Strategic Investment Planning by Using Dynamic Decision Trees

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Abstract

In this paper we shall represent strategic planning problems by dynamic decision trees, in which the nodes are projects that can be deferred or postponed for a certain period of time. Using the theory of real options an enhanced option pricing method is provided, where the relationship of investments is taken into consideration as well, and the optimal path of the decision tree, i.e. the path with the biggest real option value in the end of the planning period will be identified. The theory developed implicitly contains not only the deferral or cancellation flexibility of projects but also the possibility of considering vague informations, which needs to be taken into account when (long-time range) financial decisions are made.

1 Introduction

In this paper we shall introduce heuristic decision support methods evaluating investment opportunities, where the expected cash flows and expected costs are estimated by possibility distributions. We will see that our methods are generalizations of already existing methods, moreover, with them we can take into consideration vague informations, which should be regarded when long-time range investment decisions are made. We shall provide set up of a theory using possibilistic approach to evaluate certain types of projects. In section 2 we shall introduce the notion of fuzzy numbers and some basic properties of possibility theory. In that chapter two operators defined on fuzzy numbers will be introduced, which will play the key role in our set up. In section 3 we consider unflexible, i.e. now or never type projects, and introduce the valuation method Fuzzy Net Present Value. In section 4 we consider investment opportunities with deferral flexibility, and introduce the method Fuzzy Real Option Valuation. In section 5 using the results obtained in the previous sections we represent strategic planning problems by dynamic decison trees, in which the nodes are projects that can be deferred or postponed for a certain period of time. Using the theory of net present value analysis and real options we shall be able to identify the optimal path of the tree, i.e. the path with the biggest real option value in the end of the planning period.

This theory has been invented to help evaluating investment activities of several kind, and we have been testing it in a few existing very large industrial investments in Finnish industrial corporations. The goal of the investment analysis based on this theory is to define a concept and a methodology for planning and evaluation of very large investments (so-called giga-investments).

There exists a developed Excel-based platform, in which all of this theory has been implemented.

2 Possibilistic mean value and variance of fuzzy numbers

A fuzzy number A is a fuzzy set of the real line $\mathbb{R}$ with a normal, convex and continuous membership function of bounded support. The family of fuzzy numbers will be denoted by $\mathcal{F}$. A γ-level set of a fuzzy number A is defined by $[A]^\gamma = \{t \in \mathbb{R} | A(t) \geq t\}$ if $\gamma > 0$ and $[A]^\gamma = \text{cl}\{t \in \mathbb{R} | A(t) > 0\}$ (the closure of the support of A) if $\gamma = 0$. It is well known that if A is a fuzzy number then $[A]^\gamma$ is a compact and convex subset of $\mathbb{R}$ for all $\gamma \in [0, 1]$.

For any $A \in \mathcal{F}$ we shall use the notation $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ for γ-level sets of A. Fuzzy numbers can also be considered as possibility distributions, more precisely, they define a special class of possibility distributions [6]. If $A \in \mathcal{F}$ is a fuzzy number and $x \in \mathbb{R}$ a real number then $A(x)$ can be interpreted as the degree of possibility of the statement "x is the value of A".

Definition 2.1 *A fuzzy number $A \in \mathcal{F}$ is called trapezoidal fuzzy number with core $[a, b]$, left width α and right width β if its γ-level set has the form*

$$[A]^\gamma = [a - (1-\gamma)\alpha, b + (1-\gamma)\beta]$$

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for all $\gamma \in [0,1]$, and it can easily be shown that membership function of A has the following form

$$A(t) = \begin{cases} 1 - \dfrac{a-t}{\alpha} & \text{if } a-\alpha \le t < a, \\ 1 & \text{if } a \le t \le b, \\ 1 - \dfrac{t-b}{\beta} & \text{if } b < t \le b+\beta, \\ 0 & \text{otherwise,} \end{cases}$$

and we use the notation $A = (a, b, \alpha, \beta)$.

The values a and b are the lower and upper modal value, respectively, and the support of A is interval $(a-\alpha, b+\beta)$.

