Consensual Dynamics in Group Decision Making with Triangular Fuzzy Numbers

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Abstract

In this paper we study the modelling of consensus reaching in a 'soft' environment, i.e. when the individual testimonies are expressed as fuzzy preference relations. Here consensus is meant as the degree to which most of the experts agree on the preferences associated to the most relevant alternatives. First of all we derive a degree of dissensus based on linguistic quantifiers and then we introduce a form of network dynamics in which the quantifiers are represented by scaling functions. Next, assuming that the decision makers can express their preferences in a more flexible way, i.e. by means of triangular fuzzy numbers, we describe the iterative process of opinion changing towards consensus via the gradient dynamics of a cost function expressed as a linear combination of a dissensus cost function and an inertial cost function. Finally, some computer simulations are carried out together with a short description of a case study in progress.

keywords group decision making; consensus; fuzzy preference relations; linguistic quantifiers; fuzzy numbers; gradient dynamics.

1 Introduction

The traditional models of consensual dynamics, from DeGroot's classical consensus model [1] to various extended or alternative proposals, have been mostly formulated in the probabilistic framework [2] [3] [4] [5] [6] [7].
In the traditional approaches consensus is meant as a strict and unanimous agreement. However, since decision makers typically have different and more or less conflicting opinions, the traditional strict meaning of consensus is unrealistic. The human perception of consensus is much 'softer', and people are willing to accept that consensus has been reached when most actors agree on the preferences associated to the most relevant alternatives.
The problem of consensus modelling in a fuzzy environment was originally addressed in [8] [9] [10] [11] [12] [13] [14]. Then it was developed in [15] [16] [17] [18] [19] [20] [21] [22].
Further innovative approaches to the modelling of consensus in soft environments were developed under linguistic assessments. Among others, the interested reader is referred to [23] [24] [25] [26].
The soft consensus paradigm proposed by Kacprzyk and Fedrizzi [16] [17] [18] in the standard framework of numerical fuzzy preferences was extended to an explicit dynamical context in [27] [28] [29] [30] [31]. The consensus dynamics model combines a soft measure of collective dissensus with an inertial mechanism of opinion changing aversion. It acts on the network of single preference structures by a combination of a collective process of (nonlinear) diffusion and an individual mechanism of (nonlinear) inertia.
In relation with the crisp model of consensus dynamics described in [29], the fuzzy model introduced in [30] substitutes the standard crisp preferences with fuzzy ones, represented by triangular fuzzy numbers. Technically, the extension is based on the use of a distance measure between triangular fuzzy numbers. In analogy with the standard crisp model, the iterative process of collective opinion changing towards consensus in the fuzzy model can also be modelled via the gradient dynamics of a cost function. In this paper, finally, we present and comment on a number of computer simulations of the fuzzy model, all of which show interesting dynamical patterns of the fuzzy preferences.
The paper is organized as follows. In section 2, starting from the model developed in [16] for measuring the degree of consensus based on individual fuzzy preference relations, we derive a new degree of dissensus depending on linguistic quantifiers, and then we propose a way to transform the quantifiers into scaling functions. Starting from the soft consensus model proposed in [29], we show how to develop the dynamical process of modifying individual opinions on the basis of a cost function W, defined as a convex linear combination of the two different dynamical components we mentioned above. In section 3, assuming that the preferences of the decision makers are expressed by means of triangular fuzzy numbers, we

describe how to measure the distance between individual preferences and then we introduce the new cost function W. Section 4 contains the main contribution of the paper. After having described the extended consensual dynamics based on the gradient method, some interesting cases are studied by means of numerical simulations. The aim is to test the extended dynamics by simulating group decision problems where the decision maker's initial preferences are differently structured, as well as their opinion changing aversion. At the end of the section we briefly review a case study involving the application of the model to supplier selection in a supply chain management system under development in the healthcare department of a local government agency. In section 5 we present the conclusions and some future research.

