Linear Analysis of Multiple Outage Interaction

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Abstract

Linear sensitivities are popular methods for approximating the change in power system quantities after a change in the system. This paper examines the formulas that are used to calculate line outage distribution factors (LODFs) for both single and double outages. An examination of these formulas shows cases where they break down. In particular, the formulas fail when system islanding occurs. This happens for both single and multiple line outage sensitivities. Comments are made about the use of linear sensitivities in these cases, and a metric of interaction – a measure of how close the system is brought to islanding by a particular outage is developed. This metric is proposed as a useful method of categorizing line outages. Statistics are presented for the IEEE 300-bus case showing that the distribution of the metric indicates that most double outages are nearly decoupled.

1. Introduction

Linear methods have been used for at least 45 years to quickly approximate the change in power system state without the computational expense of solving the full ac power flow [1]. Line outage distribution factors (LODFs) are the linear sensitivities of line flows to a line outage [2]. They provide a good metric of the impact a line outage has on the other lines in the system. Using LODFs as a metric it is possible to efficiently determine which lines impact each other in the system.

Typically, LODF values tend to decay as the distance from an outaged line increases. However, this tendency does not hold in all cases. In particular, the tendency of LODFs to decay breaks down in the event of islanding. In the event of islanding, it is possible for the outage of one line to have a large impact on another distant line. Consider the case illustrated in Figure 1. Figure 1 shows two system islands connected by two tie lines. In the event that one of those lines outages, all the power must flow on the other line. This is because of the conservation of power. It does not matter how far (geographically or electrically) the two lines are from each other.

Because the net load in the two systems is constant, the outage of one line between the islands will result in the entire transfer shifting to the other line.

Figure 1. System islands connected by tie lines

The behavior of LODFs in the event of islanding is analyzed in detail for both the single and double line outage cases in Section 3. Based on this analysis, a metric is proposed to measure the interaction between the outaged elements in a two element contingency. For the single outage case, LODFs are a metric of interaction, but there is no clear metric of interaction in the multiple outage case. This paper proposes a metric of outage coupling that measures the interaction between outaged lines.

The operation of the power grid has changed dramatically over the past decade. Deregulation has changed the way the system is owned and operated [3]. Considering the impact of multiple outages has become more relevant after the introduction of new NERC standards setting performance requirements in the event of multiple outages [4]. These standards are intended to ensure the reliability of the system in the new deregulated environment.

Considering multiple outage contingencies is common for certain special cases. System operators
maintain a list of contingencies, and it is not uncommon for a contingency definition to involve more than one piece of equipment. They know from experience the outages that will threaten their system. The metric is proposed as quick way to exhaustively search for coupled contingencies.

Section 2 presents background material for linear analysis, introducing the notation and the linear factors that are used. Section 3 examines the special cases where the formulas for calculating linear sensitivities fail and the definitions must be applied to achieve reasonable results. Section 4 introduces a measure of outage coupling. Section 5 presents statistical data from the IEEE 300-bus test case [5], and conclusions are presented in Section 6.

2. Linear multiple outage analysis

Linear sensitivities are popular methods of approximating the power system states after a change occurs in the system. They are based on reducing the nonlinear power flow equations in to a linear system using the dc assumptions [6].

Power transfer distribution factors (PTDFs) are used to tell how the system line flows respond to a change in generation and load [7]. Line outage distribution factors (LODFs) are the sensitivities of line flows to line outages. LODFs tell how the flow changes on a line when some other line in the system is outaged. They are used to quickly predict the change in line flows for the outage of another line. For example, flowgates are used in an online capacity to monitor the post contingency state of critical elements in the system [8].

By definition a PTDF is the change in flow on a line for an injection and withdrawal in the system [2]

\[ p_{\alpha(i,j)} = \frac{\Delta f_\alpha}{T_{(i,j)}} \]  

(1)

Lines are represented by Greek characters, and buses are represented by Latin characters. In (1), the PTDF, \( p \), is written as the change in flow, \( f \), on line \( \alpha \) for a transfer, \( T \), from bus \( i \) to bus \( j \).

PTDFs can be expressed in terms of the inverse of \( B \), the dc Jacobian matrix

\[ p_{\alpha(i,j)} = \frac{1}{x_\alpha} a_\alpha B^{-1} a_{(i,j)} \]  

(2)

where \( x_\alpha \) is the reactance of line \( \alpha \), and the \( a \) vectors are incidence row vectors that contain a 1 and -1 as indicated by their subscripts. In the case that a Greek letter is a subscript, the incidence vector e.g., \( a_\alpha \), has a 1 and -1 at the from and to positions of the line. If the subscript is a pair of buses, e.g., \((i,j)\), then the vector \( a_{(i,j)} \) has a 1 and -1 in the \( i \) and \( j \) positions respectively.

