Abstract

In this paper, we study the contracting issues in IT outsourcing projects. Our analysis also applies to other collaborative services, such as consulting, financial planning, and auditing. We consider that the client gets utility from the project throughout the development period since the software can be used as it is being developed by the vendor(s). The service output is contingent on the effort level of each party and we allow these effort levels to change throughout the planning period. Hence, the client firm needs to optimally decide on the terms to offer in the contract so as to maximize the service output and minimize its cost. We analyze the performance of different contracts and glean useful managerial insights. Moreover, we analyze two additional settings: (a) the vendor may lower its cost through training, and (b) the vendor or the client gets utility from the output even after the project finishes.

1. Introduction

The information service sector is one of the fastest growing sectors in the U.S. [3]. These services, such as information technology consulting and projects, constitute a vital part of the U.S. economy [16]. Attaining high productivity and efficiency levels has been argued to be difficult for the information or knowledge intensive services because of the loose structured delivery processes, tailoring of the services for the customer needs, and in particular, significant customer involvement throughout the process [7].

1.1. Motivation

Business services in general [11], and IT services alike [6], have to deal with many challenges even though these sectors had impressive growth performances in the 1990s [3]. Toppin and Czerniawiska [32] argue that the specifics of the client-consultant relationship have changed towards creating value through effective collaboration, or value co-creation. By value co-creation or service co-production process, we mean that joint efforts are required from both the client and the vendor in order to create and deliver the service. Nonetheless, from an operations perspective, research on services in general has mainly focused on physical characteristics of services such as queuing models, logistics, retailing, and revenue management [26]. According to Roels et al. [26] and Hopp et al. [17], there are many promising research opportunities pertaining to the management of service operations.

Therefore, in the current study, we explore different contracts to analyze several characteristics of the service co-production process. In particular, in a value co-creation environment, we focus on the determination of the effort levels of client and vendor in different contract types, as well as the optimal payment terms. Further, we analyze which contract type is better for the client in different settings. We frame our discussion around IT consulting and projects. However, our discussion and analyses also apply to engineering projects, auditing [24], R&D projects, product development [21], consulting, and financial planning [26], among others.

Knowledge intensive services such as IT consulting or projects are complex, unstructured, and tailored to fit unique needs of a particular client [7]. Therefore, in order to attain a successful outcome, the clients must actively participate in the creation of the service [24]. More specifically, the clients need to get involved in: (i) the problem definition stage, (ii) the selection of solution, and (iii) the development and the implementation of the solution. Karmarkar and Pitbladdo [20] point out the importance of the involvement of customers in the service delivery process, and call for explicit modeling of service co-production. We respond to this call by analyzing a setting that requires joint efforts from the client and the vendor.
1.2. Contributions

The contribution of our study follows from the recognition that, first, the nature of joint-production or service co-creation implies that the effort levels from the client and the vendor are complementary. Second, the effort levels of both parties need not remain constant but can be changed throughout the planning horizon. Third, the client can receive utility from the outcome as it is being developed. This is the case in many IT projects in which the vendor develops many versions of the software that the client uses before the finalized product. Whether or not the vendor gains utility from the outcome depends on the contract type that is in effect. We consider three contractual settings, which differ in payment terms. In our model, although the effort levels are observable, they are not contractible as they are in the incomplete contracts [14].

In the next section, we review the related literature that forms the theoretical background of this study. Then, we develop three contracting models that represent service cocreation processes. Next, we present the results and insights. After the extensions, we summarize the findings and conclude with future research opportunities.

2. Literature

Our study is related to three different streams of research, i.e., value co-creation, contracting, and differential games. In this section, we briefly position our research among these studies.

2.1. Value Co-creation

Customer involvement in design and management of services has been studied in many different disciplines, such as information technology [12, 23], operations management [9, 13], and marketing [2]. These studies generally focus to improve the performance of the service in dimensions like efficiency, quality, and/or customer satisfaction [25, 34]. Spohrer et al. [30] argue that any service product involves the joint effort or co-creation of both vendor and the client. According to Bettencourt et al. [7] and Skjølsvik et al. [29], this fact is more important in a knowledge intensive context, such as in IT. Hence, the nature of the joint work in IT business settings renders the traditional top-down approach in defining the specifics of the collaboration obsolete [26].

