The Economic Production Quantity Problem with a Finite Production Rate and Fuzzy Cycle Time

Kaj-Mikael Björk
IAMSR, CIEM / Åbo Akademi University
Joukahainengatan 3-5A, FIN-20520 ÅBO
Finland
Email: kaj-mikael.bjork@abo.fi

Abstract

Managing the inventories along with carrying out the production program is essential for many companies in the producing industry. In this paper, a fuzzy EPQ (Economic Production Quantity) model is developed to address this specific problem as a theoretical study. However, this problem derives from some real world applications, in which the producing company in a supply chain had to decide the size of the production batches under uncertainty. The uncertainty will be handled with fuzzy numbers and we will find an analytical solution to the optimization problem. The paper concludes with a small example to illustrate the analytical results.

1. Introduction

The earliest models of batch-production were derived from the basic EOQ (Economic Order Quantity) model in the early 20th century. During this time, mathematical methods started emerging to optimize the size of the inventory and the orders [1] and since then, there have been an increasing number of contributions that complement the basic model in different ways. One of them is the extension of finite production rate. The EOQ-models are most often used in a continuous-review setting and it is assumed that the inventory can be monitored every moment in time.

The importance of the production aspects in supply chain management cannot be ignored in many industry applications. One such application is found within paper making industry and more general, the process industry. In order to obtain a good solution to the entire supply chain, the individual production efficiency aspects need to be addressed [2].

The applications for the EPQ- and EOQ-formulas and its variants are huge. A specific application that inspired the author to conduct this research is found in the fine paper supply chains in the Nordic countries. These chains consist of a few large paper producing companies and quite many distributors that operate independently from the producers. The production decisions for these producers are done under uncertainties that often don't allow themselves to be captured by probabilistic measures, c.f. [3] and [4]. The cycle time (or equivalently the production batch size) is often not guaranteed since the production does not stop exactly at the desired batch size. Due to uncertain market conditions and a tough competition (there has been an overcapacity on the European fine paper market during the last years, [2]) the management might increase the batch size (to produce more to stock). There are also other reasons for the uncertainty in the batch size (which results in uncertainties in the cycle times). Therefore, models to take these uncertainties into account have been created and this paper contributes to the research track of fuzzy EPQ-models.

Decision making in real life situations is often uncertain. If the uncertainty is insignificant, it may be possible to use some classic EPQ- or EOQ-formula. The uncertainties today are often significant and thus the models need to account for them. Sometimes the uncertainties can be modeled stochastically (as done in [5]), but quite often, they cannot be captured with probabilistic means, but only from expert opinions within the companies. This is typically the case with new products, and products with very large seasonal and other unknown variations. For these kinds of uncertainties it is possible to use fuzzy numbers instead of probabilistic approaches, [6] and [7]. There are many contributions within this field, for instance, [8], who worked out fuzzy modifications of the model of [9], which took the defective rate of the goods into account. [10] solved numerically an EOQ with fuzzy backorder quantities and [11] solved it with a fuzzy order quantity. [12] and [13] solved an EOQ-model with the lead times as decision variables as well as the order quantities. [14] introduced an EOQ-model, without backorders, but for two replaceable merchandizes. [15] used the signed distance method for a fuzzy demand EOQ-model without backorders.

The most similar results in the literature to the results in this paper are found in [16], where an EPQ-model was worked out, where the demand and the production quantity were allowed to be fuzzy (with triangular fuzzy numbers). They used, however, a numerical optimization method to solve the problem. This paper addresses a similar EPQ-model, where the demand is crisp but the cycle time is kept fuzzy, and as a positive trade-off, an analytical solution is found to the optimization problem. The analytical solution can be found under the assumption of symmetrical triangular fuzzy numbers (describing the cycle time) and with a defuzzification of the objective function before the optimization process begins.

The paper is organized as follows: First the crisp model is presented. Then the fuzzy model will be presented and defuzzified in a similar manner as in [8]. The defuzzification will be performed with the signed distance method so that the analytical solution can be obtained from the first order derivative (since the objective function is proven to be convex). Finally a small example is given and the paper is concluded with a discussion.

2. The crisp EPQ model with a finite production rate

The classical EPQ problem formulation consists of one decision variable, the size of the production batch. The variable can be exchanged to the maximum amount of inventory there will be (directly after the production has stopped, c.f. variable q in Figure 1), or the cycle time. Under the case with no uncertainty, the inventory will undergo seesaw behaviour, c.f. Figure 1.

