Supply Function Equilibrium: Theory and Applications

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The paper deals with the Supply Function Equilibrium (SFE) as a model of competition in electricity markets. It introduces theoretical advancement through relaxing traditional assumptions of continuity of supply functions and provides a foundation for efficient computational algorithms. Special emphasis is placed on exploring the existence of equilibria and on the players' ability to learn subject to information available to them. The theory is applied to modeling electricity markets in the United States.

Introduction

The concept of supply function equilibrium (SFE) was originally developed by Klemperer and Meyer [1] as a way of modeling how competitors could achieve profit-maximizing equilibria in the marketplace under conditions of uncertain demand. The SFE approach was then adopted by Green and Newbery [2] as a model for strategic bidding in a competitive spot market for electricity. That particular publication attracted a substantial interest to the SFE model both in the industry and in academia. The SFE concept offers a compelling model of competitive behavior of multiple suppliers of a single product in which the existence of the Nash equilibrium does not require the demand to be elastic. Instead, representation of the suppliers' behavior by means of supply functions, rather than price-quantity pairs, creates elasticity of the residual demand faced by each player and could result in a sensible equilibrium outcome even if the demand is non-responsive to price. The on-going over the last decade deregulation of the electric industry in various countries of the world prompted industry analysts and consultants to study the SFE concept as means for modeling strategic behavior in electricity markets, creating tools for the direct analysis of market power, assessment of the impact of strategic behavior on electricity prices and on the market value of generating assets. (See for example [3]).

Green and Newbery advanced the SFE theory by including capacity constraints [2] and by incorporating contracts for differences in the SFE framework (Green [4], Newbery [5]). Rudkevich *et al* [6] obtained a closed-form solution to the Klemperer-Meyer equation in a special case of zero price elasticity of demand, generalized the model for the case of non-convex step-wise marginal cost curves representing discrete generating units operating in the market. Anderson and Xu [7], following Anderson and Philpott [8] considered a similar problem but relied on a an original technique for representing supply functions as parametrical two-dimensional curves. This approach allowed them to obtain optimality conditions in a very general form allowing for discontinuous relationship between price and quantity and to prove the existence of the optimal response of an individual player. Rudkevich [9] analyzed the ability of players to adapt their behavior through market observations and learning by means of the Cournot adjustment process and proved that players characterized with linear marginal costs and unconstrained capacities are capable of converging to the linear SFE. Baldick, Grant and Khan [10] explored the applicability of this approach to piece-wise linear systems. Baldick and Hogan [11] attempted a substantial analytical and numerical explorations of SFEs in piece-wise linear systems.

Yet a wide spectrum of theoretical and applied problems remains unresolved. The original paper [1] was dealing with systems with uncertain demand and a single market clearing during the game period. Moreover, suppliers were assumed to be identical (symmetrical system). In light of these two assumptions, it is not surprising that the resulting equilibria are insensitive to the shape of the demand distribution (e.g., demand duration). Attempts to integrate Klemperer-Meyer equations in non-symmetrical cases could yield supply functions that were not always monotonically increasing or even always monotonically declining supply functions. In was not clear whether equilibrium conditions could be discontinuous and if so, what rules should be guiding their discontinuity. Finally, equilibrium conditions were explored only for the case of the so-called one-price payment rule.

In this paper, we offer a general formulation of the SFE game, consider a spectrum of payment rules ranging from the one-price to the pay-bid market design and derive necessary conditions of equilibria in those general settings. In all other parts we focus entirely on the system with one-price payment rule. We use these optimality conditions to outline an efficient algorithms for solving the problem of optimal response for piece-wise linear systems. Finally we

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discuss the application of this algorithm to finding the equilibrium in supply functions for power systems.

Description of the SFE Game

We consider a one-shot non-cooperative game of *n+1* players -- *n* competing generating firms and one market administrator (MA).

We assume that consumers' demand for the product is given in a form of a demand duration function *D(t,p)*. *D(t,p)* depends on time and price such that $D'_t(t,p) > 0,\ D'_p(t,p) \le 0$. Time is continuous and $0 \le t \le \bar{T}$. We further assume that firms are characterized by cost functions $C_j(q_j)$ that are continuous and piece-wise differentiable functions of production capacities q_j. Production capacities are bounded from below by zeros and from above by total capacity available to each firm: $0 \le q_j \le W_j;\ j = \overline{1,n}$.