2 The soft dissensus measure and the consensus dynamics

The modelling framework used for introducing the degree of dissensus is the one described in [16]. Our point of departure is a set of individual fuzzy preference relations. If $A = \{a_1, \dots, a_m\}$ is a set of decisional alternatives and $I = \{1, \dots, n\}$ is a set of individuals, then the fuzzy preference relation R_i of individual i is given by its membership function $R_i : A \times A \rightarrow [0,1]$ such that

$$\begin{aligned}
R_i(a_k,a_l) &= 1 \quad \text{if } a_k \text{ is definitely preferred over } a_l \\
R_i(a_k,a_l) &\in (0.5,1) \quad \text{if } a_k \text{ is preferred over } a_l \\
R_i(a_k,a_l) &= 0.5 \quad \text{if there is indifference} \\
& \qquad \text{between } a_k \text{ and } a_l \\
R_i(a_k,a_l) &\in (0,0.5) \quad \text{if } a_l \text{ is preferred over } a_k \\
R_i(a_k,a_l) &= 0 \quad \text{if } a_l \text{ is definitely preferred over } a_k,
\end{aligned}$$

where $i = 1, \dots, n$ and $k, l = 1, \dots, m$. Each individual fuzzy preference relation R_i can be represented by a matrix $[r^i_{kl}]$, $r^i_{kl} = R_i(a_k, a_l)$ which is commonly assumed to be reciprocal, that is $r^i_{kl} + r^i_{lk} = 1$. Clearly, this implies $r^i_{kk} = 0.5$ for all $i = 1, \dots, n$ and $k = 1, \dots, m$.
The general case $A = \{a_1, \dots, a_m\}$ for the set of decisional alternatives is discussed in [29] and [30]. Here, for the sake of simplicity, we assume that the alternatives available are only two ($m = 2$), which means that each individual preference relation R_i has only one degree of freedom, denoted by $x_i = r^i_{12}$. In such case, the degree of dissensus between individuals i and j as to their preferences between the two alternatives is

$$V(i,j) = (x_i - x_j)^2 \in [0,1], \tag{1}$$

and the degree of dissensus between Q pairs of individuals as to their preferences between the two alternatives becomes

$$V_Q(i,j) = Q(V(i,j)) = Q((x_i - x_j)^2) \in [0,1], \tag{2}$$

where the quantifier Q is defined as follows,

$$Q(x) = (f(x) - f(0))/(f(1) - f(0)). \tag{3}$$

Here f is a scaling function defined as

$$f(x) = -\frac{1}{\beta}\, ln(1 + e^{-\beta(x-\alpha)}), \tag{4}$$

where $\alpha \in (0,1)$ is a threshold parameter and $\beta \in (0,\infty)$ is a free parameter. The parameter β controls the polarization of the sigmoid function $f' : [0,1] \rightarrow (0,1)$ given by

$$f'(x) = 1/(1 + e^{\beta(x-\alpha)})\,. \tag{5}$$

Now, following the soft consensus model proposed in [29], we show how to develop the dynamical process of modifying individual opinions using the dissensus based framework. For more details refer to the original paper.
In the soft consensus model each decision maker $i = 1, \dots, n$ is represented by a pair of connected nodes, a primary node (dynamic) and a secondary node (static). The n primary nodes form a fully connected subnetwork and each of them encodes the individual opinion of a single decision maker. The n secondary nodes, on the other hand, encode the individual opinions originally declared by the decision makers, denoted $s_i \in [0,1]$, and each of them is connected only with the associated primary node. The iterative process of opinion transformation corresponds to the gradient dynamics of a cost function W, depending on both the present and the original network configurations. The value of W combines a measure V of the overall dissensus in the present network configuration and a measure U of the overall change from the original network configuration.
The various interactions involving node i are mediated by interaction coefficients whose role is to quantify the strength of the interaction. The consensual interaction between primary nodes i and j is mediated by the interaction coefficient $v_{ij} \in (0,1)$, whereas the inertial interaction between primary node i and the associated secondary node is mediated by the interaction coefficient $u_i \in (0,1)$. It turns out that the values of these interaction coefficients are given by the derivative f' of the scaling function according to

$$v_{ij} = f'((x_i - x_j)^2) \tag{6}$$

$$v_i = \sum_{j(\neq i)=1}^{n} v_{ij}/(n-1) \tag{7}$$

$$u_i = f'((x_i - s_i)^2)\,. \quad (8)$$

The average preference $\bar{x}_i$ is given by

$$\bar{x}_i = \sum_{j(\neq i)=1}^{n} v_{ij} x_j / \sum_{j(\neq i)=1}^{n} v_{ij} \quad (9)$$

and represents the average preferences of the remaining decision makers as seen by the i^{th} decision maker.
The individual dissensus cost $V(i)$ is given by