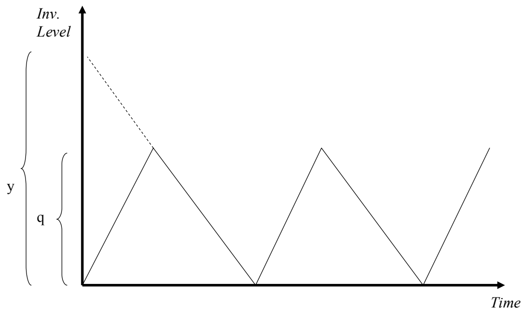

Figure 1. The representation of the EPQ model with a finite production rate.

The parameters and variables in the classical EPQ model are the following:

y is the production batch size (variable)
K is the fixed cost per production batch (parameter)
D is the annual demand of the product (parameter)
q is the maximum difference in the inventory level (variable)
R is the annual production rate (parameter)
h is the unit holding cost per year (parameter)
T is the cycle time (variable)

The total cost function TCU is given by (according to the basic result in the EPQ-theory)

$$TCU(y,q) = \frac{KD}{y} + \frac{hq}{2} \tag{1}$$

In addition, the EPQ-theory will give the following relationship between the variables y, q and the cycle time T.

$$\begin{aligned} q &= y\frac{R-D}{R} \\ T &= y/D \end{aligned} \tag{2}$$

The insertion of (2) into (1) yields the total cost function to minimize (in only one variable)

$$TCU(y) = \frac{KD}{y} + \frac{h(R-D)y}{2R} \tag{3}$$

The decision variable y can also be exchanged with the cycle time T according the second formula in Eq. (2). This will yield the following results

$$TCU(T) = \frac{K}{T} + \frac{hDT}{2} - \frac{hD^2T}{2R} \tag{4}$$

All parameters and variables can be assumed to be strictly greater than 0. Eq (4) are a version of the crisp (classical) EPQ-model and may be solved by using the derivatives. This is possible since all the terms in Eq. (4) are convex. The optimal solution will be a result from

$$\frac{dTCU}{dT} = -\frac{K}{T^2} + \frac{hD}{2} - \frac{hD^2}{2R} = 0 \tag{5}$$

Classical results give us (from Eq. 5) that the optimal order quantity is

$$T^* = \sqrt{\frac{2KR}{hDR - hD^2}} \tag{6}$$

3. The EPQ Model with a finite production rate and a Fuzzy Cycle Time

Many managers today need to make decision under uncertainties that are often inherent fuzzy. Therefore, there is a need of a set of EPQ-models in order to capture the fuzzy uncertainties accurately. In this section, a fuzzy EPQ-model is presented as well as its analytical solution. Let us assume that the cycle time is uncertain but it is possible to describe it with a triangular fuzzy number (symmetric). The cycle time T will then be

$$\tilde{T} = (T - \Delta, T, T + \Delta) \quad (7)$$

The EPQ model with fuzzy cycle times will have a cost function given by

$$T\tilde{C}U(\tilde{T}) = \frac{K}{\tilde{T}} + \frac{hD\tilde{T}}{2} - \frac{hD^2\tilde{T}}{2R}. \quad (8)$$

In order to defuzzify the cost function, the signed distances need to be defined. The signed distance of TCU and 0 is given by

$$d(T\tilde{C}U, \tilde{0}) = K \cdot d(1/\tilde{T}, \tilde{0}) + \frac{hD \cdot d(\tilde{T}, \tilde{0})}{2} - \frac{hD^2 \cdot d(\tilde{T}, \tilde{0})}{2R} \quad , \quad (9)$$

where, according to [8] and Eq. (A6) in the Appendix we find that

$$d(\tilde{T}, \tilde{0}) = T + \frac{1}{4}\Delta - \frac{1}{4}\Delta = T \quad (10)$$

and in a similar manner to [8] and Eq. (A5) in the Appendix we find that we can calculate

$$\begin{aligned} d(1/\tilde{T}, \tilde{0}) &= \\ &\frac{1}{2}\int_0^1 \left[(1/T)_L(\alpha) + (1/T)_U(\alpha)\right] d\alpha \\ &= \frac{1}{2}\int_0^1 \left[\frac{1}{T + \Delta - \Delta\alpha} + \frac{1}{T - \Delta + \Delta\alpha}\right] d\alpha \\ &= \frac{1}{2}\left[\begin{array}{l} -\frac{1}{\Delta}\ln T + \frac{1}{\Delta}\ln(T + \Delta) + \frac{1}{\Delta}\ln T \\ -\frac{1}{\Delta}\ln(T - \Delta) \end{array}\right] \\ &= \frac{1}{2\Delta}\ln\frac{T + \Delta}{T - \Delta} \end{aligned} \quad (11)$$