A trapezoidal fuzzy number with core $[a, b]$ can be seen as a context-dependent description of the property "the value of a real variable is approximately in the interval $[a, b]$", where left width α and right width β can define the context. In possibilistic setting $A(t)$, $t \in \mathbb{R}$ can be interpreted as the degree of possibility of the statement "t is in the interval $[a, b]$". We can see that if $t \in [a, b]$ then $A(t) = 1$ (i.e. t belongs to A with degree of membership one), and if $t \notin (a-\alpha, b+\beta)$ (i.e. t is *too far* from $[a, b]$) then $A(t) = 0$, and otherwise $0 < A(t) < 1$ (i.e. t is *close enough* to $[a, b]$).

Now let us recall some basic properties of fuzzy numbers. Let A and B be fuzzy numbers with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ and $[B]^\gamma = [b_1(\gamma), b_2(\gamma)]$, $\gamma \in [0,1]$, respectively, and let $\lambda \in \mathbb{R}$ be a real number. Using the sup–min extension principle [11] we can verify that the following properties hold for addition and multiplication by a scalar of fuzzy numbers

$$[A+B]^\gamma = [a_1(\gamma)+b_1(\gamma), a_2(\gamma)+b_2(\gamma)] \tag{1}$$

and

$$[\lambda A]^\gamma = \begin{cases} [\lambda a_1(\gamma), \lambda a_2(\gamma)] & \text{if } \lambda \ge 0, \\ [\lambda a_2(\gamma), \lambda a_1(\gamma)] & \text{if } \lambda < 0. \end{cases} \tag{2}$$

In the following we introduce two measures on fuzzy numbers, which will play important roles in our setting. They can be considered as the first and second momentums on possibility distributions, which correspond to the ones known from probability theory.

The possibilistic mean value has been introduced in [5] and has been generalized in [7] and is the following.

Definition 2.2 *Let $A \in \mathcal{F}$ be a fuzzy number with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0,1]$. The weighted possibilistic mean value of A is*

$$\mathrm{E}_f(A) = \int_0^1 \frac{a_1(\gamma)+a_2(\gamma)}{2} f(\gamma) d\gamma,$$

where weighting function f is non-decreasing and satisfies

$$\int_0^1 f(\gamma) d\gamma = 1,$$

i.e. $\mathrm{E}_f(A)$ is the f-weighted average of the arithmetic means of all γ-level sets, that is, the weight of the arithmetic mean of $a_1(\gamma)$ and $a_2(\gamma)$ is just $f(\gamma)$.

Setting $f(\gamma) = 2\gamma$ we obtain the possibilistic mean value of A

$$\mathrm{E}(A) = \int_0^1 [a_1(\gamma) + a_2(\gamma)] \gamma d\gamma,$$

which is nothing else but the pure level-weighted average of the arithmetic means of all γ-level sets, that is, now the weight of the arithmetic mean of $a_1(\gamma)$ and $a_2(\gamma)$ is just 2γ.

It can easily be proven that $\mathrm{E}_f : \mathcal{F} \to \mathbb{R}$ is a linear function (with respect to operations defined by (1) and (2)).

The possibilistic variance has been introduced in [5] and has been generalized in [7] and is the following.

Definition 2.3 *Let $A \in \mathcal{F}$ be a fuzzy number with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0,1]$. The weighted possibilistic variance of A is*

$$\begin{aligned} \mathrm{Var}_f(A) &= \int_0^1 \left(\frac{a_2(\gamma) - a_1(\gamma)}{2} \right)^2 f(\gamma) d\gamma \\ &= \int_0^1 \frac{1}{2} \left(\left[\frac{a_1(\gamma)+a_2(\gamma)}{2} - a_1(\gamma) \right]^2 \right. \\ &\quad \left. + \left[a_2(\gamma) - \frac{a_1(\gamma)+a_2(\gamma)}{2} \right]^2 \right) f(\gamma) d\gamma \end{aligned}$$

where weighting function f is non-decreasing and satisfies

$$\int_0^1 f(\gamma) d\gamma = 1,$$

i.e. the weighted possibilistic variance of A is defined as the weighted mean value of the arithmetic mean of the squared deviations between the arithmetic mean and the endpoints of its level sets.

