Fundamental Properties of Kleene-Stone Logic Functions

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Abstract

Kleene algebra has correspondence with fuzzy sets or fuzzy logic, so it is suitable for treating ambiguity. In contrast, Stone algebra connects with modality. Kleene-Stone algebras have been proposed as an algebra having both properties of Kleene algebra and Stone algebra. Therefore, they have connections with ambiguity and modality.

We define Kleene-Stone logic function as a function \( F : \{0, 1\}^n \rightarrow \{0, 1\} \) represented by a logic formula. Here, a logic formula is defined to be the form constructed by applying the logic operations AND(\( \cdot \)), OR(\( \lor \)), NOT(\( \neg \)), to the variable \( x_1, \ldots, x_n \) and the constants \( 0 \) and \( 1 \). Kleene-Stone logic functions are one of the models of Kleene-Stone algebras, that is, Kleene-Stone logic functions are suitable for treating ambiguity and modality. In this paper, we will show a necessary and sufficient condition for functions to be Kleene-Stone logic functions and an equation of the number of \( n \)-variable Kleene-Stone logic functions.

1 Introduction

In ordinary classical logic (binary logic), every proposition is assumed to be either true(1) or false(0) and is not permitted to take indeterminate state as truth value. Many propositions and subjects, in the so-called "real world", however, are ambiguous or not exactly definable. Thus, we can not treat ambiguous discussions or deductions in binary logic. In order to treat such ambiguous state, many multiple-valued logic functions have been studied hitherto, in which every proposition was permitted to take an intermediate value between true(1) or false(0) as its truth value. For example, the following multiple-valued logic functions have been studied in order to treat ambiguous state; B-ternary (binary ternary) logic functions[11], regular ternary logic functions[4], fuzzy logic functions(fuzzy switching functions)[4, 5, 6, 8], and so on.

Each one of these logic functions, if looked at as part of an algebraic system, satisfies the axioms of Kleene algebra[7], that is, each one of them is one of the models of Kleene algebra. Kleene algebra has correspondence with fuzzy sets or fuzzy logic and has recently studied as an algebraic system treating ambiguity or fuzziness. In contrast, Stone algebra, which has connections with modality, has properties different from Kleene algebra. Kleene-Stone algebras[9,10] have been proposed as an algebra which is both a Kleene algebra and a Stone algebra. Therefore, Kleene-Stone algebras are an algebraic system which satisfies the properties of both Kleene algebra and Stone algebra, that is, Kleene-Stone algebras have connections with ambiguity and modality. The characteristic feature of Kleene-Stone algebras is that there are two kinds of unary operations \( \neg \) and \( \sim \), where \( \neg \) means unary operation for Kleene algebra and \( \sim \) means that for Stone algebra.

In this paper, at first, we define Kleene-Stone logic functions(Kleene-Stone logic functions are one of the models of Kleene-Stone algebras, that is, they are suitable for treating modality(possibility, necessity) and ambiguity). Then we will show the fundamental properties of Kleene-Stone logic functions(especially, algebraic properties).

In section 3, a necessary and sufficient condition for functions to be Kleene-Stone logic functions will be shown, and in section 4, we will show a formula which represents the number of \( n \)-variable Kleene-Stone logic functions. Recently, the same results concerning the number of Kleene-Stone logic functions were obtained by F. Guzmán and C. C. Squier[12].

2 Kleene-Stone logic functions

Let \( V \) be a closed interval \([0, 1]\). An \( n \)-variable Kleene-Stone logic function is a mapping from \( V^n \) to \( V \); \( F: V^n \rightarrow V \), which is represented by a logic formula consisting of variables \( x_1, \ldots, x_n \), constants \( 0 \) and \( 1 \), and logical operations AND(\( \cdot \)), OR(\( \lor \)) and two kinds of NOT(\( \neg \)). Here, logical operations are defined as follows:

\[
\begin{align*}
x \cdot y &= \min(x, y) \\
x \lor y &= \max(x, y) \\
\neg x &= 1 - x, \\
\sim x &= \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0, \end{cases}
\end{align*}
\]

where \( x, y \in V \).
The notation \( \cdot \) may be often omitted. Hereafter, for simplicity, we will call an \( n \)-variable Kleene-Stone logic function a Kleene-Stone logic function, and we will identify a Kleene-Stone logic function with a logic formula which represents it. In logic formulas, we assume that symbols \( \neg \) and \( \cdot \) are stronger than \( \vee \), and \( \cdot \) is stronger than \( \vee \), for simplicity.

Originally, as shown above, Kleene-Stone logic functions were defined as infinite valued logic functions. It was known, however, in [11] that the following theorem holds;

\[ \text{Theorem 1}[11] \]
Let \( F_1 \) and \( F_2 \) be Kleene-Stone logic functions. \( F_1(A)=F_2(A) \) for all elements \( A \) of \( [0, 1/4, 1/2, 3/4, 1]^n \) if and only if \( F_1(A)=F_2(A) \) for all elements \( A \) of \( [0, 1]^n \).

(Proof is omitted)

Therefore, in this paper we will discuss Kleene-Stone logic functions on \( V_5^n \), where \( V_5=[0, 1/4, 1/2, 3/4, 1] \), because we can change the domain \( V^n \) of Kleene-Stone logic functions to \( V_5^n \) from the above fact.