Inserting Eq. (10) and (11) into (9) yields the defuzzified total cost function

$$TCU(T) \equiv d(T\tilde{C}U, \tilde{0}) = \frac{K}{2\Delta}\ln\frac{T + \Delta}{T - \Delta} + \frac{hDT}{2} - \frac{hD^2T}{2R} \quad (12)$$

The defuzzified objective function need to be examined for convexity in order to be able to solve with derivatives. Since the latter two terms in Eq. (12) are linear (and hence convex) the only term that needs to be examined is the first one. For the examination of convexity, the derivatives need to be computed (first and second grade). Also since a positive constant does not affect the convexity, we will examine the following function:

$$f(T) = \ln\frac{T + \Delta}{T - \Delta} = \ln(T + \Delta) - \ln(T - \Delta) \quad (13)$$

Then,

$$f' = \frac{1}{T + \Delta} - \frac{1}{T - \Delta} \quad (14)$$

$$\begin{aligned} f'' &= -\frac{1}{(T + \Delta)^2} + \frac{1}{(T - \Delta)^2} \\ &= -\frac{1}{T^2 + 2\Delta T + \Delta^2} + \frac{1}{T^2 - 2\Delta T + \Delta^2} \end{aligned} \quad (15)$$

In Eq (15), it is evident that the absolute value of the first (negative) term is strictly smaller than the absolute value of second term (the positive one). This implies that the function in Eq. (13) is convex and hence Eq. (12) is convex. The minimum of Eq. (13) can be found by solving the equation resulting from putting the derivative of Eq. (13) equal to zero, i.e.

$$\frac{dTCU}{dT} = \frac{K}{2\Delta(T + \Delta)} - \frac{K}{2\Delta(T - \Delta)} + \frac{hD}{2} - \frac{hD^2}{2R} = 0 \quad . \quad (16)$$

This can be solved by simplification according to

$$\frac{dTCU}{dT} = \frac{K}{2\Delta(T+\Delta)} - \frac{K}{2\Delta(T-\Delta)}$$
$$+\frac{hD}{2} - \frac{hD^2}{2R} = 0$$
$$\Leftrightarrow KR(T-\Delta) - KR(T+\Delta)$$
$$+ hD\Delta R(T+\Delta)(T-\Delta) \qquad (17)$$
$$-hD^2\Delta(T+\Delta)(T-\Delta) = 0$$
$$\Leftrightarrow (hDR\Delta - hD^2\Delta)T^2 - 2KR\Delta$$
$$-hDR\Delta^3 + hD^2\Delta^3 = 0$$
$$\Rightarrow T = \sqrt{\frac{2KR + hDR\Delta^2 - hD^2\Delta^2}{hDR - hD^2}}$$

, and simplified somewhat more we obtain

$$T^* = \sqrt{\frac{2KR}{hDR - hD^2} + \Delta^2} \qquad (18)$$

In Eq. (18) one can see that the first term within the square root is identical to the crisp EOQ-solution, c.f. Eq. (6). But in addition to the crisp solution we will have an addition of Δ^2. It is also interesting to notice that the fuzzy optimal cycle time increases with the uncertainty in the cycle time.

In Eq. (18) it can also be extracted that if there is no uncertainty, i.e. Δ equals 0, then the fuzzy case will collapse into the classical crisp case given by the Eq. (6).

4. Example

In the following, an example to illustrate the model will be given. Let us assume that a producer has an annual demand (D) of a product at 500000 kg. The paper cost 0.75 euro / kg to produce at the rate (R) of 4500000 kg / year. There is a fixed cost incurring at every production run (setup costs, K): 1000 euro. The holding costs are 33 % of the production cost, i.e.: 0.25 euro per kg and annum (the parameter h). The Δ-parameter in the fuzzy case is assumed to be 0.04. Given these parameters the optimization results for this example are given in Table 1.