Setting $f(\gamma) = 2\gamma$ we obtain the possibilistic variance of A

$$\begin{aligned} \mathrm{Var}(A) &= \frac{1}{2} \int_0^1 [a_2(\gamma) - a_1(\gamma)]^2 \gamma d\gamma \\ &= \int_0^1 \left(\left[\frac{a_1(\gamma)+a_2(\gamma)}{2} - a_1(\gamma) \right]^2 \right. \\ &\quad \left. + \left[a_2(\gamma) - \frac{a_1(\gamma)+a_2(\gamma)}{2} \right]^2 \right) \gamma d\gamma \end{aligned}$$

which is the mean value of the squared deviations between the arithmetic mean and the endpoints of the γ-level sets of A.

It can easily be verified that if $A = (a, b, \alpha, \beta)$ is a trapezoidal fuzzy number then

$$\mathrm{E}(A) = \int_0^1 [a - (1-\gamma)\alpha + b + (1-\gamma)\beta]\gamma d\gamma$$
$$= \frac{a+b}{2} + \frac{\beta - \alpha}{6}$$

and

$$\mathrm{Var}(A) = \frac{1}{2}\int_0^1 [b + (1-\gamma)\beta - a + (1-\gamma)\alpha]^2 \gamma d\gamma$$
$$= \left(\frac{b-a}{2} + \frac{\alpha+\beta}{6}\right)^2 + \frac{(\alpha+\beta)^2}{72}$$

hold.

3 Fuzzy Net Present Value Analysis

In traditional investment approaches investment activities or projects are often seen as *now or never*, which means that the main question is whether to go ahead with an investment *yes or no* [2]. Formulated in this way it is very hard to make a decision when there is uncertainty about the exact outcome of the investment. In order to help these tough decisions valuation methods as *Net Present Value* (NPV) or *Discounted Cash Flow* (DCF) have been developed.

Considering NPV analysis we can evaluate an investment opportunity, in which we may enter immediately or have to abandon it. We can compute the exact value of that project according to our forecast and make a decision whether worthwhile to enter into it or not. The data needed are the expected cash flows for each year of the project V_i, the investment costs X, and the required rate of return on the investment r, which is so-called the project's beta. The expected annual cash flows gives the operational annual profit, which actually are the difference between operational revenues and operational costs at a certain year of the project, and which quantities are aggregated according to the discounting parameter beta of the investment

$$S_0 = \sum_{i=0}^{L} \frac{V_i}{(1+r)^i},$$

where L denotes the length of the investment activity. The discounting parameter r implicitly contains the degree of the investor's or decision maker's risk aversion. Setting r higher means that the underlying investment is thought more risky, because higher future cash flows are needed to reach the same aggregated income. The investment costs X is a one-time cost, which should be paid at the beginning of the project to be able to enter into it. The value of the underlying investment is

$$NPV = S_0 - X = \sum_{i=0}^{L} \frac{V_i}{(1+r)^i} - X,$$

and the decision rule is obvious, i.e. if $NPV > 0$ then we should enter the investment, otherwise we have to abandon it.

Usually, the expected cash flows and expected investment costs cannot be characterized by single numbers. We can, however, estimate these quantities by using possibility distributions, i.e. fuzzy numbers. In the following we shall only consider trapezoidal possibility distributions, although the possibility distributions of expected cash flows and expected investment costs could also be represented by nonlinear (e.g. Gaussian) membership functions. However, from computational point of view it is easier to use linear membership functions and, more importantly, our experience shows that senior managers prefer trapezoidal fuzzy numbers to Gaussian ones when they estimate the uncertainties associated with future cash inflows and outflows.

Hence, in each year we shall estimate the expected cash flows by using trapezoidal fuzzy number of the form

$$V_i = (a_i, b_i, \alpha_i, \beta_i)$$

for $i = 0, 1, \ldots, L$, i.e. the most possible values of the expected cash flows at year i of the project lie in the interval $[a_i, b_i]$ (which is the core of the trapezoidal fuzzy number V_i), and $(a_i - \alpha_i)$ is the downward potential and $(b_i + \beta_i)$ is the upward potential for the expected cash flows at year i.