Looking at a set of Kleene-Stone logic functions as an algebraic system, the following equations, motivated by properties of Boolean algebra, Kleene algebra, and Stone algebra, are hold.

\( 1 \) the commutative laws: \( A \land B = B \land A \), \( A \lor B = B \lor A \)
\( 2 \) the associative laws: \( A \land (B \land C) = (A \land B) \land C \), \( A \lor (B \lor C) = (A \lor B) \lor C \)
\( 3 \) the absorption laws: \( A \land (A \lor B) = A \), \( A \lor (A \land B) = A \)
\( 4 \) the idempotent laws: \( A \land A = A \), \( A \lor A = A \)
\( 5 \) the distributive laws: \( A \land (B \lor C) = (A \land B) \lor (A \land C) \), \( A \lor (B \land C) = (A \lor B) \land (A \lor C) \)
\( 6 \) De Morgan's laws: \( \neg (A \land B) = \neg A \lor \neg B \), \( \neg (A \lor B) = \neg A \land \neg B \)
\( 7 \) the double negation law: \( \neg \neg A = A \)
\( 8 \) the least element: \( 0 \land A = 0 \), \( 0 \lor A = A \)
\( 9 \) the greatest element: \( 1 \land A = 1 \), \( 1 \lor A = 1 \)
\( 10 \) Kleene's laws: \( A \land (B \lor C) = A \land B \lor A \land C \), \( A \lor (B \land C) = A \lor B \land A \lor C \)
\( 11 \) \( \neg A = A \)
\( 12 \) \( A \land \neg A = 0 \), \( A \lor \neg A = 1 \)

The equations \((1)-(12)\) are identical to the axioms of Kleene-Stone algebra, given in [9], that is, a set of Kleene-Stone logic functions is one of the models of Kleene-Stone algebra. It is characteristic of a Kleene-Stone logic function that it allows two types of unary negation operations \( \neg \) and \( \neg \). These two negations are weaker Boolean negation. In particular, \( 0 \land A = A \land 0 \lor B = 0 \land B \), whereas \( 0 \lor A = A \). Also \( \neg A = A \), \( \neg A = A \), and \( \neg A = \neg A \).

Here, we will define a partial order relation \( \gg \) on the set \( V_5 \) as follows;

\[ \text{Definition 1} \]
\[ 1/2 \gg 1/4, \ 1/2 \gg 3/4, \ i \gg i, \ \text{where} \ i \in V_5. \]

The partial order relation \( \gg \) can be extended among \( V_5^n \) as follows; For two elements \( A=(A_1,...,A_n) \) and \( B=(B_1,...,B_n) \) of \( V_5^n \), \( A \gg B \) if and only if \( A_i \gg B_i \) for all \( i \in \{0,...,n\} \).

Let \( A \) and \( B \) be any elements of \( V_5 \). If \( A \) and \( B \) are comparable to each other, then there is the minimum (the lower bound) of \( A \) and \( B \) concerning the partial order relation \( \gg \), otherwise there is not. We will write the minimum of \( A \) and \( B \) as \( A \land B \), and if the minimum of \( A \) and \( B \) does not exist, then we will write it as \( A \land B = \bot \). This can be extended among \( V_5^n \) as follows; For two elements \( A=(A_1,...,A_n) \) and \( B=(B_1,...,B_n) \) of \( V_5^n \), we will define \( A \land B \) as \( (A_1 \land B_1,...,A_n \land B_n) \), and if \( A_i \gg B_i \) for some \( i \in \{0,...,n\} \), then we will define it as \( A \land B = \bot \).

For the partial order relation \( \gg \), the following theorem holds for any Kleene-Stone logic function.

\[ \text{Theorem 2}[11] \]
Let \( F \) be a Kleene-Stone logic function, and \( A \) and \( B \) be any elements of \( V_5^n \). If \( A \gg B \), then \( F(A) \gg F(B) \).

(Proof is omitted)

\section{A necessary and sufficient condition for Kleene-Stone logic functions}

The following set of four conditions is a necessary and sufficient condition for a \( 5 \)-valued logic function \( F:V_5^n \rightarrow V_5 \) to be a Kleene-Stone logic function \( F \),

\( a \) \( A \in \{0, 1\}^n \Rightarrow F(A) \in V_2 \)
\( b \) \( A \in \{0, 1/2, 1\}^n \Rightarrow F(A) \in V_3 \)
\( c \) \( A \in \{0, 1/4, 3/4, 1\}^n \Rightarrow F(A) \in V_4 \)
\( d \) \( A, B \in V_5^n \) and \( A \gg B \Rightarrow F(A) \gg F(B) \)
Before showing that the conditions (a)-(d) are a necessary and sufficient condition for a 5-valued logic function \( F \) to be a Kleene-Stone logic function \( F \), we will clarify some properties of 5-valued logic functions satisfying the conditions (a)-(d), and Kleene-Stone logic functions.