	T*	y*	q*	TCU*
Crisp	0.134	67082	59628	14907.12
Fuzzy	0.14	70000	62222	15125.11

Table 1. The result from the example calculations.

Note that the results in Table 1 indicate that the order size would increase with 4.35 % if the uncertainties are accounted for in appropriate manner. This would require an increased total cost of 1.46 %.

The result of this example will give some insight in the effect of fuzzy cycle time (equivalently to fuzzy batch sizes) will have on the batch sizes. First of all, the batch sizes should be increased slightly in order to be optimal in the fuzzy environment (or equivalently the cycle time). Secondly, the increase in total costs is somewhat marginal (1.46 % in the example). This implies that it is not very costly to allow a positive attitude to fuzzy batch-sizes, since it will not result in a significant cost increase, if, at the same time, the batch sizes are increased somewhat (in order to be close to the optimum in the fuzzy sense).

In order to make a stronger foundation for the analysis above, a more comprehensive sensitivity analysis is made for several parameters. The analysis is found in Tables 2-5. These tables are all showing the increase (in percent) from the crisp case to the fuzzy case in the optimal cycle time (inc T*) and optimal total costs (inc TCU*). In Table 2, the sensitivity of the Δ-

parameter is analyzed (in the table, the "*"-sign denotes the base case scenario value). If the Δ value is small (for instance 0.005, representing a possible alternation in the cycle time of at most 1.8 days in each direction) the increase in the cycle time as well as total costs are practically zero. However, for large Δ-values (0.1, representing 36.5 days of possible alternation) gives significant values. Still, the increase in total costs is only 8.61 % even for this very extreme case.

delta	inc T*	inc TCU*
0.005	0.00 %	0.02 %
0.01	0.75 %	0.09 %
0.02	1.49 %	0.37 %
0.03	2.24 %	0.83 %
*0.04	4.48 %	1.46 %
0.05	6.72 %	2.27 %
0.06	9.70 %	3.24 %
0.07	12.69 %	4.37 %
0.08	16.42 %	5.64 %
0.09	20.90 %	7.06 %
0.1	24.63 %	8.61 %

Table 2. The sensitivity analysis of the Δ-parameter.

The second parameter of investigation is the fixed cost, *K*. Firstly, from Table 3, we can see that with a higher the fixed cost, we will have smaller effect in the increase of both total costs as well as the cycle time because of the fuzziness introduced. Also the results in Table 3 support the conclusion drawn earlier in this section.

K	inc T*	inc TCU*
500	8.42 %	2.89 %
*1000	4.48 %	1.46 %
1500	3.05 %	0.98 %
2000	2.11 %	0.74 %

Table 3. The sensitivity analysis of the Δ-parameter.

Table 4 analyzes the sensitivity of the demand parameter in the base case example. Increasing the demand will increase the total costs, but not significantly. In fact, doubling the demand will increase the incremental total costs from 1.46 % to 2.54 %. This increase is considered quite modest.

D	inc T*	inc TCU*
100000	1.05 %	0.32 %
250000	2.17 %	0.78 %
*500000	4.48 %	1.46 %
750000	6.19 %	2.05 %
1000000	7.92 %	2.54 %

Table 4. The sensitivity analysis of the Δ-parameter.

Finally, in Table 5, the important parameter *h* (the holding costs) will increase the incremental total costs slightly more than the effect *D*. For instance, doubling the holding costs will increase the incremental total costs from 1.46 % to 2.89 %.

h	inc T*	inc TCU*
0.1	1.89 %	0.59 %
*0.25	4.48 %	1.46 %
0.5	8.42 %	2.89 %
0.75	12.99 %	4.28 %
1	16.42 %	5.64 %

Table 5. The sensitivity analysis of the Δ-parameter.

The analysis found in Table 2-5 will support the conclusion that it will not be costly to have a positive attitude towards fuzzy batch sizes (or equivalently, fuzzy cycle times). Even in the extreme case of having the Δ-value to be 0.1 (representing 36.5 days), we will only increase the incremental total costs from 1.46 % to 8.61 % and all other extreme cases in the analysis (Tables 3-5) are significantly less.