In a similar manner we can estimate the expected investment costs by using a trapezoidal possibility distribution of the form

$$X = (x_1, x_2, \alpha', \beta'),$$

i.e. the most possible values of the expected investment costs lie in the interval $[x_1, x_2]$ (which is the core of the trapezoidal fuzzy number X), and $(x_1 - \alpha')$ is the downward potential and $(x_2 + \beta')$ is the upward potential for the expected investment costs.

Definition 3.1 *In these circumstances we suggest the use of the following (heuristic) formula for computing fuzzy net present values*

$$FNPV = S_0 - X = \sum_{i=0}^{L} \frac{V_i}{(1+r)^i} - X, \qquad (3)$$

where the addition and multiplication by a scalar of fuzzy numbers is defined by the sup–min extension principle, i.e. we calculate with them using operations defined by (1) and (2).

Expanding (3) we have

$$\begin{aligned} FNPV &= \sum_{i=0}^{L} \frac{(a_i, b_i, \alpha_i, \beta_i)}{(1+r)^i} + (-x_2, -x_1, \beta', \alpha') \\ &= \left(\sum_{i=0}^{L} \frac{a_i}{(1+r)^i} - x_2, \sum_{i=0}^{L} \frac{b_i}{(1+r)^i} - x_1, \right. \\ &\quad \left. \sum_{i=0}^{L} \frac{\alpha_i}{(1+r)^i} + \beta', \sum_{i=0}^{L} \frac{\beta_i}{(1+r)^i} + \alpha' \right). \end{aligned}$$

Notice, that if every possibility distribution has support containing only one point then it will be a real number, and we get back the original method NPV analysis. Hence, this method implicitly includes the original NPV method, i.e. it is a generalization of that. We emphasize that the model just obtained is a pure possibilistic approach, since we only regard possibility distributions, which indeed make the original method be more enhanced and general.

We have to make a decision whether we should enter the underlying project or we have to abandon it, therefore we need to decide if the trapezoidal possibility distribution $FNPV$ is "greater than zero" or not. However, to define whether a fuzzy number is "greater than zero" is generally impossible (even if it is a trapezoidal one), since there is no ranking on the set of trapezoidal fuzzy numbers.

In our computerized implementation we have employed some value functions to rank fuzzy net present values. For example, having $FNPV = (c_1, c_2, \alpha, \beta)$ we can consider the following utility function

$$u(FNPV) = \frac{c_1 + c_2}{2} + r_A \frac{\beta - \alpha}{6},$$

where $r_A \geq 0$ denotes the degree of the investor's risk aversion. If $r_A = 1$ then the investor compares trapezoidal fuzzy numbers by comparing their possibilistic mean values.

Notice, that in this approach the degree of the investor's risk aversion is taken into consideration in two different ways. It is implicitly included in parameter risk-adjusted discount rate r and is related to future expected payoff, and it is considered when the final decision, i.e. go into the investment yes or no, is made, which is related to rank uncertainties associated with future cash inflows and outflows.

4 Fuzzy Real Option Valuation

In this section we shall consider investment activities, which can be deferred or postponed for a certain period of time. These kind of investment opportunities can be considered as real options, and to have a real option means to have the possibility for a certain period to either choose for or against something, without binding oneself up front.

Real options is an important way of thinking about valuation and strategic decision-making, and the power of this approach is starting to change the economic "equation" of many industries.

Real options in option thinking are based on the same principles as financial options. In real options, the options involve "real" assets as opposed to financial ones [1]. As has already been mentioned, to have a "real option" means to have the possibility for a certain period to either choose for or against something, without binding oneself up front. For example, owning a power plant gives a utility the opportunity, but not the obligation, to produce electricity at some later date.

Real options can be valued using the analogue option theories that have been developed for financial options, which is quite different from traditional net present value or discounted cash flow investment approaches. As has already been described in the previous section, in traditional investment approaches investment activities or projects are considered as now or never, and decision should be made to enter into the underlying investment yes or no [2].