2.2. Contracting

Our analysis involves continuous time decision-making in contrast to many studies in the contracting literature that generally analyze discrete decision-making. However, our study is related to the contracting literature because both investigate equilibrium behavior of different parties. Contract theory is an active research topic in economics. Collaborative business environments have been studied with the application of contract theory, e.g., cross-functional coordination [22] and supply chain coordination [8]. The contracts which are related to our work are in the domain of projects with examples in general collaborative services [26], construction [5], legal services [27], and call centers [15]. Although our methodology and results apply to a wide variety of business settings, we focus on IT consulting services and projects.

There are many empirical studies that analyze the contract choice in IT, e.g., see [19] and [31]. In these studies, it is generally argued that the project complexity, project size, resource shortage, and the firm’s reputation are the drivers of the contract choice. Among many possible contract types, two prominent contract types in professional services are effort-level dependent contracts and the shared-revenue contracts [23]. Therefore, in this study, we analyze the following three types of contracts. The first one is the effort-level dependent contract, where the payment is based on the labor-hours of vendor. In the second type of contract, the output determines the payment structure. Finally, the third type is a hybrid one in which transfer payment depends on both the effort level of vendor and the output.

Although the contracting literature is vast, the problem we are analyzing is not deeply investigated. There is little, if there is any, guidance on how to set the effort levels in continuous-time in a value co-creation setting. In this setting, because of the collaborative effort levels, the nature of contracting is different from most of the past studies [26]. Furthermore, consideration of continuous-time brings another level of difficulty and richness.

In a related study, Iyer et al. [18] analyze a setting in which buyer’s and supplier’s resources determine the production costs in a supply chain. Similarly, Xue and Field [33] study complementary effort levels in collaborative services with the effects of information stickiness. Next, Roels et al. [26] compare and contrast different contracts and show what type of contract is preferred in different service environments. Our model is different from these studies in the sense that we explicitly capture time dynamics, whereas these studies analyze static
models. The rationale for considering a dynamic model is that the client might get utility from the project throughout the planning horizon, and the effort levels may change from time to time. Besides, we study an additional setting in which vendor might have decreasing costs or higher level of effectiveness due to training. In another setting we analyze, the client gets utility not only during the collaboration with the vendor but also after the project is finished. Similarly, we also study the scenario when the vendor gets utility even after the project is finished.

2.3. Differential Games

In this study, we capture all the dynamism discussed above by utilizing a differential game approach in a single client and a single vendor setting. Differential games can be considered as a fusion of game theory and optimal control theory. Hence, they incorporate strategic decision making and continuous change simultaneously. However, because of the inherent difficulty in solving differential games in many operations and supply chain management models, there are few studies in this field [8]. To the best of our knowledge, this is the first study that considers differential games in a value or service co-creation environment. For a general discussion of differential games, the readers can refer to [10].

3. Problem Definition

As mentioned earlier, we consider a single client that offers a contract to a vendor. The output requires joint efforts from both the client and the vendor. Hence, our model is different from many supplier-selection studies where the supplier assumes whole responsibility in generation of the output. Now we discuss the inputs, the output, and the payment structure. The models are introduced in the next section.

3.1. Inputs

We denote the effort levels of client and vendor at time \( t \) by \( u[t] \) and \( v[t] \), respectively. These levels represent the resources, such as labor-hours, exerted by each party. In contrast to many contracting papers, the effort levels are continuous variables. In our model, \( t \) denotes an instance of time, and \( T \) denotes the planning horizon. The cost of spending effort for the client is modeled as \( c_{c} u[t]^{\gamma} \), which is a general cost structure. Here, \( c_{c} \) is the cost multiplier and the parameter \( \gamma \) is more than 1 to reflect the fact that the cost of exerting more effort is increasing with the level of effort itself. In other words, the vendor utilizes its least costly options first. Similarly, the cost for the vendor is \( c_{v} v[t]^{\delta} \) with multiplier \( c_{v} \). Again, the power term \( \delta \) is greater than 1.