$$V(i,j) = f((x_i - x_j)^2) \quad (10)$$

$$V(i) = \sum_{j(\neq i)=1}^{n} V(i,j)/(n-1) \quad (11)$$

and the individual opinion changing cost $U(i)$ is

$$U(i) = f((x_i - s_i)^2)\,. \quad (12)$$

Summing over the various decision makers we obtain the collective dissensus cost V and inertial cost U,

$$V = \frac{1}{4}\sum_{i=1}^{n} V(i) \quad (13)$$

$$U = \frac{1}{2}\sum_{i=1}^{n} U(i) \quad (14)$$

with conventional multiplicative factors of $1/4$ and $1/2$. The full cost function W is then $W = (1-\lambda)V + \lambda U$ with $0 \leq \lambda \leq 1$.
The consensual network dynamics, which can be regarded as an unsupervised learning algorithm, acts on the individual opinion variables x_i through the iterative process

$$x_i \rightsquigarrow x_i' = x_i - \gamma \frac{\partial W}{\partial x_i}\,. \quad (15)$$

Analyzing the effect of the two dynamical components V and U separately we obtain

$$\frac{\partial V}{\partial x_i} = v_i(x_i - \bar{x}_i) \quad (16)$$

where the coefficients v_i were defined in (7) and the average preference $\bar{x}_i$ was defined in (9), and therefore

$$x_i' = (1 - \gamma v_i)x_i + \gamma v_i \bar{x}_i\,. \quad (17)$$

On the other hand, we obtain

$$\frac{\partial U}{\partial x_i} = u_i(x_i - s_i)\,, \quad (18)$$

where the coefficients u_i were defined in (8), and therefore

$$x_i' = (1 - \gamma u_i)x_i + \gamma u_i s_i\,. \quad (19)$$

The full dynamics associated with the cost function $W = (V+U)/2$ acts iteratively according to

$$x_i' = (1 - \gamma(v_i + u_i))x_i + + \gamma v_i \bar{x}_i + \gamma u_i s_i\,. \quad (20)$$

and the decision maker i is in dynamical equilibrium, in the sense that $x_i' = x_i$, if the following stability equation holds,

$$x_i = (v_i \bar{x}_i + u_i s_i)/(v_i + u_i) \quad (21)$$

that is, if the present opinion x_i coincides with an appropriate weighted average of the original opinion s_i and the average opinion value $\bar{x}_i$.

3 The consensual dynamics with triangular fuzzy numbers

Let us now assume that the preferences of the decision makers are expressed by means of fuzzy numbers, see for instance [32] [33], in particular by means of triangular fuzzy numbers. Then, in order to measure the differences between the preferences of the decision makers, we need to calculate the distances between the fuzzy numbers representing those preferences. Let

$$\mathbf{x} = \{\varepsilon_L, x, \varepsilon_R\} \quad \mathbf{y} = \{\theta_L, y, \theta_R\} \quad (22)$$

be two triangular fuzzy numbers, where x is the central value of the fuzzy number $\mathbf{x}$ and ε_L, ε_R are its left and right spread respectively. Analogously for the triangular fuzzy number $\mathbf{y}$.
Various definitions of distance between fuzzy numbers are considered in the literature [34] [35] [36] [37]. Moreover, the question has been often indirectly addressed in papers regarding the ranking of fuzzy numbers, see [38] [39] for a detailed review. In our model we refer to a distance, indicated by $D^*(\mathbf{x},\mathbf{y})$, which belongs to a family of distances introduced in [34]. This distance is defined as follows.
For each $\alpha \in [0,1]$, the α–level sets of the two fuzzy numbers $\mathbf{x}$ and $\mathbf{y}$ are respectively

$$[x_L(\alpha), x_R(\alpha)] = [x - \varepsilon_L + \varepsilon_L\alpha\ ,\ x + \varepsilon_R - \varepsilon_R\alpha] \quad (23)$$

$$[y_L(\alpha), y_R(\alpha)] = [y - \theta_L + \theta_L\alpha\ ,\ y + \theta_R - \theta_R\alpha]\,. \quad (24)$$

The distance $D^*(\mathbf{x},\mathbf{y})$ between $\mathbf{x}$ and $\mathbf{y}$ is defined by means of the differences between the left boundaries of