However, only a few projects are now or never. Often it is possible to delay, modify or split up the project in strategic components, which generate important learning effects, and therefore reduce uncertainty. And in those cases option thinking can help [9].

The real option rule is that one shoult invest today only if the net present value is high enough to compensate for giving up the value of the option to wait, i.e. the deferral flexibility. Since the option to invest loses its value when the investment is irreversibly made, this loss is an opportunity cost of investing. Now the main question that an investor or decision maker must answer for a deferrable investment opportunity is: *How long do we postpone the investment, if we can postpone it, up to T time periods?* Considering the probabilistic approach for real option valuation we shall introduce a (heuristic) real option rule in a fuzzy setting to decide when we should enter the project (if it is worthwhile going into that at all).

The new rule, derived from option pricing theory, which was established in 1973 by Black and Scholes [4], and Merton [10], is that you should enter into the project only if the net present value is high enough to compensate for giving up the value of flexibility of the option. Following Leslie and Michaels [8] we shall compute the value of a real option by

$$\text{ROV} = S_0 e^{-\delta T} N(d_1) - X e^{-r_f T} N(d_2)$$

where

$$d_1 = \frac{\ln(S_0/X) + (r_f - \delta + \sigma^2/2)T}{\sigma\sqrt{T}},$$
$$d_2 = \frac{\ln(S_0/X) + (r_f - \delta - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T},$$

and where

ROV	= current real option value,
S_0	= the present value of expected cash flows,
X	= the nominal value of investment costs,
σ	= uncertainty of expected cash flows,
T	= time to maturity of option, in years,
δ	= value lost over duration of the option,
r_f	= the annualized continuously compounded rate on a safe asset,

and $N(d)$ stands for the cumulative normal distribution function, that is, $N(d)$ equals the probability that a random draw from a standard normal distribution will be less than d.

The main question that a firm must answer for a deferrable investment opportunity is: *How long do we postpone the investment up to T time periods?* To answer this question, we refer to Benaroch and Kauffman [3] and consider the following decision rule for optimal investment strategy: Where the maximum deferral time is T, make the investment (exercise the option) at time t^*, $0 \leq t^* \leq T$, for which the value of the option, ROV_{t^*}, is positive and attends its maximum value,

$$ROV_{t^*} = \max_{0\leq t\leq T} ROV_t = S_0^{(t)} e^{-\delta t} N(d_1) - X e^{-r_f t} N(d_2),$$

where $S_0^{(t)}$ denotes the present value of the aggregated expected cash flows belonging to time t, which can be obtained from the appropriate NPV table belonging to the case we enter the project after waiting t years. We should take into consideration that real option valuation is a heuristic method, which only produces a benchmark of the value of the real option considered, therefore we need not to be precise in setting up all the NPV tables. Moreover, we consider large investments, therefore we can assume that our time scale is discrete (and not continuous), and the investment activities can only be deferred whole years.

Hence, we have for the decision rule for optimal investment strategy that if the maximum deferral time is T then we should make the investment at time t^*, $t^* = 0, 1, \ldots, T$, for which the value of the option, ROV_{t^*}, is positive and attends its maximum value,

$$ROV_{t^*} = \max_{t=0,1,\ldots,T} ROV_t = S_0^{(t)} e^{-\delta t} N(d_1) - X e^{-r_f t} N(d_2).$$

However, there can be assumed some relationship between deferring times of the investment and the appropriate NPV tables, which can be the following. The present value of the aggregated expected cash flows belonging to time t, $S_0^{(t)}$, can be the expected cash flows in years $t, t+1, \ldots, L$ aggregated according to the project's beta, that is,

$$S_0^{(t)} = \mathrm{PV}(V_t, V_{t+1}, \ldots, V_L; r) = \sum_{i=t}^{L} \frac{V_i}{(1+r)^i},$$

where, as has been introduced in the previous section, V_i denotes the expected cash flows of the project at year i, $i = 0, 1, \ldots, L$, and r is the risk-adjusted discount rate (or required rate of return on the project, which is usually the project's beta).