By definition an LODF is the change in flow on a line as a percentage of the preoutage flow on another line. Thus, the LODF on line \( \alpha \) for the outage of line \( \beta \) is defined as

\[ d_{\alpha,\beta} = \frac{\Delta f_\alpha}{f_\beta} \]  

(3)

LODFs can be expressed in terms of PTDFs [2]

\[ d_{\alpha,\beta} = \frac{p_{\alpha,\beta}}{1 - p_{\alpha,\beta}} \]  

(4)

where \( d_{\alpha,\beta} \) is the sensitivity on line \( \alpha \) for the outage of line \( \beta \). This equation has a singularity when \( p_{\alpha,\beta} \) is 1.0. The causes of this condition and the modeling implications are discussed in detail in Section 3.

Linear factors have been extended to handle the outage of more than one element [9], [10]. This can be done by writing a linear system of equations that assumes we know the post outage flows on the outaged lines (denoted by a tilde). These are the flows that we will need to know to calculate the flow changes in other places in the system. For the outage of line \( \beta \) and line \( \delta \), we can construct the system of equations

\[ \begin{bmatrix} \tilde{f}_\beta = f_\beta + d_{\beta,\delta} \tilde{f}_\delta \\ \tilde{f}_\delta = f_\delta + d_{\delta,\beta} \tilde{f}_\beta \end{bmatrix} \]  

(5)

This system of equations accounts for the impact of the outages on each other. Putting (5) into matrix form, and solving for the post outage flows gives

\[ \begin{bmatrix} \tilde{f}_\beta \\ \tilde{f}_\delta \end{bmatrix} = \begin{bmatrix} 1 & -d_{\beta,\delta} \\ -d_{\delta,\beta} & 1 \end{bmatrix}^{-1} \begin{bmatrix} f_\beta \\ f_\delta \end{bmatrix} \]  

(6)

These are the flows that we can use to calculate flow changes on other lines in the system. To do so, we simply need to multiply by the sensitivities of the outaged lines onto a particular line of interest, e.g., line \( \alpha \).
\[ \Delta f_a = \begin{bmatrix} d_{a,\beta} & d_{a,\delta} \\ -d_{\delta,\beta} & 1 \end{bmatrix}^{-1} \begin{bmatrix} f_{\beta} \\ f_{\delta} \end{bmatrix} \]  

(7)

Thus, we can calculate the change in flow for two outages using linear sensitivities.

Extending the methodology for more than two outages is a straightforward matter. Instead of having a linear system with two equations, as is shown in (5), the number of equations will equal the number of outaged lines. Thus, for an arbitrary number of outages it is possible to write for \( c \) contingent lines

\[ \Delta f = LM^{-1}f \]  

(8)

where \( Le \mathbb{R}^{bc} \) is a row vector of LODF values relating the outaged lines onto the line of interest. \( Me \mathbb{R}^{mc} \) is a matrix containing the LODF values of the outaged lines onto each other, and \( fe \mathbb{R}^{ec} \) is a column vector of preoutage flows on the contingent lines.

3. Analysis of linear sensitivities

The application of linear sensitivities for contingency analysis has certain caveats. For example, the singularity in (4) which occurs when the self-PTDF \( (p_{\beta,\beta}) \) value is 1. Traditionally, this is handled by the application of the LODF definition (3) to arrive at sensible results. This section examines the situations in which these caveats appear for both the single and double outage cases, and uses this analysis to make statements about the use of these sensitivities in these situations.

3.1. Single outage sensitivities

In the scalar case (i.e., single line outage), the equation for the change in flow on line \( \alpha \) for the outage of line \( \beta \) is

\[ \Delta f_{\alpha} = d_{a,\beta} f_{\beta} \]  

(9)

which can be written in terms of PTDFs as

\[ \Delta f_{\alpha} = \frac{p_{a,\beta}}{1 - p_{\beta,\beta}} f_{\beta} \]  

(10)

Examining this equation it may be observed that \( \Delta f_{\alpha} \to \infty \) as \( p_{\beta,\beta} \to 1.0 \). This may be interpreted in terms of system topology for the case when the self-PTDF value is 1.0. In this case, the fact that the self-PTDF value is 1.0 indicates that a transfer from one end of the line to the other must flow entirely across that line. There is no alternate path for power to flow on. In other words, the line is radial.