Prior to the beginning of the game period, each firm submits to the MA a supply function $q_j(p)$ which is a piece-wise differentiable, monotonically non-descending function of price. The MA develops a market-wide supply function

$$Q(p) = \sum_{j=1}^{n} q_j(p)$$

and in each moment of time $0 \le t \le \bar{T}$ solves equation $Q(p) = D(t,p)$ for price. Solutions to this equation at each instantaneous moment *P(t)* form a monotonically ascending function of time, known as a price duration function. Let us compute

$$p_0 = P(0);\quad p_1 = P(\bar{T});\quad p_0 < p_1$$

and define function *T(p)* inverse to the price duration function:

$$P(T(p)) \equiv p;\quad T(p_0) = 0;\quad T(p_1) = \bar{T}.$$

The results of the MA's actions are: the price limits p_0 and p_1 and the inverse price duration function (IPDF) *T(p)*.

In general we assume that in each instant moment of time, for each infinitesimal increment of capacity $dq_j(p)$ offered to the market at prices ranging between *p* and *p+dp*, the firm is paid the bid price *p* and an incentive proportional to the difference between the market clearing price and the bid price. Thus the infinitesimal revenue flow for this capacity increment will be equal to

$$dr(t,p) = [p + \beta(P(t) - p)]dq_j(p)$$

where β is a parameter of market design ranging between zero and unity. $\beta = 1$ corresponds to the one-price market design, $\beta = 0$ corresponds to the pay-bid market.

The integrated flow of revenues for firm *j* in time moment *t* could be expressed in the following way:

$$\begin{aligned} r_j(t) = {} & q_j(p_0)[p_0 + \beta(P(t) - p_0)] \\ & + \int_{p_0}^{P(t)} [x + \beta(P(t) - x)] dq_j(x) = \\ & q_j(p_0)[p_0 + \beta(P(t) - p_0)] \\ & + \int_{p_0}^{P(t)} [x + \beta(P(t) - x)] q'_j(x)dx \end{aligned}$$

Total revenues the firm *j* will receive during the game period $0 \le t \le \bar{T}$ will equal to

$$R_j = \int_0^{\bar{T}} dt \int_{p_0}^{P(t)} [x + \beta(P(t) - x)] q'_j(x)dx + \bar{T} p_0 q_j(p_0)$$

Firm's total variable costs over the game period are equal to

$$V_j = \int_0^{\bar{T}} C_j\left[q_j\left(P(t)\right)\right]dt,$$

resulting in profit margin $\pi_j = R_j - V_j$. It is easy to derive that

$$\begin{aligned} \pi_j = {} & \int_{p_0}^{p_1} [pq_j(p) - C_j(q_j(p))]T'(p)dp \\ & - (1-\beta)\int_{p_0}^{p_1} (\bar{T} - T(p))q_j(p)dp \end{aligned} \qquad (1)$$

Let us call the integral in the right hand side of this formula the profit functional and denote it as $\Pi_j(q_j, T, p_0, p_1)$. This notation reflects key factors influencing the value of the profit margin of each firm.

Formula (1) shows that the supply function a firm submits to the MA influences its payoff in two ways – directly and indirectly. The latter is through influencing MA's actions, namely the price limits and IPDF which also appear in that formula.

Definition of the Nash Equilibrium

We assume that firms choose their supply functions in order to maximize the total profit π_j for the entire game period $0 \le t \le \overline{T}$. When firms choose supply functions $q_j(p), j = 1,2,...,n$, the MA responds with price limits p_0 and p_1 and the IPDF $T(p)$. As stated earlier, firms' profit margins are denoted as $\pi_j = \Pi_j(q_j, T, p_0, p_1)$.

If the firm number k **unilaterally changes its supply function**, i.e. it submits supply function $\tilde{q}_k(p)$ that is different from the supply function $q_k(p)$ while all other firms adhere to supply functions $q_j(p), j \ne k$ then the MA by solving equation

$$\sum_{j \ne k} q_j(p) + \tilde{q}_k(p) = D(t, p)$$

will respond with a new set of $\tilde{p}_0$ and $\tilde{p}_1$ and the IPDF $\tilde{T}(p)$. This response could change profit margins of all firms to $\tilde{\pi}_j = \Pi_j(q_j, \tilde{T}, \tilde{p}_0, \tilde{p}_1)$ if $j \ne k$ and $\tilde{\pi}_k = \Pi_k(\tilde{q}_k, \tilde{T}, \tilde{p}_0, \tilde{p}_1)$