Notice, that considering this decision method we implicitly assume that $T \leq L$, i.e. the maximum deferral time of the investment is not greater than the total length of the project.

Of course, this decision rule has to be reapplied every time new information arrives during the deferral period to see how the optimal investment strategy might change in light of the new information.

As in the previous section we can find that the present value of expected cash flows cannot be characterized by a single number. However, we can estimate the present value of expected cash flows belonging to deferral time t by using a trapezoidal possibility distribution of the form

$$S_0^{(t)} = (s_1^{(t)}, s_2^{(t)}, \alpha^{(t)}, \beta^{(t)}),$$

for $t = 0, 1, \ldots, T$, i.e. the most possible values of the present value of expected cash flows belonging to deferral time t lie in the interval $[s_1^{(t)}, s_2^{(t)}]$ (which is the core of the trapezoidal fuzzy number $S_0^{(t)}$), and $(s_1^{(t)} - \alpha^{(t)})$ is the downward potential and $(s_2^{(t)} + \beta^{(t)})$ is the upward potential for the present value of expected cash flows belonging to deferral time t, i.e. the case waiting t years before going ahead with the investment.

In a similar manner we can estimate the expected investment costs by using a trapezoidal possibility distribution of the form

$$X = (x_1, x_2, \alpha', \beta').$$

If the NPV table of expected cash flows belonging to deferral time t are given by using trapezoidal fuzzy numbers then we can compute the value $S_0^{(t)}$ by aggregating them according to the discounting parameter r. Because of properties (1) and (2) the result $S_0^{(t)}$ will obviously be a trapezoidal fuzzy number as well. More precisely, having expected cash flows

$$V_t^{(t)}, V_{t+1}^{(t)}, \ldots, V_L^{(t)},$$

which should be considered when we enter the underlying investment (which is L years long) after waiting t years ($t = 0, 1, \dots, T$, where T is the maximum deferral time of the project), and the risk-adjusted discount rate r we can estimate the present value of the aggregated expected cash flows as

$$S_0^{(t)} = \sum_{i=t}^{L} \frac{V_i^{(t)}}{(1+r)^i}.$$

However, referring again to Benaroch and Kauffman [3] we can consider a simpler aggregation method to derive the present value of expected cash flows by considering only one NPV table with cash flows

$$V_0, V_1, \dots, V_L,$$

and

$$S_0^{(t)} = \sum_{i=t}^{L} \frac{V_i}{(1+r)^i}$$

for $t = 0, 1, \dots, T$.

Definition 4.1 *Let us consider an investment opportunity, which has to be postponed T years, and let the present value of expected cash flows $S_0 = (s_1, s_2, \alpha, \beta)$ and the nominal value of expected costs $X = (x_1, x_2, \alpha', \beta')$ be estimated by using trapezoidal fuzzy numbers. Then, we suggest the use of the following (heuristic) formula for computing fuzzy real option value*

$$FROV = S_0 e^{-\delta T} N(d_1) - X e^{-r_f T} N(d_2), \qquad (4)$$

where

$$d_1 = \frac{\ln\big(\mathrm{E}(S_0)/\mathrm{E}(X)\big) + (r_f - \delta + \sigma^2/2)T}{\sigma\sqrt{T}},$$

$$d_2 = \frac{\ln\big(\mathrm{E}(S_0)/\mathrm{E}(X)\big) + (r_f - \delta - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T},$$

and where $\mathrm{E}(S_0)$ denotes the possibilistic mean value of the present value of expected cash flows, $\mathrm{E}(X)$ stands for the possibilistic mean value of expected fixed costs, and $\sigma = \mathrm{Var}(S_0)$ is the possibilistic variance of the present value of expected cash flows, that is

$$\mathrm{E}(S_0) = \frac{s_1 + s_2}{2} + \frac{\beta - \alpha}{6},$$

$$\mathrm{E}(X) = \frac{x_1 + x_2}{2} + \frac{\beta' - \alpha'}{6},$$

$$\sigma = \left(\frac{s_2 - s_1}{2} + \frac{\alpha + \beta}{6}\right)^2 + \frac{(\alpha + \beta)^2}{72},$$

and where the addition and multiplication by a scalar of fuzzy numbers is defined by the sup–min extension principle, i.e. we calculate with them using operations defined by (1) and (2).