5. Discussion and Further Research

Since the decision makers today need to operate under uncertainties that are often inherent fuzzy, the EPQ-models need to be able to capture it accurately. This paper contributes to the fuzzy EPQ- and EOQ theory by providing an analytical solution to a known problem. The problem under investigation is the EPQ-problem with a finite production rate, but uncertain cycle time (or consequently, uncertain production batches). The choice to keep the fuzzy numbers triangular is motivated by the desire to obtain the analytical solution for the problem under study. Previously this problem was solved with triangular fuzzy numbers both for the batch size and demand (in [16]), but solved numerically, while keeping the problem fuzzy during the optimization process.

This paper provides an analytical solution to the case where the cycle time is allowed to be symmetrical triangular fuzzy numbers. This is possible since the defuzzified objective function is strictly convex and the optimal solution can be obtained through the solution of an equation given by the derivative that equals zero. The optimal value was obtained and the solution was given for an illustrative example. This example showed that the uncertainties in the cycle times affect the optimal production batch size in such a way that it should be approx. 4 % higher (in the illustrative example) than in the crisp case. In addition, the increase in the total costs when we move from the optimum in the crisp case to the optimum in the fuzzy case is somewhat marginal. This implies that the fuzzy batch-sizes will not come with a high increase in the total costs, but on the contrary, it is such a marginal increase that the management can give the production planning some degrees of freedom to move away from the crisp optimum. This flexibility may be very useful in the turbulent and dynamic environment many process industries dwell within.

Future research includes the investigation of a model that should be extended to cover more membership functions than the symmetrical triangular one. Also the difference between different defuzzification methods should be investigated within these settings. Finally a complete sensitivity analysis of both the cost of fuzzy cycle times (which has been investigated to some extent in this paper) and the savings such flexibility can impose would need to be done. This track of further research is important in order to give the industry a more complete picture of the actual implications of fuzzy batch sizes.

6. Acknowledgement

The financial support from the Technology Agency in Finland (Tekes) through the SmartOpt project is gratefully acknowledged.

7. References

[1] Harris F.W., 1913. How many parts to make at once. *The Magazine of Management*. **10**, pp. 135-136.

[2] Björk K-M, Carlsson C., 2007. The Effect of Flexible Lead Times on a Paper Producer. *International Journal of Production Economics*, **107**, No. 1, pp. 139-150

[3] Björk K-M, Carlsson C., 2005. The Outcome of Imprecise Lead Times on the Distributors. *Hicss*, p. 81-90, *Proceedings of the 38th Annual Hawaii International Conference on System Sciences (HICSS'05) - Track 3*.

[4] Carlsson C. & Fuller R., 1999. Soft computing and the Bullwhip effect. *Economics and Complexity*. **2**, No 3, p. 1-26.

[5] Liberatore, M.J., 1979.The EOQ model under stochastic lead time. *Operations Research*, **27**, pp. 391-396.

[6] Zadeh L.A. 1965. Fuzzy Sets. *Information and Control* **8**, p 338-353

[7] Zadeh L. 1973. Outline of a new Approach to the Analysis of Complex Systems and Decision Processes. *IEEE Transactions on Systems, Man and Cybernetics.* **Vol SMC-3**, pp. 28-44.

[8] Chang H-C., 2004. An application of fuzzy sets theory to the EOQ model with imperfect quality items. *Computers & Operations Research*, **31**, pp. 2079-2092

[9] Salameh M.K. and Jaber M.Y., 2000. Economic production quantity model for items with imperfect quality. *Int. Journal of Production Economics*, **64**, pp.59-64.

[10] Chang S-C.,. Yao J-S and Lee H-M., 1998. Economic reorder point for fuzzy backorder quantity. *European Journal of Operational Research*, **109**, pp. 183-202.

[11] Yao J-S. and Lee H-M., 1996. Fuzzy inventory with backorder for fuzzy order quantity. *Information Sciences*, **93**, pp. 283-319.

[12] Ouyang L-Y. and Wu K-S., 1998. A minimax distribution free procedure for mixed inventory model with variable lead time, *Int. Journal of Production Economics*, **56-57**, pp. 511-516.

[13] Ouyang L-Y. and Yao J-S., 2002. A minimax distribution free procedure for mixed inventory model involving variable lead time with fuzzy demand, *Computers & Operations Research,* **29**, pp. 471-487.

[14] Yao J-S., Ouyang L-Y. and Chiang J., 2003. Models for a fuzzy inventory of two replaceable merchandises without backorders based on the signed distance of fuzzy sets. *European Journal of Operational Research*, **150**, pp. 601-616.