Definition 1. *We will say that the set of supply functions $q_j(p), j = 1,2,...,n$, the price limits p_0 and p_1 and the IPDF $T(p)$ form a Nash equilibrium in the above described game if*

1) *All supply functions are defined in the interval $[p_0, p_1]$ and on that interval they satisfy the following feasibility conditions:*

$$0 \le q_j(p) \le W_j; j = 1,2,...,n \tag{2}$$

$$\frac{dq_j(p)}{dp} \ge 0; j = 1,2,...,n \tag{3}$$

2) *No unilateral change of the supply function by any firm k from $q_k(p)$ to a function $\tilde{q}_k(p)$ satisfying feasibility conditions* (2) *and* (3) *and resulting in a change in price limits to $\tilde{p}_0$ and $\tilde{p}_1$ and in the IPDF to $\tilde{T}(p)$ could increase this firm's payoff:*

$$\Pi_k(\tilde{q}_k, \tilde{T}, \tilde{p}_0, \tilde{p}_1) \le \Pi_k(q_k, T, p_0, p_1) \tag{4}$$

The Problem of Optimal Response

The key to formulating equilibrium conditions in the SFE game is to understand the optimal response of one firm given the supply functions of its competitors. Let us assume that the firm number *j* knows the aggregate supply function of its competitors $S_j(p)$ and chooses its own supply function $q_j(p)$ in order to maximize its profit functional (1). In general, the domain of the aggregate supply function $S_j(p)$ may be limited to the finite interval of prices $\left[p_j^{\min}, p_j^{\max}\right]$. However, this domain could be extended over $[0, \infty)$ by setting $S_j(p) = 0$ for $p \in [0, p_j^{\min})$ and $S_j(p) = S_j(p_j^{\max})$ for $p \in [p_j^{\max}, \infty)$. $S_j(p)$ is assumed to be non-negative, continuous and differentiable almost everywhere on $\left[p_j^{\min}, p_j^{\max}\right]$ function of price and that $S_j'(p) \ge 0$ where the derivative exists. The extension of the domain will keep it continuous and differentiable almost everywhere on $[p_j^{\min}, \infty)$ but allows for a discontinuity at $p_j^{\min}$ if $S_j(p_j^{\min} + 0) > 0$. As shown below, this discontinuity is not essential, because the optimal response of firm *j* would be to offer no supply at prices below $p_j^{\min}$.

Theorem 1. *If $q_j(p)$ is an optimal response of firm j resulting in price limits p_0, p_1 and IPDF $T(p)$ then there exists a continuous and differentiable almost everywhere on $[p_0, p_1]$ adjoint to $q_j(p)$ function $\psi_j(p)$ such that:*

1) *$\psi_j(p) \le 0$ and satisfies the adjoint differential equation*

$$\frac{d\psi_j}{dp} = (1-\beta)(\overline{T} - T) - + \frac{\beta q_j - [p - C_j'(q_j)][S_j' - D_p'(T, p)]}{D_t'(T, p)} \tag{5}$$

2) *If $q_j'(p) > 0$, then $\psi_j(p) = 0$* (6)
and $q_j(p)$ obeys the following equation

$$S_j'(p) = \frac{\beta q_j + (1-\beta) D_t'(T(p), p)\left[\overline{T} - T\right]}{p - C_j'(q_j)} + D_p'(T(p), p) \tag{7}$$

3) *If $\psi_j(p) < 0$, over some interval of prices, then the supply function $q_j(p)$ must remain constant on that interval, $q_j'(p) = 0$.*

4) *Supply meets demand at all prices:*

$$q_j(p) + S_j(p) = D(T(p), p) \tag{8}$$

5) *The price limits and boundary values satisfy the following transversality conditions:*

$$\psi_j(p_0) q_j(p_0) = 0; \ \psi_j(p_0) \le 0; \ q_j(p_0) \ge 0 \tag{9}$$

$$T(p_0) = 0 \quad (10)$$

$$p_0 \ge p_j^{\min} \quad (11)$$

$$\begin{aligned} &\psi_j(p_1)[q_j(p_1) - W_j] = 0;\ \psi_j(p_1) \le 0; \\ &q_j(p_1) \le W_j \end{aligned} \quad (12)$$

$$T(p_1) = \overline{T} \quad (13)$$

6) *The optimal supply function may have a vertical jump at price* p *from* q_- *to* q_+; where $q_+ > q_-$ *only if* $S_j'(p)$ *also jumps at that price and the following condition holds*

$$\frac{S_j'(p+) - D_p'}{S_j'(p-) - D_p'} - \frac{p - C_j'(q_-)}{p - C_j'(q_+)} > 0 \text{ (if } \beta{>}0) \quad (14)$$
$$\ge 0 \text{ (if } \beta{=}0)$$

The proof of this theorem appears in the Appendix.