3.2. Output

In many consulting and IT services, the client gets utility from the service output as the project is being developed. For example, the service might consist of many deliverables that serve the needs of the client individually by themselves. Another example is IT projects in which the client uses many versions of the software that is not finalized yet. Similar to the past studies, we consider that the output level, which we denote by \( q[t] \), is continuous, double differentiable, strictly concave, and non-decreasing in effort levels. Furthermore, the instantaneous increase in the output level is due to the collaborative work between the client and the vendor. Therefore, both parties need to exert effort in order to generate output.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>Planning horizon</td>
</tr>
<tr>
<td>( t )</td>
<td>Index for time; ( 0 \leq t \leq T )</td>
</tr>
<tr>
<td>( c_{c} )</td>
<td>Cost multiplier for client’s effort</td>
</tr>
<tr>
<td>( c_{v} )</td>
<td>Cost multiplier for vendor’s effort</td>
</tr>
<tr>
<td>( k )</td>
<td>Utility conversion parameter for client</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>Output elasticity of client’s effort level</td>
</tr>
<tr>
<td>( \beta )</td>
<td>Output elasticity of vendor’s effort level</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>Parameter for marginal cost of client’s effort level; ( \gamma \geq 1 )</td>
</tr>
<tr>
<td>( \delta )</td>
<td>Parameter for marginal cost of vendor’s effort level; ( \delta \geq 1 )</td>
</tr>
<tr>
<td>( \bar{u} )</td>
<td>Reservation utility of the vendor</td>
</tr>
<tr>
<td>( q[t] )</td>
<td>Output at time ( t ) (State Variable)</td>
</tr>
<tr>
<td>( u[t] )</td>
<td>Level of client’s effort at time ( t ) (Decision Variable)</td>
</tr>
<tr>
<td>( v[t] )</td>
<td>Level of vendor’s effort at time ( t ) (Decision Variable)</td>
</tr>
<tr>
<td>( p )</td>
<td>Transfer payment per unit vendor’s effort (Decision Variable)</td>
</tr>
<tr>
<td>( l )</td>
<td>Transfer payment per unit output (Decision Variable)</td>
</tr>
</tbody>
</table>

Table 1: Summary of Notations

Next, the instantaneous change in the output is considered to be a supermodular function. This implies that if one party exerts higher levels of effort, then the other party will have incentive to do so as well. This is in line with the nature of the
Collaborative services or value co-creation. Collaborative services are not only complex and unstructured, but also they are tailored to fit unique needs of a particular client [7]. Hence, as discussed earlier, both parties need to participate for a successful outcome. In order to capture all these facts, similar to [26], we utilize a Cobb-Douglas functional form to model the instantaneous increase in the output, i.e., \( q[t] = \frac{dq[t]}{dt} = u[t]^\alpha v[t]^\beta \) with \( \alpha, \beta > 0 \), and \( \alpha + \beta < 1 \). The notations are summarized in Table 1.

The client gets utility from the project while it is being developed, so we define it as \( \int_0^T k q[t] dt \). Here, \( k \) is used to convert the output to a utility measure. One possible interpretation of \( k \) is dollar value per output. From the discussion above, we can write \( q[t] \) as:

\[
q[t] = \int_0^T u[s]^\alpha v[s]^\beta ds \quad \text{with} \quad q[0] = 0.
\]

### 3.3. Payment Structure

As mentioned earlier, the payment between the client and the vendor is modeled to be a function of either the effort level of the vendor, or the output, or both. This is in line with the business practices as noted in [23]. In the first model, the payment is based on the effort level spent by the vendor. In this case, the payment between the parties is modeled as \( \int_0^T p v[t] dt \), where \( p \geq 0 \) denotes the payment per unit effort. Such linear payment is the prevalent payment structure in business-to-business settings. As will be explained later, we optimize the parameter \( p \) in order to maximize the value for the client.

In the second model, the client transfers a portion of the revenues (or reduced costs) to the vendor. Hence, the transfer between the parties is modeled as \( \int_0^T l q[t] dt \) with \( l \geq 0 \) and \( k \geq l \). We optimize the payment parameter \( l \) later. Finally, in the third model, the payment is based on both the effort level and the output. In this case, the payment is \( \int_0^T p v[t] dt + \int_0^T l q[t] dt \) with \( p, l \geq 0 \) and \( k \geq l \).