[15] Yao J-S and Chiang J., 2003. Inventory without backorder with fuzzy total cost and fuzzy storing cost defuzzified by centroid and signed distance. *European Journal of Operational Research*, **148**, pp. 401-409.

[16] Lee H-M. & Yao J-S., 1998. Economic production quantity for fuzzy demand quantity and fuzzy production quantity. *European Journal of Operational Research*. **109**, pp 203-211

[17] Yao J.S. and Wu K., 2000. Ranking fuzzy numbers based on decomposition principle and signed distance, *Fuzzy Sets and Systems*, **116**, pp. 275-288.

[18] Kaufmann A. and Gupta M.M., 1991. *Introduction to fuzzy arithmetic: theory and applications*, New York: Van Nostrand, Reinhold.

Appendix

In this appendix the basics of fuzzy numbers as well as the signed distance method is given in order to make the modeling effort self-contained.

Definition 1. Consider the fuzzy set $\tilde{A} = (a, b, c)$ where $a < b < c$ and defined on R, which is called a triangular fuzzy number, if the membership function of $\tilde{A}$ is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} (x-a)/(b-a), & a \le x \le b, \\ (c-x)/(c-b), & b \le x \le c \\ 0, & otherwise. \end{cases} \qquad \text{(A1)}$$

Definition 2. Let $\tilde{B}$ be a fuzzy set on R and $0 \le \alpha \le 1$. The α-cut of $\tilde{B}$ is all the points x such that $\mu_{\tilde{B}}(x) \ge \alpha$, i.e.

$$B(\alpha) = \{x | \mu_{\tilde{B}}(x) \ge \alpha\} \qquad \text{(A2)}$$

The interval operations are as follows according to [18]. For any $a, b, c, d, k \in R \text{ and } a, c > 0$

(i) $[a,b](+)[c,d] = [a+c, b+d]$

(ii) $[a,b](-)[c,d] = [a-c, b-d]$

(iii) $k(\cdot)[c,d] = \begin{cases} [kc, kd], & k > 0 \\ [kd, kc], & k < 0 \end{cases}$ (A3)

(iv) $[a,b](\cdot)[c,d] = [a \cdot c, b \cdot d]$

(v) $[a,b](\div)[c,d] = \left[\frac{a}{d}, \frac{b}{c}\right]$

In order to find non-fuzzy values for the model, we need to use some distance measures, and as in [8] we will use the signed distance [17].

Defininition 3. For any a and $0 \in R$, the signed distance from a to 0 is $d_0(a,0) = a$. And if $a < 0$, the distance from a to 0 is $-a = -d_0(a,0)$.

Let Ω be the family of all fuzzy sets $\widetilde{B}$ defined on R for which the α-cut $B(\alpha) = [B_L(\alpha), B_U(\alpha)]$ exists for every $\alpha \in [0,1]$, and both $B_L(\alpha)$ and $B_U(\alpha)$ are continuous functions on $\alpha \in [0,1]$. Then, for any $\widetilde{B} \in \Omega$, we have (c.f. [8])

$$\widetilde{B} = \bigcup_{0 \le \alpha \le 1} [B_L(\alpha)_\alpha, B_U(\alpha)_\alpha] \qquad \text{(A4)}$$

From [8] it can be finally stated (originally by results from [17]) how to calculate the signed distances.

Definition 4. For $\widetilde{B} \in \Omega$ define the signed distance of $\widetilde{B}$ to $\widetilde{0}_1$ as

$$d(\widetilde{B}, \widetilde{0}_1) = \frac{1}{2} \int_0^1 [B_L(\alpha) + B_U(\alpha)] d\alpha. \qquad \text{(A5)}$$

The Definition 4 will give us several properties of which the most important is

Property 1. Consider the triangular fuzzy number $\widetilde{A} = (a,b,c)$: the α-cut of $\widetilde{A}$ is $A(\alpha) = [A_L(\alpha), A_U(\alpha)]$, for $\alpha \in [0,1]$, where $A_L(\alpha) = a + (b-a)\alpha$ and $A_U(\alpha) = c - (c-b)\alpha$, the signed distance of $\widetilde{A}$ to $\widetilde{0}_1$ is

$$d(\widetilde{A}, \widetilde{0}_1) = \frac{1}{4}(a + 2b + c). \qquad \text{(A6)}$$