Conditions (5)-(14) formulate a two-point boundary problem for the adjoint differential equation (5) and functional equations (7) and (8) in terms of $q_j(p)$ and $T(p)$, combined with boundary conditions *(9)-(13)* and "switching" rules 2) and 3). Later in this paper we will outline the solution algorithm in the case of a step-wise marginal cost function $C_j'(q_j)$ and a piece-wise linear aggregate supply function $S_j(p)$. From the "common sense" point of view, the two-point boundary problem is well defined: it has a sufficient number of boundary conditions to expect either a unique solution or a discrete set of solutions. Moreover, as shown in Anderson and Xu [7], the feasible set in this problem is compact under the Hausdorff metric, the objective function is continuous and therefore the optimal response always exists. That implies that the two-point boundary problem (5)-(14) too has a solution.

Piece-Wise Linear Systems

In this section we further analyze the problem of optimal response by making some simplifying assumptions. Thus, we assume that the cost function of the responding firm is monotonically increasing, continuous piece-wise linear function of supply and therefore, marginal cost function is step-wise. This assumption reasonably well reflects the economics of power generation with firms operating discreet generating units.

$$\begin{aligned} &C'(q) = z_k \text{ if } Y_{k-1} < q \le Y_k;\ k = \overline{1,u} \\ &Y_0 = 0;\ Y_m = W \end{aligned} \quad (15)$$

where monotonically increasing quantities $Y_1,...,Y_{u-1},W$ are cumulative capacities of u generating units and monotonically increasing values $z_1,...,z_u$ are their running costs.

Let us assume that the rival's supply function $S(p)$ is also a piece-wise linear function of price and therefore its derivative which exists at all prices except $P_1,...,P_N$ is also a step-wise function:

$$S'(p) = s_k \text{ if } P_{k-1} < p \le P_k;\ k = \overline{2,N} \quad (16)$$

We can also assume that demand is a piece-wise linear function of time and a linear function of price with zero cross-elasticity:

$$\begin{aligned} &D(t,p) = \overline{D}(t) + \delta\overline{M}(t) - \delta p = D_\delta(t) - \delta p \\ &\overline{D}(t) = \overline{D}(t_{k-1}) + d_k(t - t_{k-1}) \text{ if } t_{k-1} \le t \le t_k \\ &\overline{M}(t) = \overline{M}(t_{k-1}) + m_k(t - t_{k-1}) \text{ if } t_{k-1} \le t \le t_k \\ &k = \overline{1,F}; t_0 = 0, t_F = \overline{T} \end{aligned} \quad (17)$$

where $\overline{D}(t)$ is system demand measured at reference prices $\overline{M}(t)$ and δ is the slope of demand responsiveness to price.

If we substitute (15) and (16) into equation (7), the latter would yield the following bidding rule:

$$B_{jk} = z_j + \frac{Y_{j-1} + x}{s_k + \delta} \quad (18)$$

suggesting the price at which the quantity *x* of the unit *j* should be offered to compete with the *k*-th segment of competitors' supply function. Thus, in order to construct an optimal response supply function we can visualize an $N \times u$ grid on the $p \times q$ plane with the formula (18) directing the potential growing segment of the supply functions within each rectangle in that grid. Vertical gridlines are defined by price points $P_1,...,P_N$. A part of the plane between two vertical gridlines defined by prices P_{k-1}, P_k we will call *rivals' segment k*. Horizontal gridlines are at levels $Y_1,...,Y_{u-1},W$ and a part of the plane between horizontal gridlines Y_{j-1}, Y_j corresponds to unit *j*. Thus, every rectangle on this grid corresponds to a combination of unit *j* and rivals' segment *k*. The algorithm of forming the optimal response must yield a monotonically non-descending supply function such that if it is strictly monotonic within a particular rectangle it obeys the equation (18) bidding rule. The supply function could move vertically (quantity could increase in a jump) and/or horizontally (flat segment). At each end, the flat segment could be of a different kind: 1) connecting with a growth segment inside a rectangle; 2) connecting with a vertical jump along the gridline; 3) connecting with the problem boundary (left or right). With three possibilities at each end, there are overall nine types of flat segments. Lemma 1 below establishes additional rules and formulas allowing for an efficient search algorithm. In order to state it properly, we need to make several definitions.