### 4. Models

In the following subsections, we present each of the three models discussed earlier. In the differential game settings we analyze, the client is the principal and the vendor is the agent. The client offers a contract to the vendor and the vendor accepts it if the gain is more than its reservation utility \( \bar{u} \). Without any loss of generality, we normalize \( \bar{u} \) to zero. If the contract is accepted, then both parties start to work together and select and adjust their effort levels continuously.

#### 4.1. Effort-Dependent Payment Structure

As mentioned earlier, the transfer payment between the parties in this model is \( \int_0^T p v[t] dt \). The value for the client is given by \( \int_0^T k q[t] dt \). The costs accrued by client and vendor are \( \int_0^T c_c u[t]' dt \) and \( \int_0^T c_v v[t]' dt \), respectively. The client maximizes its value by optimizing the effort trajectory \( u[t] \) throughout the planning horizon. Similarly, the vendor optimizes \( v[t] \) for all \( t \). Hence, the objective functions of client and vendor, and the constraints are

\[
\max_{u[t]} \int_0^T k q[t] dt - \int_0^T c_c u[t]' dt - \int_0^T p v[t] dt
\]

\[
\max_{v[t]} \int_0^T p v[t] dt - \int_0^T c_v v[t]' dt
\]

s.t. \( q[t] = u[t]^\alpha v[t]^\beta ; u[t] \geq 0; v[t] \geq 0 \)

\[
\int_0^T p v[t] dt - \int_0^T c_v v[t]' dt \geq \bar{u}
\]

The Hamiltonian for the client \( H_c \), and for the vendor \( H_v \), can be written as [28]:

\[
H_c[t] = k q[t] - c_c u[t]' - p v[t] + \lambda_1[t] u[t]^\alpha v[t]^\beta,
\]

\[
H_v[t] = p v[t] - c_v v[t]' + \alpha \lambda_1[t] u[t]^\alpha v[t]^\beta.
\]

Here, \( \lambda_1[t] \) and \( \lambda_2[t] \) are the adjoint variables for the client and the vendor, respectively. These variables can be interpreted as the change in the corresponding objective functions for a small change in the state variable \( q[t] \). In effect, the adjoint variables are the marginal value of changes in the output for the client and the vendor.

The Hamiltonian for the client \( H_c \) consists of two parts. The first part is the integrand of the objective function for the client, and the second part is the adjoint variable \( \lambda_1[t] \) times the right hand side of the state equation for the output. For a given time instance \( t \), the first part represents the direct contribution to the objective function of the client. The second part represents an indirect contribution of increase in output level on the objective function. Consequently, the Hamiltonian needs to be maximized at each instance of time. The maximum principle decouples the dynamic problem into a set of static maximization problems in which the Hamiltonian acts as a surrogate objective [28].
Hamiltonian for the vendor can be interpreted in a similar manner.

Now we present the equilibrium effort levels in the following proposition. All the proofs are omitted due to space limitation.

**Proposition 1:** The equilibrium effort levels for the client and the vendor are:

\[
u[t] = \left( k(T - t) \alpha / (\gamma r_c) (p / (\delta c_v))^{\beta / (\gamma + \beta)} \right)^{1 / (\gamma + \beta)},
\]

\[
v[t] = (p / (\delta c_v))^{\beta / (\gamma + \beta)}.
\]

Next, the client chooses the payment parameter \( p \) such that its objective function is maximized. The result is presented below.

**Proposition 2:** The optimal value of the payment per unit effort (i.e., \( p \)) is

\[
p^* = \left( \frac{\beta (a - \gamma)c_v}{a(a - 2\gamma \delta)} \frac{r}{\gamma + \beta} \right)^{\frac{\gamma + \beta}{\gamma - a}} \frac{\gamma - a - \beta}{\gamma - a - \beta r}. \]

### 4.2. Output-Dependent Payment Structure

In this contractual setting, the client does not pay based on the efforts of the vendor. Rather, the client offers a portion of the output, i.e., \( \int_0^T q[t]dt \), to the vendor. The costs for both the client and the vendor remain unchanged compared to the first model. Hence, the objective functions of client and vendor and the constraints can be written as

\[
\max_{u[t]} \int_0^T k q[t]dt - \int_0^T c_v u[t]v dt - \int_0^T l q[t]dt
\]

\[
\max_{v[t]} \int_0^T l q[t]dt - \int_0^T c_v v[t]v dt
\]

s.t. \( \dot{q}[t] = u[t]a v[t]^{\beta} ; u[t] \geq 0; v[t] \geq 0 \)

\[
\int_0^T l q[t]dt - \int_0^T c_v v[t]v dt \geq \bar{u}
\]

The equilibrium effort levels are now presented in the following proposition.