(23), (24) and the differences between the right boundaries of (23), (24). More precisely, the left integral I_L is defined as the integral, with respect to α, of the squared difference between the left boundaries of (23) and (24),

$$I_L = \int_0^1 (x_L(\alpha) - y_L(\alpha))^2 d\alpha \qquad (25)$$

and the right integral I_R is defined as the integral, with respect to α, of the squared difference between the right boundaries of (23), (24),

$$I_R = \int_0^1 (x_R(\alpha) - y_R(\alpha))^2 d\alpha\,. \qquad (26)$$

Finally, the distance $D^*(\mathbf{x}, \mathbf{y})$ is defined as

$$D^*(\mathbf{x}, \mathbf{y}) = \left(\frac{1}{2}(I_L + I_R)\right)^{1/2}. \qquad (27)$$

The distance (27) is obtained by choosing $p = 2$ and $q = 1/2$ in the family of distances introduced in [34]. In order to avoid unnecessarily complex computations, we skip the square root and we use, in our model, the simpler expression

$$D(\mathbf{x}, \mathbf{y}) = (D^*(\mathbf{x}, \mathbf{y}))^2 \; = \; \frac{1}{2}(I_L + I_R)\,. \qquad (28)$$

Note that expression (28), except for the numerical factor $1/2$, has been introduced, independently from [34], also in [40]. It has been then pointed out in [42] that (28) is not a distance, as it does not always satisfy the triangular inequality. Nevertheless, as long as optimization is involved, expression (28) can be equivalently used in place of the distance (27) [41]. In any case, for simplicity, in the following we shall use the term distance when referring to (28). Solving (25) and (26), we obtain

$$D(\mathbf{x}, \mathbf{y}) = d^2 + \frac{1}{6}\delta_L^2 + \frac{1}{6}\delta_R^2 + \frac{d}{2}(\delta_R - \delta_L), \qquad (29)$$

where $d = x - y$, $\delta_L = \varepsilon_L - \theta_L$ and $\delta_R = \varepsilon_R - \theta_R$.
As we assumed in section 2, the preferences of the n decision makers are expressed by pairwise comparing the alternatives $a_1, a_2, ..., a_m$. Given a pair of alternatives, we assume that the preference of the first over the second alternative is represented, for decision maker i, by a triangular fuzzy number indicated by

$$\mathbf{r}^i = \{\varepsilon_L^i, r^i, \varepsilon_R^i\}, \qquad (30)$$

where, as in (22), r^i is the central value of the fuzzy number $\mathbf{r}^i$, whereas ε_L^i and ε_R^i are its left and right spreads respectively. Analogously, let $\mathbf{r}^j$ be the triangular fuzzy number of type (30) representing the preference of the first alternative over the second given by decision maker j.
Following definition (28), the distance between the fuzzy preference of decision maker i and the one of decision maker j becomes

$$D(\mathbf{r}^i, \mathbf{r}^j) = d^2 + \frac{1}{6}\delta_L^2 + \frac{1}{6}\delta_R^2 + \frac{d}{2}(\delta_R - \delta_L), \qquad (31)$$

where $d = r^i - r^j$, $\delta_L = \varepsilon_L^i - \varepsilon_L^j$ and $\delta_R = \varepsilon_R^i - \varepsilon_R^j$.

As in section 3, we consider, for the sake of simplicity, a problem with $m = 2$ alternatives and we define the dissensus measure between two decision makers by applying the scaling function f to $D(\mathbf{r}^i, \mathbf{r}^j)$,

$$V(i, j) = f(D(\mathbf{r}^i, \mathbf{r}^j))\,. \qquad (32)$$

The dissensus measure of decision maker i with respect to the rest of the group is given by the arithmetic mean of the various dissensus measures $V(i, j)$,

$$V(i) = \sum_{j(\neq i)=1}^{n} V(i, j)/(n-1)\,. \qquad (33)$$

Finally, the global dissensus measure of the group is defined by

$$V = \frac{1}{4}\sum_{i=1}^{n} V(i)\,, \qquad (34)$$

thus obtaining

$$V = \frac{1}{4}\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n} f(D(\mathbf{r}^i, \mathbf{r}^j))/(n-1)\,. \qquad (35)$$

Denoting by $\mathbf{s}^i = \{\theta_L^i, s^i, \theta_R^i\}$ the triangular fuzzy number describing the initial preference of decision maker i, the cost for changing the initial preference $\mathbf{s}^i$ into the actual preference $\mathbf{r}^i$ is given by

$$U(i) = f(D(\mathbf{r}^i, \mathbf{s}^i))\,. \qquad (36)$$

The global opinion changing aversion component U of the group is given by

$$U = \frac{1}{2}\sum_{i=1}^{n} U(i)\,. \qquad (37)$$

The numerical factors $\frac{1}{4}$ and $\frac{1}{2}$ have been introduced in (34) and in (37) respectively in order to simplify the results in the following and are not relevant in defining the components V and U. As mentioned before, the global cost function W is defined as a convex linear combination of the components V and U,

$$W = (1 - \lambda)V + \lambda U, \qquad (38)$$

and the exogenous parameter $\lambda \in [0,1]$ represents the relative importance of the inertial component U with respect to the dissensus component V.