Definition 2. *Using formula (18) we can define the upper and lower bid limits for the unit within each rectangle*:

$$B^-_{jk} = z_j + \frac{Y_{j-1}}{s_k+\delta}; B^+_{jk} = z_j + \frac{Y_j}{s_k+\delta} \quad (19)$$

We will say that the rival's segment k *<u>excites the unit j</u> if* $[B^-_{jk}, B^+_{jk}] \cap [P_{k-1}, P_k] \neq \varnothing$

Definition 3. *We will say that rival's segment k <u>initializes units up to j</u> if each of the following is true:*

$P_{k-1} < z_j < P_k$

$[Y_{j-1}, Y_j] \cap [D_\delta(0) - S(P_{k-1}) - \delta P_{k-1}, D_\delta(0) - S(P_k) - \delta P_k] \neq \varnothing$

either $z_{j+1} > P_k$ *or* $D_\delta(0) - S(z_{j+1}) - \delta z_{j+1} < Y_k$

Definition 4. *Let* $P^*(q)$ *be a peak price in the system in the event of withholding all capacity above the level q:* $S(P^*(q)) + q = D_\delta(\bar{T}) - \delta p$ *and therefore* $P^*(q)$ *is a monotonically non-descending function of q. We will say that rival's segment k is <u>in agreement with withholding</u> all units above unit j if*

$\exists q \in [Y_{j-1}, Y_j]$ such that $P_{k-1} \le P^*(q) \le P_k$ *and*

$\forall x > 0 \;\; P^*(Y_j + x) > P_j$

Lemma 1. *Let q(p) be the optimal response supply function*

a) *If* $Y_{j-1} < q(p_0) \le Y_j$ and $P_{k-1} < p_0 \le P_k$, *then rivals' segment k initializes units up to j;*

b) *If* $Y_{j-1} < q(p_1) \le Y_j$ and $P_{k-1} < p_1 \le P_k$ *then rival's segment k is in agreement with withholding all units above unit j;*

c) *q(p) is growing within a rectangle k,j if and only if rival's segment k excites unit j and in this case*

$$q(p) = (p - z_j)(s_k + \delta) \quad (20)$$

d) *If rival's segment k excites unit j but does not excite unit j+1, then it excites no units above j+1;*

e) *If rival's segment k excites unit j but does not excite unit j-1, then it does not excite units below j-1;*

f) *Vertical jumps are possible only long a vertical gridline;*

g) *if q(p) jumps from level A corresponding to unit j to level B corresponding to (possibly a different) unit f along the vertical gridline separating rivals' segments k and k+1, then*

$$\frac{s_{k+1}+\delta}{s_k+\delta} > \frac{P_k - z_j}{P_k - z_f} \;\; and \;\; \begin{matrix} A \ge (s_k+\delta)(P_k - z_j) \\ B \le (s_{k+1}+\delta)(P_k - z_f) \end{matrix}$$

h) *if q(p) has a flat segment at level A corresponding to unit j that stretches from price* p_-, *corresponding to rivals' segment k to* p_+, *corresponding to rivals' segment r, then*

$$\int_{p_-}^{p_+} \frac{A - (S'(p)+\delta)(p - z_j)}{D'_t(T(p))} dp = 0 \quad (21)$$

if q(p) has no vertical jumps at p_- *and* p_+, *then the unit j is excited by both segments k and r and* $s_r < s_k$*;*

i) *if the flat segment terminates at the end-price, i.e.* $p_+ = p_1$ *and A=W (no capacity withheld), then*

$$\int_{p_-}^{p_1} \frac{W - (S'(p)+\delta)(p - z_j)}{D'_t(T(p))} dp \le 0 \quad (22)$$

This lemma is a direct interpretation of Theorem 1 applied to the case of piece-wise linear functions.