In practice when this situation is encountered, the typical approach is to resort to the definition of an LODF (3) and use the fact that the line is radial to say that the self-LODF must be -1 and that the other LODFs are 0. Using -1 is equivalent to saying that the outage of this line will zero its flow. Setting the other LODF values to zero says that this line will not impact any other lines when it outages. By doing this, we have created a situation in which equation (9) may still be used to be used to calculate the change in flows.

There is a problem with doing this. Namely, we are not considering the fact that the system load or generation is changing when the radial line is opened. Except in the special case that the flow on the radial line is zero, the outage of the radial line will result in a generation imbalance within the power system island. Generation will no longer equal load.

Simply setting the self-LODF to -1 and the others to 0 will not account for the imbalance. In order to correctly predict the effects of the outage on the line flows, the generation response must be modeled. The important characteristic of this case is not that a line has been outaged. Instead, the important characteristic is that some load or generation has been lost. This means that modeling flow changes cannot be done realistically without modeling the generation’s response to an outage.

In the normal use of distribution factors (i.e., without islanding), the conservation of power is maintained. The net power within an island does not change. In reality there will be a small change caused by the change in losses as power flows are redistributed. However, this change is typically small, which is one reason the dc sensitivities tend to work well.

In the case a radial line is outaged, a load or generator is disconnected from the system. In this event, there is a net change in the islands load or generation and conservation of power is no longer satisfied. This suggests that modeling the system redispatch must be done to simulate the outage of a radial line.

3.2. Double outage sensitivities

In the double outage case we consider (8) expanded for the outage of line \( \beta \) and line \( \delta \).
If we expand this expression using (4) to put it in terms of PTDFs

$$\Delta f_\alpha = \begin{bmatrix} d_{\alpha,\beta} & d_{\alpha,\delta} \end{bmatrix} \begin{bmatrix} 1 & -d_{\beta,\delta} \\ -d_{\delta,\beta} & 1 \end{bmatrix}^{-1} \begin{bmatrix} f_\beta \\ f_\delta \end{bmatrix}$$  \hspace{1cm} (11)$$

we can repeat the analysis preformed for the scalar case. Examining (12) it is clear that strange behavior will arise in the event that \(p_{\beta,\beta} = 1.0\) or \(p_{\delta,\delta} = 1.0\). In these cases, the matrix \(M\) becomes extremely ill conditioned. In fact, as the self-PTDF approaches 1.0, the matrix approaches singularity. Thus, we can conclude that this formula will not apply in the event one of the outaged lines is radial. It is not too surprising to discover that double outage sensitivities do not work when single outage sensitivities fail.

A more interesting result may be had by examining the case when two lines connecting two islands in the system outage as illustrated in Figure 1. In this case, the conservation of power requires that the net flow between the islands remain the same. This requires that the line that remains in service pick up the amount of power flow necessary to maintain conservation of power when the other is outaged.

In terms of LODFs, the requirement is that the LODFs of two lines in Figure 1 onto each other will be 1.0. If the matrix \(M\) is evaluated with these values we get

$$M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$  \hspace{1cm} (13)$$

which is clearly singular. Right multiplying by any vector of the form \([1 \ 1]\) will result in a zero. In this case, (11) may not be evaluated because \(M^{-1}\) does not exist.

The outage of these two lines is analogous to the outage of a radial line for the single outage case. When both lines outage, there will be an imbalance in load and generation in the island (except in the special case that the net flow across the lines is zero).

We can put the analysis in terms of PTDFs

$$\Delta f_\alpha = \begin{bmatrix} p_{\alpha,\beta} & p_{\alpha,\delta} \\ 1-p_{\beta,\beta} & 1-p_{\delta,\delta} \end{bmatrix} \begin{bmatrix} 1 & -p_{\beta,\delta} \\ -p_{\delta,\beta} & 1 \end{bmatrix}^{-1} \begin{bmatrix} f_\beta \\ f_\delta \end{bmatrix}$$  \hspace{1cm} (12)$$

to examine why the matrix \(M\) becomes singular when an island forms in the system. After putting \(M\) in terms of PTDFs, we can make use of the fact that the sum of PTDFs is 1.0 for a cutset [11], [9]. For our analysis, the cutset is the two islands in the system.

Because the conservation of power must hold, a transfer from one island to the other will flow entirely on one line if the other is outaged. Since we are simulating outages using transfers, this is important to keep in mind. It means that the transfer simulating the outage of line \(\beta\) can only flow on line \(\delta\) and vice versa as illustrated in Figure 1. This gives

$$p_{\delta,\beta} + p_{\beta,\delta} = 1$$  \hspace{1cm} (15)$$

which is simply a statement that a transfer from one island into the other will flow on the lines connecting the two. This was originally observed in [11].