Let \( F \) be a 5-valued logic function satisfying the conditions (a)-(d). Then, we consider specific five subsets \( F^{-1}(0) \), \( F^{-1}(1/4) \), \( F^{-1}(1/2) \), \( F^{-1}(3/4) \) and \( F^{-1}(1) \) of \( V^5 \) (refer to Fig. 2). It is clear that they are direct sums, that is, \( F^{-1}(i) \cup F^{-1}(j) = \varnothing \) (increase and \( F^{-1}(0) \cup F^{-1}(1/4) \cup F^{-1}(1/2) \cup F^{-1}(3/4) \cup F^{-1}(1) = V^5 \). Suppose \( A \) be an element of \( F^{-1}(1/4) \) \( (F^{-1}(3/4)) \). Then, all elements \( A' \) such as \( A \gg A' \) am also elements of \( P^{-1}(1/4) \) \( (P^{-1}(3/4)) \) from the condition (d). If \( A \) is an element of \( P^{-1}(1/2) \), then all elements \( A' \) such as \( A' \gg A \) are also elements of \( P^{-1}(1/2) \). Moreover, if \( A \) is an element of \( P^{-1}(0) \) \( (P^{-1}(1)) \), then all elements \( A' \) such as \( A' \gg A \) or \( A \gg A' \) are also elements of \( P^{-1}(0) \) \( (P^{-1}(1)) \). Therefore, the subsets \( P^{-1}(0) \), \( P^{-1}(1/4) \), \( P^{-1}(1/2) \), \( P^{-1}(3/4) \) and \( P^{-1}(1) \) each form a partial order finite sets concerning the relation \( \gg \). Therefore, for any given 5-valued logic function \( F \) satisfying the conditions (a)-(d), the set of maximal elements of \( P^{-1}(0) \), \( P^{-1}(1/4) \), \( P^{-1}(1/2) \), \( P^{-1}(3/4) \) and \( P^{-1}(1) \), denoted by \( F^{-1}(0) \), \( F^{-1}(1/4) \), \( F^{-1}(1/2) \), \( F^{-1}(3/4) \) and \( F^{-1}(1) \), and the set of minimal elements of \( F^{-1}(1/2) \), denoted by \( F^{-1}(1/2) \), are uniquely determined, respectively.

Next, we will define a specific subset \( T(C) \) of \( V^5 \).

[Definition 2] Let \( C \) be any element of \( V^5 \). Then, a set \( T(C) \) is defined as follows: \( T(C) = \{ A \mid A \gg C, A \in V^5 \} \).

\( T(C) \) is a partial order set concerning the relation \( \gg \), and \( C \) is the maximum element of \( T(C) \).

[Example 1] Let \( C_1 \) and \( C_2 \) be elements of \( V^5 \) such as \( (1/2, 1) \) and \( (0, 1/2) \), respectively. Then, \( T(C_1) = (1/2, 1) \) and \( (0, 1/2) \), \( (1/4, 1) \) and \( (0, 1/4) \), \( (3/4, 1) \) and \( (0, 3/4) \), and \( T(C_2) \cap T(C_2) = \varnothing \) holds.

[Lemma 1] Let \( F \) be any 5-valued logic function satisfying the conditions (a)-(d), and \( A \) be any element of \( V^5 \). If \( C \) is the element of \( V^5 \) such as \( C \gg A \) and \( T(C) \) is the set corresponding to \( C \), then the following five propositions hold:

1. If \( A \in F^{-1}(0) \), then \( A' \in T(C) \) for all elements \( A' \) of \( F^{-1}(1/4) \), \( F^{-1}(1/2) \), \( F^{-1}(3/4) \) and \( F^{-1}(1) \). Moreover, if \( A \in F^{-1}(0) \), then \( A \in V^5 \).
2. If \( A \in F^{-1}(1) \), then \( A' \in T(C) \) for all elements \( A' \) of \( F^{-1}(1/4) \), \( F^{-1}(1/2) \), \( F^{-1}(3/4) \) and \( F^{-1}(0) \). Moreover, if \( A \in F^{-1}(1) \), then \( A \in V^5 \).
3. If \( A \in F^{-1}(1/2) \), then \( A' \subseteq T(C) \) holds for all elements \( A' \) of \( F^{-1}(0) \) and \( F^{-1}(1) \). Moreover, let \( A' \) be any element of \( F^{-1}(1/4) \) or \( F^{-1}(3/4) \). Then, \( A' \gg A \) for all elements \( A \) of \( F^{-1}(1/2) \).
4. If \( A \in F^{-1}(1/4) \), then \( A \gg A' \) for all elements \( A' \) of \( F^{-1}(0) \), \( F^{-1}(3/4) \) and \( F^{-1}(1) \). Moreover, there is at least one element \( A' \) of \( F^{-1}(1/4) \) such as \( A' \gg A \) and \( A \neq A' \) for all elements \( A \) of \( F^{-1}(1/4) \).
5. If \( A \in F^{-1}(3/4) \), then \( A \gg A' \) for all elements \( A' \) of \( F^{-1}(0) \), \( F^{-1}(1/4) \) and \( F^{-1}(1) \). Moreover, there is at least one element \( A' \) of \( F^{-1}(1/4) \) such as \( A' \gg A \) and \( A \neq A' \) for all elements \( A \) of \( F^{-1}(3/4) \).

(Proof is omitted)

Next, we will show some properties of Kleene-Stone logic functions.