**Proposition 3:** The equilibrium effort levels for the client and the vendor are:

\[
u[t] = \left( (T - t)^{\delta / (\beta / (\delta c_v))} ((k - \gamma c_v) / (k - \alpha / (\gamma r_c)))^{\beta / (\gamma + \beta)} \right)^{\frac{1}{\gamma + \beta}},
\]

\[
v[t] = \left( (T - t)^{\beta / (\delta / (\delta c_v))} ((k - \gamma c_v) / (k - \alpha / (\gamma r_c)))^{\beta / (\gamma + \beta)} \right)^{\frac{1}{\gamma + \beta}}.
\]

Recall that the effort level of the vendor is constant over time in the first model (see Proposition 1). However, in this model, the effort level of the vendor changes with time. Furthermore, we obtain the following result.

**Corollary 1:** Both of the equilibrium effort levels are strictly concave and decreasing in time.

Finally, the client can optimize the payment parameter \( l \) to maximize its value. This optimal level is presented below.

**Proposition 4:** The optimal transfer payment per unit output is given by:

\[
l^* = k \beta / \delta.
\]

### 4.3. Hybrid Payment Structure

In this payment structure, the objective functions of client and vendor and the constraints are

\[
\max_{u[t]} \int_0^T k q[t]dt - \int_0^T c_v u[t]v dt - \int_0^T l q[t]dt
\]

\[
- \int_0^T pv[t]dt
\]

\[
\max_{v[t]} \int_0^T l q[t]dt + \int_0^T pv[t]dt - \int_0^T c_v v[t]v dt
\]

s.t. \( \dot{q}[t] = u[t]a v[t]^{\beta} ; u[t] \geq 0; v[t] \geq 0 \)

\[
\int_0^T l q[t]dt + \int_0^T pv[t]dt - \int_0^T c_v v[t]v dt \geq \bar{u}
\]

Deriving a closed form solution for this problem is not possible because this structure is similar to that in [1]. However, we can derive the closed form solution for special cases for example when \( \delta = 1 \). For these cases, we can derive the equilibrium effort levels analytically. However, even for these special cases, it is not possible to optimize the payment parameters \( p \) and \( l \).

Nonetheless, we are able to present some interesting analytical results for the hybrid case without any restrictions on the parameter values. We also compare and contrast the performances of the three different contracts numerically accompanied by sensitivity analysis on the model parameters. We present these results, along with many others, in the next section.

### 5. Discussion and Managerial Insights

In this section, we discuss our findings regarding different aspects of the problem and outline the managerial insights.

#### 5.1. Effort-Dependent Contract

As shown in Proposition 1, the equilibrium level of client’s effort is strictly concave and decreasing.
with time. Furthermore, it becomes zero at the end of the planning horizon because it does not have any utility after the project ends. Proposition 1 also shows that \( v[t] \) strictly increases with \( k, p, \alpha, \) and \( \beta \), and strictly decreases with \( c_e, \gamma, \) and \( \delta \). This implies that the client has incentive to exert more effort as it increases the payment per unit effort to the vendor. This result might seem unintuitive, but can be explained as follows. If the payment per unit effort for the vendor is increased, this creates a direct incentive for the vendor to increase its effort level. This increased level of \( v[t] \) is an incentive for the client to increase its effort level as well. Hence, the client paying more to the vendor, in turn, is an incentive to increase its own effort level.

On the other hand, the effort level of the vendor, i.e., \( v[f] \), is constant throughout the planning horizon. This equilibrium level is essentially derived by equating the marginal gain to the marginal cost of effort. Because neither of these marginal terms have time components, the equilibrium level of \( v[f] \) is time-independent. Now, in the following corollary, we study the behavior of the optimal payment parameter \( p \).

**Corollary 2:** The optimal payment per unit effort, i.e., \( p \), strictly increases with \( c_v, k, \) and \( T \), and strictly decreases with \( c_e \).