Now, we can examine the determinant of \(M\) to explore the origins of singularity.

$$\det(M) = \frac{1}{1-d_{\beta,\delta}d_{\delta,\beta}}$$  \hspace{1cm} (16)$$

In terms of PTDFs, the determinant of \(M\) is

$$\det(M) = \frac{1}{1-p_{\beta,\delta}p_{\delta,\beta}}$$  \hspace{1cm} (17)$$

We can find the origin of the singularity by looking at the denominator of the determinant and using the expressions (15) above. The denominator of the determinant is

$$1 - \left( p_{\beta,\delta} \right) \frac{p_{\delta,\beta}}{1-p_{\beta,\delta}} \frac{p_{\beta,\delta}}{1-p_{\delta,\beta}}$$  \hspace{1cm} (18)$$

Singularity will arise whenever this quantity is zero. If we solve the expressions in (15) for \(p_{\beta,\delta}\) and \(p_{\delta,\beta}\) and substitute into (18), we arrive at
which will give us zero in the denominator of the determinant. This will clearly cause $M$ to be singular.

From this analysis, we have shown that in the event of islanding the determinant of $M$ will be zero, indicating that the matrix is singular. Also, we have examined the reason for this. The reason the determinant becomes zero is because of the conservation of power constraint. Enforcing this constraint requires that a transfer from one island to another flow through the lines connecting them. In the special case that the lines connecting the two islands are outaged, cancellation occurs in the determinant.

4. Metric of outage coupling

The analysis in Section 3 indicates that the matrix $M$ is singular in the event that a system island forms. This section extends this observation to develop a measure of coupling between outages. The coupling between outages refers to the impact the outaged lines have upon each other. This information is reflected in the off diagonal values of $M$.

This section proposes the condition number of $M$ as a metric of outage interaction. The condition number can be thought of as a distance to singularity [12], [13]. Singularity of the matrix $M$ can be thought of in power system terms as the distance to islanding. In other words, the larger the condition number of $M$, the closer the contingency brings the system to islanding.

The condition number of $M$ is defined as

$$\kappa(M) = \frac{\|M\|}{\|M^{-1}\|}$$ \hspace{1cm} (20)

If the condition number is evaluated using the 2-norm, then (20) becomes

$$\kappa_2(M) = \frac{\sigma_{\text{max}}(M)}{\sigma_{\text{min}}(M)}$$ \hspace{1cm} (21)

where $\sigma_{\text{max}}$ and $\sigma_{\text{min}}$ are the maximum and minimum singular values. For the case that $M$ is a 2-by-2 matrix, there is a simple closed form expression for the condition number evaluated using the 2-norm

$$\kappa_2(M) = \frac{1+\sqrt{d_{\delta,\beta}d_{\beta,\delta}}}{1-\sqrt{d_{\delta,\beta}d_{\beta,\delta}}}$$ \hspace{1cm} (22)

Evaluating this expression is quite efficient. It only requires one multiplication and one square root calculation.

As mentioned above, $M$ becomes singular when a contingency causes the formation of an island in the system. This results in the maximum possible condition number for the matrix $M$, $\kappa(M) = \infty$.

The lowest possible condition number for the matrix $M$ corresponds to the case when the outages are completely decoupled, i.e., when the off diagonal terms of $M$ are zeros. In this case, $M$ is the identity matrix, and the condition number is 1.0.

It can be noted that in the event that $M$ is the identity matrix (11) reduces to

$$\Delta f = \left[\begin{array}{c} d_{a,\beta} \\ d_{a,\delta} \end{array}\right] \left[\begin{array}{c} f_{\beta} \\ f_{\delta} \end{array}\right]$$ \hspace{1cm} (23)

This is the special condition under which superposition works to calculate flow changes. When the outages are completely decoupled, $M$ and $M^{-1}$ are the identity matrix, so the change in flow can be calculated by simply adding the LODF-flow products.

$$\Delta f = d_{a,\beta}f_{\beta} + d_{a,\delta}f_{\delta}$$ \hspace{1cm} (25)

The off-diagonal terms in $M$ will only be identically zero in the event of the outage of two radial lines. However, they will be close to zero for outages that have small LODF values onto each other.