Let, \( F \) be any Kleene-Stone logic function. Then, a canonical disjunctive form for a Kleene-Stone logic function \( F \), which enables us to represent any Kleene-Stone logic function uniquely, was given by [11]. The canonical disjunctive form is a disjunction of the following three types of minterms \( \alpha \) which are specific product terms existing all variables:

- type 1; A minterm \( \alpha \) consisted of only \( \neg x \), \( \neg x \) or \( \neg \neg x \) for a variable \( x \).
- type 2; A minterm \( \alpha \) consisted of only \( \neg x \), \( \neg x \), \( \neg x \) or \( \neg x \) for a variable \( x \), and includes at least a variable \( y \) such as \( y \) for a variable \( y \) and \( y \).

[Example 2] Let \( C \) be any element of \( V^5 \). Then, \( T(C) = \{ A \mid A \gg C, A \in V^5 \} \).

\( T(C) \) is a partial order set concerning the relation \( \gg \), and \( C \) is the maximum element of \( T(C) \).

[Lemma 2] Let \( F \) be any 5-valued logic function satisfying the conditions (a)-(d), and \( A \) be any element of \( V^5 \). If \( C \) is the element of \( V^5 \) such as \( C \gg A \) and \( T(C) \) is the set corresponding to \( C \), then the following five propositions hold:

1. If \( A \in F^{-1}(0) \), then \( A' \in T(C) \) for all elements \( A' \) of \( F^{-1}(1/4) \), \( F^{-1}(1/2) \), \( F^{-1}(3/4) \) and \( F^{-1}(1) \). Moreover, if \( A \in F^{-1}(0) \), then \( A \in V^5 \).
2. If \( A \in F^{-1}(1) \), then \( A' \in T(C) \) for all elements \( A' \) of \( F^{-1}(1/4) \), \( F^{-1}(1/2) \), \( F^{-1}(3/4) \) and \( F^{-1}(0) \). Moreover, if \( A \in F^{-1}(1) \), then \( A \in V^5 \).
3. If \( A \in F^{-1}(1/2) \), then \( A' \subseteq T(C) \) holds for all elements \( A' \) of \( F^{-1}(0) \) and \( F^{-1}(1) \). Moreover, let \( A' \) be any element of \( F^{-1}(1/4) \) or \( F^{-1}(3/4) \). Then, \( A' \gg A \) for all elements \( A \) of \( F^{-1}(1/2) \).
4. If \( A \in F^{-1}(1/4) \), then \( A \gg A' \) for all elements \( A' \) of \( F^{-1}(0) \), \( F^{-1}(3/4) \) and \( F^{-1}(1) \). Moreover, there is at least one element \( A' \) of \( F^{-1}(1/4) \) such as \( A' \gg A \) and \( A \neq A' \) for all elements \( A \) of \( F^{-1}(1/4) \).
5. If \( A \in F^{-1}(3/4) \), then \( A \gg A' \) for all elements \( A' \) of \( F^{-1}(0) \), \( F^{-1}(1/4) \) and \( F^{-1}(1) \). Moreover, there is at least one element \( A' \) of \( F^{-1}(1/4) \) such as \( A' \gg A \) and \( A \neq A' \) for all elements \( A \) of \( F^{-1}(3/4) \).

(Proof is omitted)

Next, we will show some properties of Kleene-Stone logic functions.
type 3: A minterm \( \alpha \) consisted of only \(-x, \neg x, x \neg x, \neg x x\) or \(x x\) for a variable \(x\), and includes at least a variable \(y\) such as \(y y\).

(Note; \(x\) and \(-x\) are represented by sum of minterms of type 1 and type 2, that is, \(x = x_1 \neg x x \neg x_2 \neg x_3\) and \(-x = x_1 x_2 x_3\).)

Here, we define a correspondence between 3 types of minterms and elements of \(V^3\).

[Definition 3] Let \(A = (A_1, \ldots, A_n)\) be an element of \(V^3\). Then, the element \(A\) corresponds to a minterm \(\alpha = x_1^{A_1} \ldots x_n^{A_n}\) of type 1 if the following relations holds:

\[
\begin{align*}
x_i^{A_i} & = \begin{cases} 
x_i & \text{if } A_i = 1 \\
\neg x_i \neg x_j & \text{if } A_j = 1/2 \\
\neg x_i & \text{if } A_i = 1/4
\end{cases}
\end{align*}
\]

Then, obviously \(\alpha(A) = 1\) and there is one-to-one correspondence between minterms of type 1 and elements of \(V^3\).

[Definition 4] Let \(A = (A_1, \ldots, A_n)\) be an element of \(V^3\). Then, the element \(A\) corresponds to a minterm \(\alpha = x_1^{A_1} \ldots x_n^{A_n}\) of type 2 if the following relations holds:

\[
\begin{align*}
x_i^{A_i} & = \begin{cases} 
x_i & \text{if } A_i = 3/4 \\
\neg x_i \neg x_j & \text{if } A_j = 1/2 \\
\neg x_i & \text{if } A_i = 1/4
\end{cases}
\end{align*}
\]

Then, obviously \(\alpha(A) = 3/4\) and there is one-to-one correspondence between minterms of type 2 and elements of \(V^3\).