4 The extended algorithm and some numerical simulations

In [30] the consensual dynamics described in section 3 was extended to the case where the preferences are expressed by means of triangular fuzzy numbers. In the consensual dynamics, the global cost function $W = W(\mathbf{r}^i) = W(\varepsilon_L^i, r^i, \varepsilon_R^i)$ is minimized through the gradient method. This implies that in every iteration the new preference $\mathbf{r}'$ is obtained from the previous preference $\mathbf{r}$ in the following way (we skip the index i for simplicity)

$$\mathbf{r} \to \mathbf{r}' = \mathbf{r} - \gamma \nabla W . \tag{39}$$

The consensual dynamics (39) will gradually update the three parameters $(\varepsilon_L, r, \varepsilon_R)$ characterizing the preferences according to

$$\begin{aligned} r \to r' &= r - \gamma \frac{\partial W}{\partial r} \\ \varepsilon_L \to \varepsilon_L{}' &= \varepsilon_L - \gamma \frac{\partial W}{\partial \varepsilon_L} \\ \varepsilon_R \to \varepsilon_R{}' &= \varepsilon_R - \gamma \frac{\partial W}{\partial \varepsilon_R} . \end{aligned} \tag{40}$$

We can consider separately the effect of the two components V and U of W, since ∇W is a linear combination of ∇V and ∇U,

$$\nabla W = (1-\lambda)\nabla V + \lambda \nabla U . \tag{41}$$

Let us first consider the component V. Taking again into account the index i, we have

$$\frac{\partial V}{\partial r^i} = v_i \left(\left(r^i - \bar{r}^i \right) + \frac{1}{4} \left(\varepsilon_R^i - \bar{\varepsilon}_R^{\,i} - \varepsilon_L^i + \bar{\varepsilon}_L^{\,i} \right) \right) \tag{42}$$

where

$$\begin{aligned} v_i &= \sum_{j(\neq i)=1}^{n} v_{ij}/(n-1) \, ; \quad v_{ij} = f'(D(\mathbf{r}^i, \mathbf{r}^j)) \\ \bar{r}^i &= \frac{\sum_{j(\neq i)=1}^n v_{ij}\, r^j}{\sum_{j(\neq i)=1}^n v_{ij}} \\ \bar{\varepsilon}_L^{\,i} &= \frac{\sum_{j(\neq i)=1}^n v_{ij}\, \varepsilon_L^j}{\sum_{j(\neq i)=1}^n v_{ij}} \\ \bar{\varepsilon}_R^{\,i} &= \frac{\sum_{j(\neq i)=1}^n v_{ij}\, \varepsilon_R^j}{\sum_{j(\neq i)=1}^n v_{ij}} . \end{aligned} \tag{43}$$

Analogously, we calculate

$$\frac{\partial V}{\partial \varepsilon_L^i} = v_i \left(\frac{1}{6} (\varepsilon_L^i - \bar{\varepsilon}_L^{\,i}) - \frac{1}{4} (r^i - \bar{r}^i) \right) \tag{44}$$

and

$$\frac{\partial V}{\partial \varepsilon_R^i} = v_i \left(\frac{1}{6} (\varepsilon_R^i - \bar{\varepsilon}_R^{\,i}) + \frac{1}{4} (r^i - \bar{r}^i) \right) . \tag{45}$$

Let us now consider the inertial component U. We obtain

$$\frac{\partial U}{\partial r^i} = u_i \left((r^i - s^i) + \frac{1}{4} (\varepsilon_R^i - \theta_R^{\,i} - \varepsilon_L^i + \theta_L^{\,i}) \right) \tag{46}$$

where

$$u_i = f'(D(\mathbf{r}^i, \mathbf{s}^i)) , \tag{47}$$

$$\frac{\partial U}{\partial \varepsilon_L^i} = u_i \left(\frac{1}{6} (\varepsilon_L^i - \theta_L^{\,i}) - \frac{1}{4} (r^i - s^i) \right) \tag{48}$$

and

$$\frac{\partial U}{\partial \varepsilon_R^i} = u_i \left(\frac{1}{6} (\varepsilon_R^i - \theta_R^{\,i}) + \frac{1}{4} (r^i - s^i) \right) . \tag{49}$$