The denominator of equation (22) can provide a quick way to check for islanding in a double outage. In the event that the product of the off diagonal LODF values is 1.0,

$$d_{\delta,\beta}d_{\beta,\delta} = 1$$ \hspace{1cm} (26)

the denominator of (22) is zero, which results in a condition number of infinity, which is associated with islanding. Thus, to check for islanding, we can simply check to see if the product of the off diagonal LODF values is 1.0.
The condition number gives a metric indicating the degree of coupling of the individual outages involved in a double outage contingency. The larger the condition number the more the outages interact. A condition number close to 1.0 indicates that the outages only have a small impact on each other. As the condition number of the matrix $M$ approaches infinity, the system comes closer to islanding.

5. Condition number statistics

Using the IEEE 300-bus test case [5], condition numbers were calculated using the 2-norm for every double outage contingency that does not result in island formation. For the 411 lines in the IEEE 300 bus test case, there are 84,255 double outage contingencies.

Some statistics for the IEEE 300 bus double outage condition numbers are given in Table 1. The statistics show that the minimum condition number is 1.0, and the average value is very close to 1.0, indicating most matrices are nearly perfectly conditioned. This is significant because it means that even for a relatively small system most outages are essentially decoupled. In larger systems, the amount of interaction could be expected to be even smaller.

![Figure 2. Condition number distributions](image)

Table 1: Condition number statistics

<table>
<thead>
<tr>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max. $\kappa_2$</td>
<td>29.63</td>
</tr>
<tr>
<td>Min. $\kappa_2$</td>
<td>1.000</td>
</tr>
<tr>
<td>Avg. $\kappa_2$</td>
<td>1.060</td>
</tr>
<tr>
<td>Std. dev. $\kappa_2$</td>
<td>0.4313</td>
</tr>
</tbody>
</table>

The probability density function (PDF) and cumulative distribution function (CDF) were also calculated for the data set. Plots of the CDF and PDF are shown in Figure 2. The plots show that the condition numbers are clustered near 1.0, indicating most outages have weak interactions. A detail of the PDF around 1.0 is shown in Figure 3. The detail illustrates how quickly the condition number distribution decays. In fact, 92.3% of the data points are between 1.0 and 1.1.

![Figure 3. PDF near $\kappa = 1.0$](image)

Table 2. Condition number distribution

<table>
<thead>
<tr>
<th>$P(x \leq \kappa)$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.64</td>
<td>1.001</td>
</tr>
<tr>
<td>0.80</td>
<td>1.016</td>
</tr>
<tr>
<td>0.90</td>
<td>1.064</td>
</tr>
<tr>
<td>0.95</td>
<td>1.193</td>
</tr>
<tr>
<td>0.99</td>
<td>2.258</td>
</tr>
</tbody>
</table>

The bottom graph in Figure 3 shows the CDF (the integral of the PDF). The CDF answers the question: what is the probability that a condition number is greater than a given value $x$? Several points from the CDF are listed in Table 2. The data in this table can be used to study how condition numbers are distributed versus $(M)$. For a selection of $(M)_I = 258$, 99% of contingencies will have a condition number that is smaller.

The maximum value of $\kappa(M)$ occurs for the outage of the lines 3-150 and 7-131. The topology around these two outages is shown in Figure 4. The two outaged lines are tie lines between System 1 and System 2.
The $M$ matrix created by the outage of 3-150 and 7-131 is

$$
M = \begin{bmatrix}
1 & -0.942 \\
-0.927 & 1
\end{bmatrix}
$$

(27)

which has the maximum condition number of any outage, $\kappa(M) = 29.63$. An examination of the topology (Figure 4) around the outages reveals the reason. The two lines 3-150 and 7-131 connect System 1 and System 2, two areas within the IEEE 300 bus test case. There are connections between System 1 and System 2 through the other areas, but 3-150 and 7-131 are the only two direct connections. The two outaged lines serve as the primary connections between the two areas, so power from one of the outaged lines will redistribute onto the other and vice versa. The two lines are operating in a parallel capacity.

The empirical results show that the condition numbers are distributed very close to 1.0. This means that the vast majority of outages are essentially decoupled. For systems larger than the IEEE 300 bus case, the distribution can be expected to be even more heavily weighted around 1.0.

6. Conclusion

This paper examines the cases when the formulas for linear sensitivities break down and fail to give reasonable results. The cause of the breakdown can be traced to islanding of the system. Using this observation as an inspiration, a metric of outage interaction was proposed. The metric measures the distance of the matrix $M$ to singularity. The metric was calculated for every double outage contingency in the IEEE 300-bus test case, and the distribution of the condition numbers was found to be very heavily weighted around 1.0, the lowest possible condition number. This indicates that most of the double outage contingencies behave almost as independent outages. Work is continuing to develop the condition number as a contingency screening tool and extend the results to an arbitrary number of outages.

10. References