[Definition 5] Let \(A = (A_1, \ldots, A_n)\) be an element of \(V^3\). Then, the element \(A\) corresponds to a minterm \(\alpha = x_1^{A_1} \ldots x_n^{A_n}\) of type 3 if the following relations holds:

\[
\begin{align*}
x_i^{A_i} & = \begin{cases} 
x_i & \text{if } A_i = 0 \\
\neg x_i \neg x_j & \text{if } A_j = 1/4 \\
\neg x_i & \text{if } A_i = 1/2
\end{cases}
\end{align*}
\]

Then, obviously \(\alpha(A) = 1/2\) and there is one-to-one correspondence between minterms of type 3 and elements of \(V^3\).

[Example 2] (0, 1/2, 1) corresponds to a minterm of type 1, \(-x_1 \neg x_2 x_3\) and (0, 1/4, 1/2) corresponds to that of type 2, \(-x_1 x_2 x_3\) and it corresponds that of type 3, \(-x_1 x_2 \neg x_3\).

[Lemma 2] Let \(A = (A_1, \ldots, A_n)\) be any element of \(V^3\), and \(\alpha\) be the corresponding minterm of type 1. If \(\alpha = (A_1', \ldots, A_n')\) is an element of \(V^3\), then

(1) \(A \rightarrow \alpha \Rightarrow \alpha(A) = 1\)
(2) \(A \neg A \Rightarrow \alpha(A) = 0\)

(Proof is omitted)

[Lemma 3] Let \(A = (A_1, \ldots, A_n)\) be any element of \(V^3\). Then, the element \(A\) corresponds to a minterm \(\alpha = x_1^{A_1} \ldots x_n^{A_n}\) of type 2 if the following relations holds:

\[
\begin{align*}
x_i^{A_i} & = \begin{cases} 
x_i & \text{if } A_i = 3/4 \\
\neg x_i \neg x_j & \text{if } A_j = 1/2 \\
\neg x_i & \text{if } A_i = 1/4
\end{cases}
\end{align*}
\]

Then, obviously \(\alpha(A) = 1/4\) and there is one-to-one correspondence between minterms of type 2 and elements of \(V^3\).

[Lemma 4] Let \(A = (A_1, \ldots, A_n)\) be any element of \(V^3\). Then, the element \(A\) corresponds to a minterm \(\alpha = x_1^{A_1} \ldots x_n^{A_n}\) of type 3 if the following relations holds:

\[
\begin{align*}
x_i^{A_i} & = \begin{cases} 
x_i & \text{if } A_i = 0 \\
\neg x_i \neg x_j & \text{if } A_j = 1/4 \\
\neg x_i & \text{if } A_i = 1/2
\end{cases}
\end{align*}
\]

Then, obviously \(\alpha(A) = 1/2\) and there is one-to-one correspondence between minterms of type 3 and elements of \(V^3\).

Next, we will show that the conditions (a)-(d) are a necessary and sufficient condition for a 5-valued logic function to be a Kleene-Stone logic function.

[Theorem 3] If \(F\) is a Kleene-Stone logic function, then \(F\) is a 5-valued logic function satisfying the conditions (a)-(d).

(Proof) It is evident from the definition of each logic operations (AND, OR, NOT) and from Theorem 2.

Q.E.D.

[Theorem 4] If \(F\) is a 5-valued logic function satisfying the conditions (a)-(d), then \(F\) is a Kleene-Stone logic function.

(Proof) For any given 5-valued logic function \(F\) satisfying the conditions (a)-(d), we will construct a Kleene-Stone logic function which represents \(F\). Let \(F_1\) be the logic formula constructed by the disjunction of all minterms of type 1 corresponding to elements of \(\mathcal{D}^{-1}(1)\) (all elements \(A\) of \(\mathcal{D}^{-1}(1)\) are also elements of \(V^3\) from Lemma 1-(1)), \(F_2\) be the logic formula constructed by the disjunction of all
minterms of type 2 corresponding to elements of \(\partial F^{-1}(3/4)\) (All elements \(A\) of \(\partial F^{-1}(3/4)\) are also elements of \(V_2^n\) \(V_3^n\), because \(F(A)\in V_2\Rightarrow A\in V_2^n\) stands always true from the condition (b), and \(F_3\) be the logic formula constructed by the disjunction of all minterms of type 3 corresponding to elements of \(\partial F^{-1}(1/2)\) (All elements \(A\) of \(\partial F^{-1}(1/2)\) are also elements of \(V_2^n\), because \(F(A)\in V_2\Rightarrow A\in V_2^n\) stands always true from the condition (c)). Then, we can show that \(F_1=F_1\lor F_2\lor F_3\) and \(F\) are equivalent 5-valued logic functions as follows;

1. Suppose \(F(A)\)=1. Then, there is an element \(A'\) in \(\partial F^{-1}(1)\) such as \(A'\gg A\). Therefore, since there is the minterm \(\alpha\) of type 1 corresponding to \(A'\) in \(F_1\), we obtain \(\alpha(A)\)=1 from Lemma 2-(1). Therefore, \(F_1(A)\)=1 holds, that is, we have \(F(A)\)=1. Conversely, suppose \(F(A)\)=1. Then, \(F_1(A)\)=1 holds, because \(F_2(B)\leq 1/4\) and \(F_3(B)\leq 1/2\) stand always true for all elements \(B\) of \(V_5^n\).