At this point we can summarize the effects of the two components obtaining

$$\frac{\partial W}{\partial r^i} = ((1-\lambda)v_i + \lambda u_i)\Delta r^i - (1-\lambda) v_i \Delta \bar{r}^i - \lambda u_i \Delta s^i \tag{50}$$

where

$$\begin{aligned} \Delta r^i &= r^i + \frac{1}{4} (\varepsilon_R^i - \varepsilon_L^i) && (51) \\ \Delta \bar{r}^i &= \bar{r}^i + \frac{1}{4} (\bar{\varepsilon}_R^{\,i} - \bar{\varepsilon}_L^{\,i}) && (52) \\ \Delta s^i &= s^i + \frac{1}{4} (\theta_R^{\,i} - \theta_L^{\,i}) . && (53) \end{aligned}$$

The derivative of W with respect to the left spread becomes

$$\frac{\partial W}{\partial \varepsilon_L^i} = ((1-\lambda)v_i + \lambda u_i)\Delta \varepsilon_L^i - (1-\lambda) v_i \Delta \bar{\varepsilon}_L^{\,i} - \lambda u_i \Delta \theta_L^{\,i} \tag{54}$$

where

$$\begin{aligned} \Delta \varepsilon_L^i &= \frac{1}{6} \varepsilon_L^i - \frac{1}{4} r^i && (55) \\ \Delta \bar{\varepsilon}_L^{\,i} &= \frac{1}{6} \bar{\varepsilon}_L^{\,i} - \frac{1}{4} \bar{r}^i && (56) \\ \Delta \theta_L^{\,i} &= \frac{1}{6} \theta_L^{\,i} - \frac{1}{4} s^i . && (57) \end{aligned}$$

The derivative of W with respect to the right spread becomes

$$\frac{\partial W}{\partial \varepsilon_R^i} = ((1-\lambda)v_i + \lambda u_i)\Delta\varepsilon_R^i - (1-\lambda)v_i\Delta\bar{\varepsilon}_R^i - \lambda u_i \Delta\theta_R^i \quad (58)$$

where

$$\Delta\varepsilon_R^i = \frac{1}{6}\varepsilon_R^i + \frac{1}{4}r^i \quad (59)$$

$$\Delta\bar{\varepsilon}_R^i = \frac{1}{6}\bar{\varepsilon}_R^i + \frac{1}{4}\bar{r}^i \quad (60)$$

$$\Delta\theta_R^i = \frac{1}{6}\theta_R^i + \frac{1}{4}s^i. \quad (61)$$

Let us now present some numerical simulations in order to illustrate the behavior of the dynamics in some interesting cases. All computations are performed with the following values of the parameters: $\gamma = 0.005$, $\alpha = 0.3$ and $\beta = 10$. In the general case reported in Figure 1 the two initial preferences are fuzzy and different in shape, although the centers are symmetrical with respect to the mid point 0.5. When $\lambda = 0$ the two preferences converge asymptotically to a single fuzzy preference whose overall spread clearly corresponds to a trade-off between the two initial ones. When $\lambda = 0.5$, on the other hand, the two preferences are asymptotically distinct but the previous trade-off effect is also present. Moreover, we note that the consensual dynamics tends to induce an area overlapping effect between the two fuzzy preferences.

In the special case reported in Figure 2 the two initial preferences are crisp and symmetrical with respect to the mid point 0.5. In time, the two crisp preferences become fuzzy, creating not only internal spreads (as expected) but also external spreads (which is more interesting). When $\lambda = 0$ the two preferences converge asymptotically to a single fuzzy preference with non zero spread, which means that the consensual dynamics does not preserve the crispness of the preferences, not even asymptotically. When $\lambda = 0.5$, on the other hand, the two preferences are asymptotically distinct but crisp, which is a very interesting effect due to the inertial component of the consensual dynamics.