Expanding (4) we have

$$\begin{aligned} FROV &= (s_1, s_2, \alpha, \beta) e^{-\delta T} N(d_1) \\ &\quad + (-x_2, -x_1, \beta', \alpha') e^{-r_f T} N(d_2) \\ &= \Big(s_1 e^{-\delta T} N(d_1) - x_2 e^{-r_f T} N(d_2), \\ &\quad s_2 e^{-\delta T} N(d_1) - x_1 e^{-r_f T} N(d_2), \\ &\quad \alpha e^{-\delta T} N(d_1) + \beta' e^{-r_f T} N(d_2), \\ &\quad \beta e^{-\delta T} N(d_1) + \alpha' e^{-r_f T} N(d_2)\Big). \end{aligned}$$

Notice, that simple substitution with possibility distributions into formula (4) is problematical, since S_0/X in formula of d_1 and d_2 may not be a fuzzy number (as a matter of fact, it is impossible to define quotient of two arbitrary fuzzy numbers).

Obviously, the investment costs having trapezoidal possibility distribution can be obtained from the Fuzzy NPV table as well, and we can see the relationship between Fuzzy NPV Analysis and Fuzzy Real Option Valuation.

In the following we shall generalize the decision rule for optimal investment strategy described above in probabilistic setting (by using real numbers) to fuzzy setting: Where the maximum deferral time is T, make the investment (exercise the option) at time t^*, $0 \le t^* \le T$, for which the value of the fuzzy real option, $FROV_{t^*}$, is positive and attends its maximum value,

$$\begin{aligned} FROV_{t^*} &= \max_{0 \le t \le T} FROV_t \\ &= S_0^{(t)} e^{-\delta t} N(d_1) - X e^{-r_f t} N(d_2), \end{aligned}$$

where $S_0^{(t)}$ denotes the trapezoidal possibility distribution of the present value of the aggregated expected cash flows belonging to time t, which can be obtained from the appropriate Fuzzy NPV table belonging to the case we enter the project after waiting t years, and X stands for the trapezoidal possibility distribution of the expected investment costs, and $\sigma = \mathrm{Var}(S_0^{(t)})$ is the possibilistic variance of the present value of the aggregated expected cash flows belonging to deferral time t.

Assuming that our time scale is discrete (and not continuous) and the investment activities can only be deferred whole years, we have for the decision rule for optimal investment strategy that if the maximum deferral time is T then we should make the investment at time t^*, $t^* = 0, 1, \dots, T$, for which the value of the option, ROV_{t^*}, is positive and attends its maximum value,

$$\begin{aligned} FROV_{t^*} &= \max_{t=0,1,\dots,T} FROV_t \\ &= S_0^{(t)} e^{-\delta t} N(d_1) - X e^{-r_f t} N(d_2). \end{aligned}$$

We should take into consideration that fuzzy real option valuation is a heuristic method, which only produces a benchmark of the value of the real option considered, therefore we need not to be precise in setting up all the Fuzzy NPV tables. Therefore, some relationship can be assumed between deferring times of the investment and the appropriate Fuzzy NPV tables, which can be the following. The present value of the aggregated expected cash flows belonging to time t, $S_0^{(t)}$, can be the expected cash flows in years $t, t+1, \ldots, L$ aggregated according to the project's beta, that is,

$$S_0^{(t)} = \mathrm{PV}(V_t, V_{t+1}, \ldots, V_L; r) = \sum_{i=t}^{L} \frac{V_i}{(1+r)^i},$$

where, V_i denotes the trapezoidal possibility distribution of expected cash flows of the project at year i, $i = 0, 1, \ldots, L$, and r stands for the risk-adjusted discount rate, and the arithmetic operations on trapezoidal fuzzy numbers are defined according to formulas (1) and (2).