It is interesting to note that as the cost for the client per unit effort, i.e., \( c_e \), increases, the optimal payment term decreases. The increase in \( c_e \) is a direct incentive for the client to decrease its effort level. Furthermore, since the payment parameter decreases with \( c_e \), this is an indirect incentive for the vendor to decrease its effort as well. On the other hand, if the cost for the vendor per unit effort, i.e., \( c_v \), increases, it is optimal for the client to increase the payment per unit effort to the vendor. Increase in \( c_v \) is actually an incentive for the vendor to decrease its effort level. However, since the increase in \( c_v \) also results in the increase in \( p \), this is an incentive for the vendor to increase its effort.

### 5.2. Output-Dependent Contract

Here, the equilibrium effort level of the vendor changes with time, unlike that in the effort-dependent contract. The reason is that, in this case, the transfer payment depends on the output which changes with time. Hence, there is incentive for the vendor to adjust its effort level continuously. Moreover, effort levels of both parties in this contract are strictly concave and decreasing in time. Besides, both of the effort levels converge to zero at the project completion time. Interestingly, we find that the optimal transfer payment per unit output, i.e., \( l \), does neither depend on the cost parameters \( c_e \) or \( c_v \), nor the output elasticity or the marginal cost parameters of the client (i.e., \( \alpha \) and \( \gamma \)).

If both the client and the vendor operate in the same industry, then it might be reasonable to assume that the marginal cost parameters \( \gamma \) and \( \delta \) are equal to each other. In this case, if the relationship between the parties is managed with an output-dependent contract, one party exerts more effort than the other throughout the planning horizon. In the next corollary, we study which party exerts more effort than the other.

**Corollary 3:** The ratio of client’s effort to the vendor’s effort is given by \( \left( \frac{(\delta - \beta)\alpha c_v}{\beta^2 c_e} \right)^{1/8} \). Although the effort levels are time-dependent, the proportionality of \( m \) does not depend on time.

From the above corollary we also see that, ceteris paribus, the ratio of the effort levels is at maximum when \( \alpha \) approaches to the value of 1. However, as the output depends more on the efforts of the vendor, i.e., \( \alpha < \frac{\beta^2}{(\delta - \beta)\alpha c_v} \), the vendor exerts more effort than the client.

### 5.3. The Hybrid Contract

Even though we are unable to obtain the closed form solution in this contract, we can present the following result regarding the equilibrium level of vendor’s effort.

**Proposition 5:** The equilibrium level of vendor’s effort in the hybrid contract is always greater than in the output-dependent contract, and both converge to the same level at the end of the planning horizon.

### 5.4. Comparison of the Contracts

Since the analytical solution of the hybrid contract is not derivable, we numerically determine the optimal payment terms \( p \) and \( l \) for that case. Before discussing the results, we should note that both effort-dependent and output-dependent contracts are special cases of the hybrid contract. If we set \( p \) or \( l \) to zero, we induce the hybrid contract to be a pure output-dependent contract or a pure effort-dependent contract. Therefore, the hybrid contract cannot be dominated by any of the other contracts. However, the effort-dependent and output-dependent contracts are more prominent in practice. Besides, managing a business collaboration with a hybrid contract is more
challenging because of the more complicated nature of the contract itself. Hence, we study under what circumstances (i) the hybrid contract dominates both of the other contracts at the same time so that it is preferred, and (ii) the effort-dependent contract dominates the output-dependent contract.

For a given instance of the problem, the numerical determination of optimal payment terms $p$ and $l$ for the hybrid contract involves numerical approximation of the (i) optimal effort trajectories for both parties, i.e., $u[t]$ and $v[t]$, and (ii) given these effort trajectories, the value of the objective function of the client. We do this analysis for many different combinations of $p$ and $l$ and denote the pair that results in the maximum objective function value for the client as the optimal $(p, l)$ pair.

For the majority of the parameter values, we observe that the hybrid contract converges to the effort-dependent contract so that both dominate the output-dependent one. However, if $\beta$ is more than 0.7 or 0.8, then the output-dependent contract dominates the effort-dependent contract. Interestingly, the hybrid contract never converged to the output-dependent contract in the entire set of cases we analyzed.