In Figure 3 it is also reported a particular case, where the two initial preferences are fuzzy with (only) external spreads and symmetrical centers with respect to the mid point 0.5. In time, the two fuzzy preferences move towards one another, driven by the area overlapping effect of the consensual dynamics. The interesting feature observed in this case is the fact that the center trajectories intersect, once with $\lambda = 0$ and twice with $\lambda = 0.5$. This intersection effect is present only in the extended version of the consensual dynamics model, not in the classical (crisp) one. When $\lambda = 0$ the two preferences converge asymptotically to a single fuzzy preference, with equal spreads. When $\lambda = 0.5$, on the other hand, the two preferences are asymptotically distinct and with unequal spreads, even though they show both internal and external spreads.

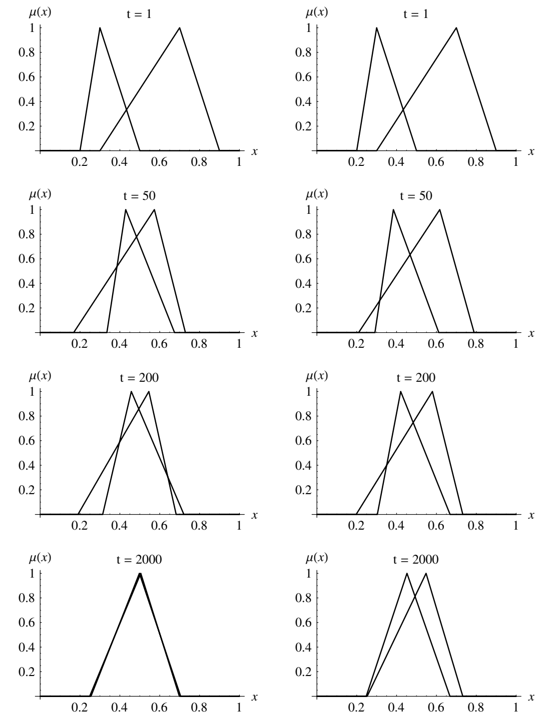

Figure 1: Case with $\lambda = 0$ (left) and $\lambda = 0.5$ (right)

In the case reported in Figure 4 we have six initial fuzzy preferences divided in two coalitions, in each of which the end points of the triangles are common. The collective preference configuration corresponds essentially to the two extreme groups of a set of linguistic terms. When $\lambda = 0$ the six preferences converge asymptotically to a single fuzzy preference, with equal spreads. When $\lambda = 0.5$, on the other hand, the six preferences are asymptotically distinct but still, within each coalition, the end points of the triangles are common. Interestingly, the asymptotic configuration of the two preference coalitions can be characterized by the end points but no longer by the centers, whose discriminatory value has been lost.

At the end of the section we would like to summarize how we started to approach the implementation of our model in a supply chain management system to be developed by our research unit under a two years project involving the heath care department of a local government agency. It is well known that management of suppliers, which represent an integral part of the supply chain of an

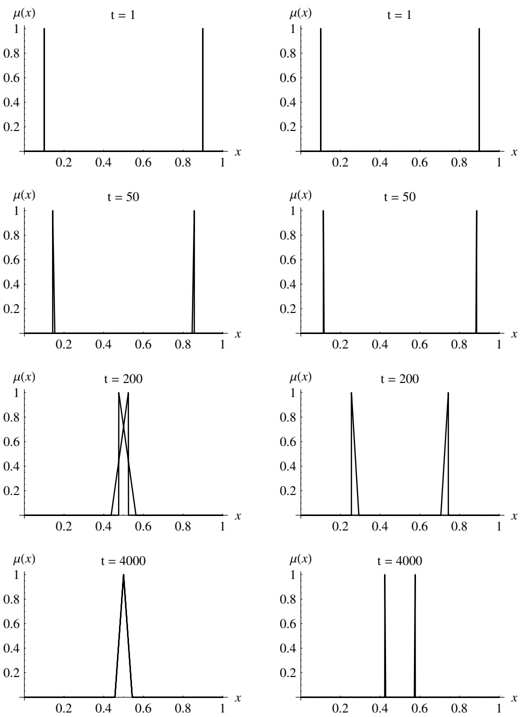

Figure 2: Case with $\lambda = 0$ (left) and $\lambda = 0.5$ (right)