2. Suppose \(F(A)\)=3/4. Then, \(F(A)\leq 3/4\) holds, because \(F_1(A)\)=1 holds, because \(A'\gg A\) for the element \(A'\) corresponding to \(\alpha\) from Lemma 2-(1). That is, \(F(A)\)=1 holds, because \(A'\) is an element of \(\partial F^{-1}(1)\). Therefore, we have \(F(A)\) in the condition (d).

3. Suppose \(F(A)\)=3/4. Then, \(F(A)\leq 3/4\) holds, because if \(F(A)\)=1, then \(F(A)\) in the discussion of (1), but this fact is contradictory to the hypothesis \(F(A)\)=3/4. Moreover, in this case, there is an element \(A'\) in \(\partial F^{-1}(3/4)\) such as \(A'\gg A\). Therefore, since there is the minterm \(\alpha\) of type 2 corresponding to \(A'\) in \(F_2\), we obtain \(\alpha(A)\)=3/4 from Lemma 3-(1). Therefore, \(F_2(A)\)=3/4 holds, that is, we have \(F(A)\)=3/4. Conversely, suppose \(F(A)\)=3/4. Then, \(F(A)\leq 3/4\) from the discussion of (1), \(F(A)\leq 1/2\) does not hold in this case. Because if \(F(A)\leq 1/2\) holds, then \(A'\gg A\) for all elements \(A'\) of \(\partial F^{-1}(1)\) from Lemma 1-(2). Therefore, \(\alpha(A)\)=0 for all minterms \(\alpha\) of type 1 in \(F_1\), that is, \(F_1(A)\)=0. Since \(A'\gg A\) for all elements \(A'\) of \(\partial F^{-1}(3/4)\) from Lemma 1-(1), (3) and (4), \(\alpha(A)\leq 1/2\) for all minterms \(\alpha\) of \(F_2\) from Lemma 3-(1). And \(F_3(A)\leq 1/2\) stands always true for all elements \(A\) of \(V_5^n\). From the above, we have \(F(A)\leq 1/2\), however, this fact is contradictory to the hypothesis \(F(A)\)=3/4. Therefore, we have \(F(A)\)=3/4.

4. Suppose \(F(A)\)=1/2. Then, \(F(A)\leq 1/2\) holds from the discussion of (1) and (2), and there is an element \(A'\) in \(\partial F^{-1}(1/2)\) such as \(A'\gg A\). Therefore, since there is the minterm \(\alpha\) of type 3 corresponding to \(A'\) in \(F_3\), we obtain \(\alpha(A)\)=1/2 from Lemma 4-(1). Therefore, \(F_3(A)\)=1/2 holds, that is, we have \(F(A)\)=1/2. Conversely, suppose \(F(A)\)=1/2. Then, \(F(A)\leq 1/2\) holds from the discussion of (1) and (2), and \(F(A)\leq 1/4\) does not hold in this case. Because if \(F(A)\leq 1/4\) holds, then \(A'\gg A\) for all elements \(A'\) of \(\partial F^{-1}(1)\) from Lemma 1-(2). Therefore, \(\alpha(A)\)=0 for all minterms \(\alpha\) of type 1 in \(F_1\) from Lemma 2-(2), that is, \(F_1(A)\)=0. Therefore, \(\alpha(A)\)=1/2 for all elements \(A'\) of \(\partial F^{-1}(3/4)\) from Lemma 1-(5). Therefore, \(\alpha(A)\leq 1/4\) for all minterms \(\alpha\) of type 2 in \(F_2\) from Lemma 3-(2) and (3), that is, \(F_2(A)\leq 1/4\). And \(A'\gg A\) for all elements \(A'\) of \(\partial F^{-1}(1/2)\) from Lemma 1-(1) and (3). Therefore, \(\alpha(A)\leq 1/4\) for all minterms \(\alpha\) of type 3 in \(F_3\) from Lemma 4-(1), that is, \(F_3(A)\leq 1/4\). From the above, we have \(F(A)\leq 1/4\), however, this fact is contradictory to hypothesis \(F(A)\)=3/4. Therefore, we have \(F(A)\)=3/4.

5. Suppose \(F(A)\)=1/4 if and only if \(F(A)\)=1/4 is derived directly from (1), (2), (3) and (4).

Q.E.D.

The condition (a)–(d) are not independent to each other. That is, the condition (a) is driven from (b) and (c), because the set \([0, 1]\) is the intersection of the sets \([0, 1, 2, 3, 4]\) and \([0, 1, 4, 3, 0, 1]\). It is easy to show that the condition (b), (c), and (d) are independent to each other.

4. The number of \(n\)-variable Kleene-Stone logic functions

Let \(C_1\) and \(C_2\) be any elements of \(V_3^n\), and \(T(C_1)\) and \(T(C_2)\) be the sets corresponding to \(C_1\) and \(C_2\), respectively. Then, it is evident from Definition 2 that the following relations hold:

- \(T(C_1)\cap T(C_2)=\emptyset\), whenever \(C_1\neq C_2\)
- \(\bigcup \{C \in V_3^n : T(C) = V_3^n\} = V_3^n\)

Therefore, the set \(V_3^n\) is decomposed into the partial order sets under the relation \(\gg\), and the number of these partial order sets is \(|V_3^n| = 3^n\).