We performed a numerical experiment for the case $\{T = 10, \gamma = 2.5, \delta = 1.5, k = 3.6, c_c = 2, c_v = 3\}$ with different values of $\alpha$ and $\beta$. In the following figure, we see that the hybrid contract converges to the effort-dependent contract when $\beta$ is roughly less than 0.5. On the other hand, if $\beta$ is more than 0.5, then the hybrid contract dominates both of the other contracts. In the following figures, $\{H, E, O\}$ stands for the objective function values of hybrid, effort-dependent, and output-dependent contracts, respectively.

6. Extensions

In this section, we present two extensions. In the first extension, we allow the parties to get utility from the output even after the project is finished. In the second case, we allow the vendor to carry out training programs in order to decrease its cost per unit effort, i.e., $c_v$.

6.1. Salvage Values

Depending on the nature of the output, either party may have benefits from the project even after it is finished. For example, in the IT context, the client might use the software for an extended period after the development has been completed. On the other hand, the salvage value can be considered as reputation benefits to the vendor. We begin with a case where the client has a salvage value in an effort-dependent payment structure. Next, we analyze a scenario that considers salvage value to the vendor.

6.1.1. Salvage value for the client. Here, we consider that the client gets utility from the output both during the development time and after it is finished. The contract we analyze is the effort-dependent contract. Hence, given the salvage value parameter $S$, the objective functions of client and vendor and the constraints can be written as
The equilibrium effort levels for the client and vendor are:
\[
\begin{align*}
    u[t] &= \left( S + k(T-t) \alpha \right) \left( p / (\delta c_v) \right)^{\beta / \alpha + \gamma}, \\
    v[t] &= \left( p / (\delta c_v) \right)^{1 / \alpha + \beta}.
\end{align*}
\]

Next, the client optimizes the value of \( p \). For brevity, we do not present this result, but we discuss it below. As might be expected, if we allow \( S \to 0 \), then the optimal payment term as well as the equilibrium effort levels converge to the case with no salvage value. We also find that as long as \( S > 0 \), the optimal payment parameter in this scenario is greater than its no salvage value counterpart.

In this scenario, the equilibrium level of client’s effort, i.e., \( u[T] \), increases as compared to that in the no salvage value case. Another difference from the no salvage value case is that the terminal effort level, i.e., \( u[T] \), is positive. In contrast, it equals to zero when there is no salvage value for the client. In both cases, \( v[t] \) is constant throughout the planning period. However, it increases compared to its no salvage value counterpart.

We also observe that if we set \( k = 0 \), which implies that the client does not get utility throughout the planning horizon but only at the project completion time, then our model converges to a game theoretical model and there is no sense of continuous analysis except the accumulation of output that has value only at the end of the planning period.

### 6.1.2. Salvage value for the vendor

In this scenario, the client is assumed to build reputation or history from what it has achieved in its previous business collaborations. In the setting we analyze below, we consider an output-dependent contract. Given the salvage value parameter \( S \), the objective functions of client and vendor and the constraints are

\[
\begin{align*}
    & \max_{u[t]} \int_0^T q[t] dt - \int_0^T c_u[t] v[t] dt - \int_0^T p v[t] dt + S q[T] \\
    & \max_{v[t]} \int_0^T p v[t] dt - \int_0^T c_v v[t] dt \\
    \text{s.t.} & \quad q[t] = u(t)^{\alpha} v(t)^{\beta}; u[t] \geq 0; v[t] \geq 0 \\
    & \quad \int_0^T p v[t] dt - \int_0^T c_v v[t] dt \geq \bar{u}
\end{align*}
\]

The Hamiltonians are same as before, but now \( \lambda_1[T] = S \). The solution is presented in the following proposition.

**Proposition 6:** The equilibrium effort levels for the client and the vendor are:
\[
\begin{align*}
    u[t] &= \left( S + k(T-t) \alpha \right) \left( p / (\delta c_v) \right)^{\beta / \alpha + \gamma}, \\
    v[t] &= \left( p / (\delta c_v) \right)^{1 / \alpha + \beta}.
\end{align*}
\]

One direct observation from these equilibrium effort levels is that they both are greater than their no salvage value counterparts. The determination of the optimal payment parameter \( l \) is not possible analytically due to the necessity of integrating involved functions like Gauss hypergeometric function \( \int F_1 \). Although, it is possible to conduct numerical experiments. We do not discuss the findings of numerical experiments because of the space limitation.