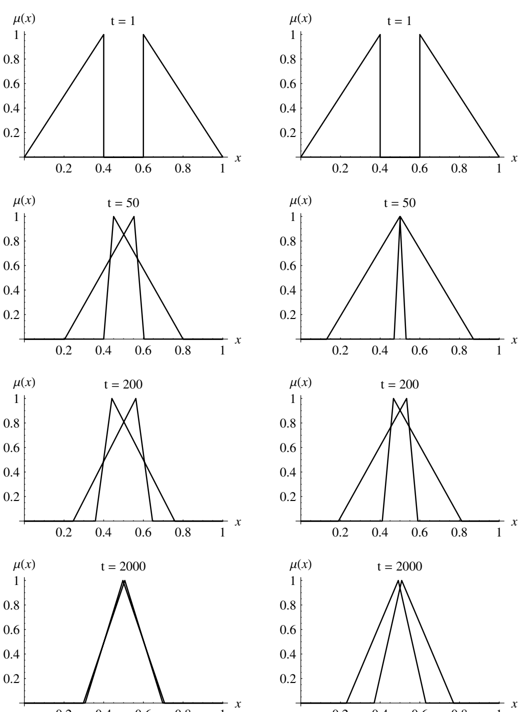

Figure 3: Case with $\lambda = 0$ (left) and $\lambda = 0.5$ (right)

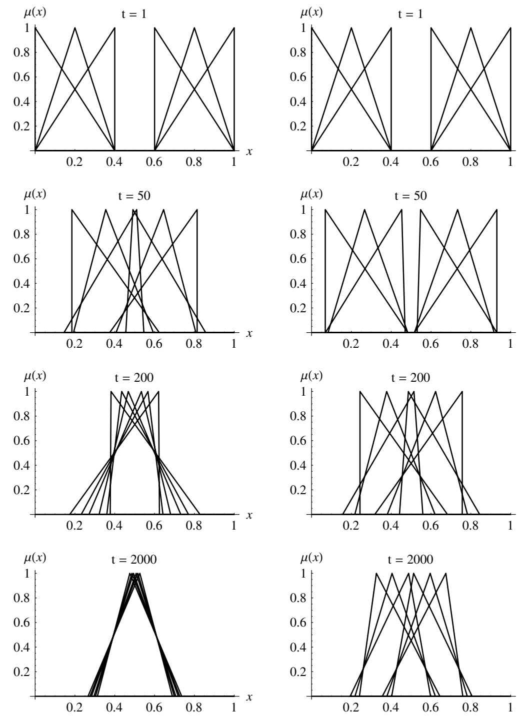

Figure 4: Case with $\lambda = 0$ (left) and $\lambda = 0.5$ (right)

organization, requires an adequate selection framework based on multiple criteria and effective negotiation skills involving more then one expert. To find the best supplier it is necessary to make a trade-off between conflicting quantitative and qualitative factors whose evaluation depends on the opinions shared by the individuals belonging to a group of experts (decision makers). The methods proposed in the literature for solving the supplier selection problem, mostly based on operations research techniques, usually involve linear weighting methods, analytic hierarchy process, mathematical programming techniques (for a selected list of references see [43]) but the problem of pooling the opinions/judgements of the experts, even if mentioned, is not formally and satisfactory addressed in the literature. Moreover, since supplier selection involves ratings and weights of the criteria that are assessed by means of linguistic labels and the expert judgements are often vague, the fuzzy approach seems to be quite suitable (see [44] [45] [46] [47] for reasons for supporting the fuzzy approach). In essential, the supplier selection problem in supply chain systems could be addressed using a group decision model in which the evaluation (consensual) of the suppliers is based on fuzzy relations estimated using fuzzy numbers (linguistic labels). Therefore, the model described in our paper seems to be suitable in supporting this kind of decision process, as some evidence from our experimental approaches to healthcare supply chain management have been so far demonstrated. We carried out several experimentations with a team of experts belonging to the healthcare department of a local autonomous government agency and involved in the joint selection of the suppliers of the regional hospital system. This is a problem that deeply affects the system, since about 30 percent of hospital management costs are supply-related and the top management of the department estimated using the time series of data in the last 10 years, that the impact of suppliers selection is crucial.

5 Conclusions and future research

We have studied by means of numerical simulations the behavior of the fuzzy model of consensual dynamics, in which the individual preferences are represented by triangular fuzzy numbers. A selection of these simulations is presented in section 4. The computer simulations provide clear evidence that the fuzzy consensual dynamics model exhibits interesting non standard opinion changing behavior in relation to the crisp version of the model. Future research should at first explore more general cases, in order to clarify the role of the various parameters involved in the consensual dynamics, and to demonstrate the potential of the methodology as an effective support for the modelling of consensus in multicriteria and multi-expert decision making. Secondly, the case study will be further improved through the development of a group decision support module to be embedded in the distributed healthcare supply chain system.

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