Notice, that considering this decision method we implicitly assume that $T \leq L$, i.e. the maximum deferral time of the investment is not greater than the total length of the project.

Of course, this decision rule has to be reapplied every time new information arrives during the deferral period to see how the optimal investment strategy might change in light of the new information.

We should note that the model just obtained is a hybrid (possibilistic–probabilistic) approach, since we regard possibility distributions as well as probability distributions, hence, we are able to model the explicit (possibility approach) and the implicit (probability approach) uncertainties of an investment activity.

We have to make a decision how long we should postpone the underlying investment up to T time periods before entering into that (if it is worthwhile to go ahead with that at all), therefore we need to find a maximizing element from the set

$$\{FROV_0, FROV_1, \ldots, FROV_T\}.$$

However, considering two arbitrary fuzzy numbers it is impossible to decide which one is "greater" (even if they are trapezoidal ones), since there is no ranking on the set of trapezoidal fuzzy numbers.

In our computerized implementation we have employed some value functions to be able to rank fuzzy real option values. For example, having $FROV_t = (a_t, b_t, \alpha_t, \beta_t)$ we can consider the following utility function

$$u(FROV_t) = \frac{a_t + b_t}{2} + r_A \frac{\beta_t - \alpha_t}{6}$$

where $r_A \geq 0$ denotes the degree of the investor's risk aversion. If $r_A = 1$ then the investor compares trapezoidal fuzzy numbers by comparing their possibilistic mean values.

We note that in our computerized implementation there are several samples how each Fuzzy NPV table belonging to some deferral time of the project can be obtained. (Considering the most precise way we can separately provide each Fuzzy NPV table belonging to each deferral year of the project. However, this method will require a large amount of data, and there is no sure that we obtain much better estimation of the value of real option.)

5 Fuzzy Dynamic Decision Tree

Many industries are experiencing changes that require large investments with substantial risk. Phasing and scheduling of projects which are related to each other can make a huge impact on the value of that set of projects. By phasing and scheduling projects, every step in a project opens or closes the possibility for further options. This is called a chain of growth options, or a compound growth option. Creating options can buy us time to think and gain information to decide whether or not go ahead with a certain bigger investment.

Decision trees are excellent tools for making financial decisions where a lot of vague information needs to be taken into account. They provide an effective structure in which alternative decisions and the implications of taking those decisions can be laid down and evaluated. They also help us to form an accurate, balanced picture of the risks and rewards that can result from a particular choice.

In our empirical cases we have represented strategic planning problems by dynamic decision trees, in which the nodes are projects that can be deferred or postponed for a certain period of time. Using the theory of fuzzy real option valuation set up in the previous sections we have been able to identify the optimal path of the tree, that is, the path with the highest fuzzy real option value (according to some utility function) in the end of the planning period.

Creating the structure of investments we can identify the compound growth real options, i.e. the chain of projects with deferral flexibility, and it can buy us time to think and gain information to decide whether or not go ahead with a certain investment. Obviously, the decision rule has to be reapplied every time new information arrives or a time period has passed during the planning period to analyze how the optimal investment strategy might change in light of the new circumstances.

6 Conclusions

Despite its appearance, the fuzzy real options model is quite practical and useful. Standard work in the field uses probability theory to account for the uncertainties involved in future cash flow estimates. This may be defended for financial options, for which we can assume the existence of an efficient market with numerous players and numerous stocks for trading, which may justify the assumption of the validity of the laws of large numbers and thus the use of probability theory. The situation for real options is quite different. The option to postpone an investment (which in our case is a very large – so-called giga-investment) will have consequences, differing from efficient markets, as the number of players producing the consequences is quite small.

The imprecision we encounter when judging or estimating future cash flows is not stochastic in nature, and the use of probability theory gives us a misleading level of precision and a notion that consequences somehow are repetitive. This is not the case, the uncertainty is genuine, i.e. we simply do not know the exact levels of future cash flows. Without introducing fuzzy real option models it would not be possible to formulate this genuine uncertainty. The proposed model that incorporates subjective judgments and statistical uncertainties may give investors a better understanding of the problem when making investment decisions.

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