Next, let \(C_1\) and \(C_2\) be any element of \(V_3^n\) such that \(C_1\neq C_2\), and \(T(C_1)\) and \(T(C_2)\) be the sets corresponding to \(C_1\) and \(C_2\), respectively. Let \(A_1\) and \(A_2\) be any element
of \( T(C_1) \) and \( T(C_2) \), respectively, then \( A_1 \) and \( A_2 \) always are not comparable to each other. Therefore, in this case, the antecedent of the condition (d) stands always false, that is, the condition (d) stands always true whether the consequent of the condition (d) is true or false. So, there is not any relation between \( F(A_1) \) and \( F(A_2) \) in this case. In other words, \( F(A_1) \) and \( F(A_2) \) can take truth values independently to each other.

**Example 3** Let \( F \) be a 2-variable Kleene-Stone logic function, and \( C_1 \) and \( C_2 \) be elements of \( V_3^2 \) such as \((1/2, 1)\) and \((1, 1)\), respectively. Then, \( T(C_1)=((1/2, 1), (1/4, 1), (3/4, 1)) \) and \( T(C_2)=((1, 1)) \). If \( F(1/2, 1)=1 \), then \( F(1/4, 1)=F(3/4, 1)=1 \) from the condition (d). But we can determine the value \( F(1, 1) \) independently from the value \( F(1/2, 1), F(1/4, 1) \) and \( F(3/4, 1) \), because \((1, 1)\) is not comparable to \((1/2, 1), (1/4, 1) \) and \((3/4, 1) \).

From the above, we can take the following equation concerning the number of \( n \)-variable Kleene-Stone logic functions;

\[
|F_{KS}(n)| = \prod_{i=1}^{3^n} |KS(C_i)| \tag{1}
\]

where \( C_i \) \((i=1, ..., n)\) is an element of \( V_3^n \), \( F_{KS}(n) \) and \( KS(C) \) are defined as follows;

- \( F_{KS}(n) \) is a set of all Kleene-Stone logic functions. That is,
  \[
  F_{KS}(n) = \{ F \mid V_j^n \rightarrow V_5, F \text{ satisfies conditions (a)-} (d) \}
  \]

- Let \( C \) be an element of \( V_j^n \) and \( F \) be a Kleene-Stone logic function. Then, a set \( KS(C) \) is defined as follows;
  \[
  KS(C) = \{ F \mid \forall i F_{T(C)} : T(C) \rightarrow V_5, F \text{ satisfies conditions (a)} \rightarrow (d) \}
  \]

where \( F_{T(C)} \) is a restriction of \( F \) to \( T(C) \).

Therefore, if we can calculate the value \( |KS(C_i)| \) \((i=1, ..., 3^n)\) of the equation (1), then we have the number of \( n \)-variable Kleene-Stone logic functions. In fact, the value \( |KS(C_i)| \) is closely connected with the number of \( B \)-ternary logic functions. Here, an \( n \)-variable \( B \)-ternary logic function \( f \) defined by [1] is a mapping from \( V_j^n \) to \( V_3 \); \( f : V_j^n \rightarrow V_3 \), satisfying the following two conditions;

\[
\begin{align*}
\text{(A)} \ a \in V_j^2 & \Rightarrow f(a) \in V_2 \\
\text{(B)} \ a, b \in V_3^n \text{ and } a \rightarrow b & \Rightarrow f(a) > f(b),
\end{align*}
\]

where the notation \( \rightarrow \) means a partial order relation over \( V_3 \), and the relation is defined as follows;

\[
\begin{align*}
1/2 & \rightarrow 0, 1/2 \rightarrow 1, i \rightarrow i, \text{ where } i \in V_3.
\end{align*}
\]

Moreover, the partial order relation \( \rightarrow \) can be extended among \( V_3^n \) as follows: For two elements \( a=(a_1, ..., a_n) \) and \( b=(b_1, ..., b_n) \) of \( V_3^n \), \( a \rightarrow b \) if and only if \( a_i > b_i \) for all value \( i \) \((i=1, ..., n)\).

Hereafter, we will discuss the relation between Kleene-Stone logic functions and \( B \)-ternary logic functions, and then we will show the equation concerning the number of \( n \)-variable Kleene-Stone logic functions.

Let a mapping \( \lambda_i : V_j^n \rightarrow V_2 \), which takes a parameter \( i \) \((i=1, ..., n)\), define as follows;

\[
\lambda_i(C) = \begin{cases} 
1 & \text{if } C_i=1/2 \\
0 & \text{if } C_i=0 \text{ or } 1
\end{cases}
\]

where \( C=(C_1, ..., C_n) \) is an elements of \( V_3^n \). Then,

\[
\sum_{i=1}^{n} \lambda_i(C) \text{ means the number of the truth value } 1/2
\]

which exist in the element \( C \) of \( V_3^n \). Hereafter, \( A(C) \) is defined as \( \sum_{i=1}^{n} \lambda_i(C) \) for short.

Here, we can show the relation

\[
|KS(C)| = |F_K(k)| \quad \text{(2)}
\]

where \( F_K(k) \) is the set of all \( k \)-variable \( B \)-ternary logic functions and \( k=A(C) \). That is, there is a one-to-one and onto mapping between \( KS(C) \) and \( F_K(k) \). The proof of \( |KS(C)| = |F_K(k)| \) is, however, omitted, because there is limited in space.