Sketch of the Algorithm

The rules outlined in the lemma set the foundation for the efficient algorithm of inspecting the grid in search of the optimal supply function. It is important to note that in formulas (21) and (22) one has to integrate a piece-wise linear function of price which can be accomplished in closed form. The algorithm itself could be interpreted as a search of a feasible path connecting a rectangle initializing one or more units with a rectangle consistent with withholding of one or more units or ending at the highest vertical gridline. If no path is found, the optimal response is to offer no supply. Feasible path may obey equation (20) only if the unit corresponding to this rectangle is excited by the corresponding rivals' segment. If feasible path does not obey equation (20), it can only be flat or jumps vertically upward along a vertical gridline. If the flat segment connects two growing segments, equation (20) relates prices at each end with the quantity level of the flat segment and equation (21) then has to be solved for that level only. If the flat segment merges with the vertical jump, it is again not difficult to solve equation (21), because one of the integration bounds is fixed at the price determining the jump gridline.

Nash Equilibrium in Supply Functions

Theorem 1 could be generalized to formulate necessary conditions of the Nash equilibrium.

Theorem 2. *If supply functions* $q_j(p), j=1,2,...,n$, *price limits* p_0 *and* p_1 *and the IPDF* $T(p)$ *form a Nash equilibrium, then there exists a set of continuous and differentiable almost everywhere on* $[p_0, p_1]$ *adjoint functions* $\psi_j(p)$, $j=1,2,...,n$ *such that each of the following is true:*

1) $\psi_j(p) \le 0$, $j=1,2,...,n$ *and satisfy adjoint differential equations*

$$\frac{d\psi_j}{dp} = (1-\beta)(\overline{T}-T) - + \frac{\beta q_j - [p - C_j'(q_j)][q_{-j}' - D_p'(T,p)]}{D_t'(T,p)} \tag{23}$$

2) If a supply function $q_j(p)$ *is growing at price p,* $q_j'(p) > 0$, *then* $\psi_j(p) = 0$ *and*

$$\frac{dq_{-j}}{dp} = \frac{\beta q_j + (1-\beta)D_t'(T,p)\left[\overline{T}-T\right]}{p - C_j'(q_j)} + D_p'(T,p) \tag{24}$$

where $q_{-j}(p)$ *denotes the aggregate supply function of all competitors of the firm j:*

$$q_{-j}(p) = \sum_{k \ne j} q_k(p) \tag{25}$$

3) If $\psi_j(p) < 0$ *on some interval of prices, then the supply function* $q_j(p)$ *must remain constant on that interval,* $q_j'(p) = 0$.

4) Supply meets demand at all prices:

$$\sum_{j=1}^{n} q_j(p) = D(T(p), p) \tag{26}$$

5) The price limits and boundary values of supply and adjoint functions satisfy the following transversality conditions:

$$\psi_j(p_0)q_j(p_0) = 0;\ \psi_j(p_0) \le 0;\ q_j(p_0) \ge 0 \tag{27}$$

$$T(p_0) = 0 \tag{28}$$

$$\psi_j(p_1)[q_j(p_1) - W_j] = 0;\ \psi_j(p_1) \le 0;\ q_j(p_1) \le W_j \tag{29}$$

$$T(p_1) = \overline{T} \tag{30}$$

6) Supply function q_j *may have a vertical jump at price p from* q_- *to* q_+; where $q_+ > q_-$ *only if* $q_{-j}'(p)$ *also jumps at that price and if* $\beta > 0$, *then the following condition holds*

$$\frac{q_{-j}'(p+) - D_p'}{q_{-j}'(p-) - D_p'} - \frac{p - C_j'(q_-)}{p - C_j'(q_+)} > 0 \text{ (if } \beta>0) \quad \ge 0 \text{ (if } \beta=0) \tag{31}$$

This theorem directly follows from Theorem 1. The problem (23)-(31) is also a two-point boundary problem, but unlike the problem of optimal response that has only adjoint differential equations, equilibrium conditions also contain differential equations in terms of supply functions $q_j(p)$. However no boundary conditions could be added to (23)-(31). Therefore, as a two-point boundary problem the latter is under-defined and could have an infinite set solutions. On the other hand, the existence of the equilibrium in the SFE game remains to be proven. If it were possible to prove that the problem of optimal response always has a unique solution, one could construct a map from the set of feasible supply functions into itself as a combination of optimal response strategies of each firm. It would be a matter of technique and some not too restrictive assumptions to establish the continuity of this map in a suitable metric. Finally, given that the set of monotonic function is convex, one could use the Schauder fixed point theorem to prove the existence of the equilibrium. However, it seems that to prove this uniqueness would be very difficult of ever possible. Indeed, an example of the problem of optimal response considered by Anderson and Xu [7] shows the existence of two local optima with very close values of the objective function. It might be just a matter of changing a few parameters in order to equate those objective function values. Moreover, given that the problem is essentially non-convex, it is not possible to use a convex shell of all solutions to the problem of optimal response rendering the use of the Kakutani theorem non-applicable. The latter could be used to prove the existence of the equilibrium in mixed strategies, but the practical applicability of a mixed strategy is rather unclear.

