to a pattern $B$ iff there exist Chrestenson or Zhang-Hartley self-similarity matrices $L$ and $R$, such that:

$$A = L \otimes B \otimes R$$

Definition 12.
Let $E(m,n)$ be a pattern with a 1-entry at the first element be a constant of the first row and 0 everywhere else and pattern all whose elements are 1.

It becomes apparent that both patterns are strong even. Moreover $T = p - (m + n) U(m,n)$ and $T U = E(m,n)$. It follows that both $E(m,n)$ and $U(m,n)$ are ZHSS and CSS.

Properties of Patterns

Theorem 3. [MOSE 85], [MORA 85]
For a given $p$, let $A(m,n), B(m,n), C(n,s), D(m,n)$, and $F(n,t)$ be patterns and let $J_{(n,m)} = (A_{(m,n)})^{-1}$. Moreover let $u, v \in \mathbb{R}$. Then we have:

$$T_{(A \ast \ast B)} = u T_A + v T_B$$
$$T_{AC} = T_A \otimes T_C \ast F^n$$
$$T = p \cdot m \cdot n \cdot (T_A)^{-1}$$
$$T_{AD} = T_A \otimes T_D$$
$$T_{(A \otimes D) (C \otimes F)} = T_{AC} \otimes T_{DF}$$

The first three properties also hold for the Zhang-Hartley spectrum. If $A$ or $D$ is strong even, the fourth and finally, if $AC$ or $DF$ is strong even, the fifth.

Theorem 4: [MOSE 85]
Let $B_{(m,n)}$ be a pattern and let $B^*$ denote the 2D $p$-adic shifted version of pattern $B$. Moreover let $B^*$ denote pattern $B$ rotated by $\pi$. Then we have:

$$T_{B \otimes Q} = Q_{(m)} T_B Q_{(n)}$$
$$T_B = T_B^* = T_B$$

If we call $QBQ$ the reversed of $B$ then the spectrum of a reversed pattern equals the reversed spectrum of the same pattern. Moreover the magnitude of the spectrum (also the power spectrum)- of a pattern is invariant to $p$-adic shift or $\pi$-rotation of this pattern.

Theorem 5. [LUMO 90]
Let $p = qr$. Then $S_{(1)} = (I_{(1)} \otimes (p) E(1))$ is CSS and $S_T = q \cdot p^2 (U_{(1)} \otimes (p) I_{(1)})$.

Theorem 6.
For a given $p$, all elements of the free Semigroup $(I_{(q)}, (p) E_{(r,s)}, (p) U_{(t,u)})$ with respect to the Kroneker product are CSS and all elements of the free Semigroup $(p) E_{(r,s)}, (p) U_{(t,u)} | q, r, s, t, u \in \mathbb{N}_0^+$ with respect to the Kroneker product are ZHSS. Moreover if $p = qr$ then all elements of the free Semigroup $(I_{(r)}, (p) S_{(1)}, (p) E_{(r,s)}, (p) U_{(t,u)}) | q, r, s, t, u, v \in \mathbb{N}_0^+$ with respect to the Kroneker product are CSS.

Theorem 7.
Let $p = 2q$ and define $(p) W_{(1)} = (2I_{(1)} \otimes (p) E_{(1)})$. Then $(p) W_{(1)}$ is ZHSS and $T_{W} = 2p^2 (U_{(1)} \otimes (p) I_{(1)})$. Moreover all elements of the free Semigroup $(p) W_{(1)}, (p) E_{(r,s)}, (p) U_{(t,u)} | q, r, s, t, u \in \mathbb{N}_0^+$ with respect to the Kroneker product are also ZHSS.

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Examples

Fig. 1 shows two matrices which are ZHSS and (up to a scaling factor) form a transform pair. On the other hand, the identity matrix for \( p > 2 \) is the simplest example of a pattern which is CSS but not ZHSS since \( I_{(p)} \) is even, but not strong-even. Only \( (2)I_{(p)} \) is strong-even.

Let \( B = (4)W_{(1)} \otimes (4)A_{(m,0)} \otimes (4)E_{(r,s)} \)

From Theorems 3, 5 and 7 we obtain

\[
T_B^* = ((2 \cdot 4^2) U_{(1)} \otimes (2)I_{(1)}) \otimes T_A \otimes 4^{r-s} U_{(r,s)}
\]

and

\[
T_B = ((2 \cdot 4^2) U_{(1)} \otimes (2)I_{(1)}) \otimes T_A \otimes 4^{r-s} U_{(r,s)}
\]

Now let \( D = (4)U_{(1)} \otimes (4)A_{(m,0)} \otimes (4)E_{(r,s)} \)

Then we have:

\[
T_D = (4^{-1}) (4)U_{(1)} \otimes T_A \otimes 4^{r-s} U_{(r,s)}
\]

It follows that \( B \) is in the Chrestenson and the Zhang-Hartley self-similarity relation to \( A \), meanwhile \( D \) is only in the Chrestenson self-similarity relation to \( A \).

Closing remarks

Spectral Techniques offer a convenient formal frame to study structural properties of patterns. In this paper we have reported mainly on self-similarity of patterns. Statistical self-similarity is a basic property appearing in nature and it addresses the fact that many natural patterns seem to repeat themselves when observed at different levels of magnification. Here we use an abstract concept of self-similarity for patterns which involves magnification, dispersion and self-reproduction (which is a further development of our idea of liveness [MOSE 85]). Definition 11 formalizes this in a rather restricted way, but nevertheless allows the analysis or synthesis of very large classes of self-similar patterns.
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