Next example shows a corresponding between \( F | T(C) \) e \( KS(C) \) and \( F | T(k) \).
Let \( F \) be a 2-variable Kleene-Stone logic function such as \( F = x_1 \overline{x_2} + x_1 \overline{x_2} \overline{x_1} \overline{x_2} + x_1 x_2 - x_1 \overline{x_2} - x_1 x_2 \), and \( C = (1/2, 1/2) \). Then, Table 1 shows truth table of \( F \), and Table 2 shows truth table of 2-variable B-ternary logic function \( f \) corresponding to \( F \).

Moreover, the following relation holds;
\[
\left| \{ C \mid \langle C \rangle = k \; \forall \; x \in V_3 \} \right| = nCk \cdot 2^{n-k}
\]
that is, the number of \( V_3 \)'s elements in which the truth value 1/2 holds \( k \) places is \( nCk \cdot 2^{n-k} \). Because, the number of combinations \( n \) places taken \( k \) at a time is \( nCk \), moreover, we have \( 2^{n-k} \) since we can take 0 and 1 arbitrary for the remaining \( n-k \) places. Therefore, we have \( nCk \cdot 2^{n-k} \).

Finally, from the equation (1), (2) and (3) we have the following equation concerning the number of \( n \)-variable Kleene-Stone logic functions;
\[
\left| F_{KS}(n) \right| = \prod_{k=0}^{n} \left| F_k(k) \right| nCk \cdot 2^{n-k}
\]
\[
= \left| F_B(n) \right| \prod_{k=1}^{n} \left| F(k) \right| nCk \cdot 2^{n-k}
\]
(4),
where \( F_B(n) \) is the set of all \( n \)-variable Boolean functions, that is,
\[
F_B(n) = \{ F \mid F:V_2^n \rightarrow V_2 \}
\]

Table 1 Truth table of \( F \) in Example 4

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>0</th>
<th>1/4</th>
<th>1/2</th>
<th>3/4</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1/4</td>
<td>0</td>
<td>1/4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1/2</td>
<td>0</td>
<td>1/4</td>
<td>1/2</td>
<td>3/4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3/4</td>
<td>0</td>
<td>1/4</td>
<td>1/2</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2 Truth table of \( f \) in Example 4

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>0</th>
<th>1/2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>1</td>
</tr>
<tr>
<td>1/2</td>
<td>0</td>
<td>1/2</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

and the equation (4) is changed from the fact that
\[
\left| F_K(0) \right| nC0 \cdot 2^n = 2^n = \left| F_B(n) \right| \]
where \( F_K(0) = \{ 0, 1 \} = 2 \). We say that the equation (4) depends on the number of \( B \)-ternary logic functions \( \left| F_K(1) \right| \) and the number of Boolean functions \( \left| F_B(n) \right| \). Especially, \( \left| F_K(1) \right| = \left| F_B(1) \right| \times \left| F_K(1) \right| = 2^1 \times 6 = 24 \). Moreover, we know the accurate number of \( n \)-variable Kleene-Stone logic functions until \( n \leq 4 \), because the number of \( n \)-variable \( B \)-ternary logic functions was calculated until \( n \leq 4 \), by [8].

(Note; There is an isomorphism between \( B \)-ternary logic functions and fuzzy switching functions[fuzzy logic functions][4]. Therefore, the number of \( n \)-variable \( B \)-ternary logic functions and \( n \)-variable fuzzy switching functions are same. Table 3 shows the number of \( n \)-variable \( B \)-ternary logic functions until \( n \leq 4 \))

5 Conclusion

In this paper, we defined Kleene-Stone logic functions which are one of the models of Kleene-Stone algebras, and then, we showed the fundamental properties of them. Main results of this paper were that the following four conditions (a)-(d) are a necessary and sufficient condition for Kleene-Stone logic functions;

(a) \( A \in (0, 1)^n \Rightarrow F(A) \in \{ 0, 1 \} \)
(b) \( A \in (0, 1/2, 1/n) \Rightarrow F(A) \in \{ 0, 1/2, 1 \} \)
(c) \( A \in (0, 1/4, 3/4, 1/n) \Rightarrow F(A) \in \{ 0, 1/4, 3/4, 1 \} \)
(d) \( A, B \in \{ 0, 1/4, 1/2, 3/4, 1 \} \) and \( A \Rightarrow B \Rightarrow F(A) \Rightarrow F(B) \)

and the equation which represents the number of \( n \)-variable Kleene-Stone logic functions was given as follows;

Table 3 The number of \( n \)-variable \( B \)-ternary logic functions

<table>
<thead>
<tr>
<th>( n )</th>
<th>The number of ( B )-ternary logic functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>84</td>
</tr>
<tr>
<td>3</td>
<td>43,918</td>
</tr>
<tr>
<td>4</td>
<td>160,297,985,276</td>
</tr>
</tbody>
</table>
The number of n-variable Kleene-Stone logic functions is given by the formula:

$$|F_{KS}(n)| = \prod_{k=0}^{n} |F_k(k)|^{nC_k2^{n-k}}$$

where $F_{KS}(n)$ means the set of all n-variable Kleene-Stone logic functions and $F_k(k)$ means the set of all k-variable B-ternary logic functions (or fuzzy switching functions).

References


** They are available in English in System-Computers-Controls (Scripta publishing Co.); same date.