Differential Equations for Supply Functions

Differential equations (24) seem to make the search for the equilibrium very promising. It is easy to notice that if $\beta = 1$, it becomes a familiar Klemperer-Meyer equation

$$\frac{dq_{-j}(p)}{dp} = \frac{q_j(p)}{p - C_j'\left(q_j(p)\right)} + D_p'(T(p), p) \qquad (32)$$

Klemperer and Meyer derived their equation assuming zero cross-elasticity of demand, $D_{pt}'' = 0$. Equation (32) is an important generalization accounting for an arbitrary cross-elasticity. It is important to note that as system of simultaneous equations, not all supply functions must obey such equation at the same price. Indeed, as indicated by condition 3), supply functions could stay flat over some interval of prices, e.g., not offering any incremental supply until "the price is right." In a system comprised of suppliers with different capacities and cost characteristics, the appearance of such flat segments over some price intervals should be more of a rule than an exception. Therefore, it should be expected that in a given price interval, only a subset of functions would obey (32). Let's assume that firms with the set of indices $A = (j_1, ..., j_k)$ and only those firms obey differential equations (32) over the same interval of prices. Define $Q_A(p) = \sum_{r \in A} q_r$. Since all other supply functions are flat over that interval of prices, $q_{-j}' = Q_A' - q_j'$. Observing that $(k-1)Q_A' = \sum_{r \in A} q_{-r}'$, equations (32) could be reduced to the normal form:

$$q_m' = \frac{1}{k-1} \sum_{r \in A} \frac{q_r}{p - C_r'(q_r)} - \frac{q_m}{p - C_m'(q_m)} + \frac{D_p'}{k-1} \qquad (33)$$

where $m \in A$

This system of simultaneous equations yields feasible solutions (monotonically increasing supply functions) as long as all right hand sides stay positive. Thus for a subset of firms $A = (j_1, ..., j_k)$ and price p we can define a feasible subset in the phase space:

$$\frac{1}{k-1} \sum_{r \in A} \frac{q_r}{p - C_r'(q_r)} - \frac{q_m}{p - C_m'(q_m)} + \frac{D_p'(T, p)}{k-1} \geq 0$$

where $m \in A$

$$0 \leq q_j \leq W_j;\ j = \overline{1, n}$$

$$D(T, p) = \sum_{j=1}^{n} q_j$$

With the total of n suppliers, there are $2^n - (n+1)$ different subsets containing two or more suppliers. Thus, for any price p there potentially could be that many feasible subsets and therefore that many different systems of differential equations defining supply functions. Thus, unlike the symmetrical case considered by Klemperer and Meyer and the case of asymmetrical duopoly explored by Green and Newbery [2], the variety of potential equilibria could be decided not only by the set of boundary conditions, but also by the variety of differential equations that should be considered at every price p. Thus, if a trajectory at a price p is passing through a point in the phase space (let us say in the direction of increasing prices) the behavior of this trajectory could potentially switch into any subset of differential equations whose feasibility set contains that point in the phase space at that price. The switch between differential equations however cannot be arbitrary, given conditions on adjoint functions specified in Theorem 2. Indeed, a switch corresponds with at least one supply function becoming flat and/or at least one supply function initiating a growth segment. Again, assuming that integration of the entire system occurs in order of increasing prices, a condition for the supply function to initiate growth will be determined by the adjoint differential equation through finding the price at which the corresponding adjoint function vanishes. However, it is not obvious at which point and which particular supply function must become flat. That determination could only be verified upon reaching the final point of integration by checking transversality conditions. Thus, we are dealing with a two-point boundary problem with added complexity when not only the boundary conditions on one end must be chosen to satisfy conditions on the other end, but also at every point a decision has to be made whether and how the composition of differential equations to integrate further should be changed. With up to $2^n - (n+1)$ possibilities to consider at each point, the complexity of this problem appears overbearing.