### 6.2. Learning/Training

In practice, the clients might engage in activities like supplier development programs or training their vendors in order to increase vendor productivity or lower participation costs. In return, the client expects better value from the project or the output. One real-world example is Boeing that trains its suppliers how to live “lean,” increase efficiency, and reduce cost [4].

In this section, we consider that due to training or learning, the cost per unit effort for the vendor, i.e., \( c_v \), decreases with time. For simplicity, we consider that training effort is constant throughout the planning horizon. For example, the number of personnel who are responsible for vendor training might stay constant throughout the planning horizon.

Specifically, for a given training effort \( w \), we model the change in the cost per unit effort for the vendor as \( \dot{c}_v = \frac{dc_v}{dt} = -wc_v \). This implies that \( c_v = c_0 e^{-wt} \). Here, \( c_0 \) denotes the beginning level of \( c_v \). In this scenario, the cost for training throughout
the planning horizon is set to the per level per time training parameter \( c_T \) times the training effort \( w \) to the power \( \theta > 1 \). The fact that \( \theta \) is greater than 1 implies that there is decreasing returns to scale in the training efforts or the cheaper resources are utilized first. Now the objective functions of client and vendor and the constraints can be written as

\[
\max_{u[t]} \int_0^T k q[t] dt - \int_0^T c_c u[t] \gamma dt - \int_0^T p v[t] dt \\
\quad - c_T w^\theta T
\]

\[
\max_{v[t]} \int_0^T l q[t] dt - \int_0^T c_v v[t] \delta dt + S q(T)
\]

s.t. \( q[t] = u[t]^\alpha v[t]^\beta \); \( u[t] \geq 0 \); \( v[t] \geq 0 \).

The equilibrium effort levels for both parties are given in the following proposition.

**Proposition 9:** The equilibrium effort levels for the client and the vendor are:

\[
u[t] = \left( \frac{k(T-t)}{\alpha c_c} \left( \frac{p}{\delta c_v} \right)^{\frac{1}{\gamma+\delta}} \right)^{\frac{1}{1-\gamma}} \]

\[
v[t] = \left( \frac{p}{\delta c_v} \right)^{\frac{1}{1-\delta}}
\]

In contrast to the no-learning case, \( v[t] \) is no longer strictly concave in time. The first and second order conditions with respect to time reveal that for \( t < T - (\delta - 1)/(w\beta) \), \( u[t] \) strictly increases in time, reaches its maximum when \( t = T - (\delta - 1)/(w\beta) \), and strictly decreases for \( t > T - (\delta - 1)/(w\beta) \). If \( T \leq (\delta - 1)/(w\beta) \), then as in the no-learning case, \( u[t] \) strictly decreases with time.

Analytical derivation of the optimal payment parameter \( p \) and the training effort level \( w \) is not possible because of the simultaneous appearance of exponential terms as well as power terms in the same set of equations. However, for a given set of parameter values, it is possible to optimize \( p \) and \( w \) numerically.

**7. Conclusions**

In many business settings, such as IT services, consulting, and auditing, the traditional view of analyzing supplier-client or the consultant-client relationships need to be re-visited in light of the fact that the output of these relationships depend on the effort levels of both parties, not just the vendors. Therefore, in this paper, we analyze a value co-creation environment in which the output necessitates the efforts of both parties.

In this paper, we also consider the fact that both vendor and client may change their effort levels over time. Besides, we allow the client to get utility from the output as it is being developed. Taking into account all these facts, we examine three different contracts in a differential game setting. We derive the equilibrium effort levels of the client and the vendor as well as the optimal payment parameters in these contracts. We also compare the performances of these contracts to find the best contract for the client. Finally, we extend our study in two additional settings. In the first one, there is salvage value of the output for either vendor or the client. In another setting, we investigate lowering the cost of vendor by providing the training.

As future research directions, these contracts can be investigated with a focus on the vendor. This might be especially valuable in the business settings in which the vendor is more powerful than the client. Another research direction might be to look at a setting in which there are multiple vendors. Besides, solving the problem in a project management context with multiple tasks that are interdependent in the presence of budget and other constraints might reveal useful managerial insights. Last but not least, other training scenarios can be explored such as self-learning that does not incur training costs to either party.

**8. References**