Cournot Adjustment Process and Conjectured Equilibrium

Following Green [12], Rudkevich [9] considered a simplified case of linear marginal cost functions of competitors which allows for a unique equilibrium in linear supply functions. Assuming further that each player starts the game with offering its linear marginal cost curve, observes the market outcome, in particular, the aggregate supply function of competitors, and makes the next move by solving the problem of optimal response given those observation. This is known as a Cournot adjustment process (see for example, Fudenberg and Levine [13]). As shown in [9], at every step of this process, the supply function of optimal response is linear and the process rapidly converges to the unique linear SFE. Thus, in this case, there appears to be no need to solve a two-point boundary problem and there is no ambiguity in choosing the equilibrium.

It appears logical to try the same approach in the general case. As noticed earlier, the optimal response to a piece-wise linear supply function of competitors is a piece-wise linear function. Thus the Cournot adjustment process would always remain in that class of functions. However, numerical experiments implemented under the assumption that players observe the entire aggregate supply function of competitors show that the Cournot adjustment process often does not converge for piece-wise linear functions.

A promising approach was suggested by Day, Hobbs and Pang [14] who suggested to search for the Conjectured Function Equilibrium rather than for the SFE. In particular, they introduce into the model an analytical representation of what each player assumes with respect to the behavior of its competitors, rather than merely expecting the overall system to be at equilibrium. Indeed, players in the electricity market have no exact information with respect to running costs of their rivals' generating units. Therefore, even if they were capable of computing the exact equilibrium, information required to perform such computations is not readily available. On the other hand, players can rely upon market observation. In particular, each player can observe hourly prices in the market (locational and/or regional) as well as total and/or regional quantity of electricity consumed. Each player also knows hourly quantity it produced and can compute the total quantity produced by its rivals at any point in time. Thus, a player j observes $p(t)$ and estimates $S_j(t)$ and therefore can develop an estimate of $S_j(p)$ using these data. It is only logical to approximate $S_j(p)$ with a piece-wise linear function. Next, each player can solve a problem of optimal response given that estimated rivals' supply function, generate a new market outcome prompting for the next iteration. Apparently, using thus estimated rivals' supply function instead of directly computed one makes the process converge. Although we were not able to obtain a formal proof of convergence, we so far have not encountered convergence problems in numerical experiments.

Applications

We implemented a first version of this algorithm as the basis for the TCA COMPEL-21 Model. We use COMPEL-21 in conjunction with GE MAPS in the following manner. First, we make a standard GE MAPS run assuming that all generators are dispatched based on short-run marginal costs. We use the results of this run for an initial calibration of competitors' supply functions. Next, COMPEL-21 simulates a Cournot adjustment process using the algorithm described above. The results of COMPEL-21 are then used to adjust generators' bids on a unit-by-unit basis. Adjusted bids are then used as an input for a new GE MAPS run. We apply this logic to modeling the Northeatern power system combining New England, New York, PJM, Ontario and New Brunswick.

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Appendix

Proof of Theorem 1

Consider the optimal control problem arising from the problem of optimal response.

$$\min(-\pi_j) = -\int_{p_0}^{p_1} [pq_j(p) - C_j(q_j(p))]T'(p)dp$$
$$+(1-\beta)\int_{p_0}^{p_1} (\bar{T} - T(p))q_j(p)dp$$

s.t. [1]

$$\begin{aligned}
&\frac{dq_j(p)}{dp} = u_j(p);\\
&u_j(p) \geq 0;\\
&q_j(p) + S_j(p) = D(T(p), p)\\
&T(p_0) = 0;\\
&-q_j(p_0) \leq 0;\\
&q_j(p_1) \leq W_j;\\
&T(p_1) = \bar{T};
\end{aligned} \tag{OR}$$

We will use the Lagrangian form of optimality conditions [15]. First, we define the Lagrangian of problem (OR) which is equal to

$$L(q_j, q'_j, T, T', u_j, p) = T'\left[C_j(q_j) - pq_j\right]$$
$$+(1-\beta)\left[\bar{T} - T\right]q_j + \psi_j\left[q'_j - u_j\right]$$
$$+\lambda\left[q_j + S_j(p) - D(T, p)\right]$$

where ψ_j, λ are Lagrange multipliers. Next, we define the terminal function

$$l(q_j(p_0), q_j(p_1)) = -\mu q_j(p_0) + \eta\left[q_j(p_1) - W_j\right]$$

The optimality conditions are then could be expressed in the following way:

Lagrange-Euler Equations

$$\begin{aligned}
&\frac{d}{dp}\frac{\partial L}{\partial q'_j} - \frac{\partial L}{\partial q_j} = 0;\\
&\frac{d}{dp}\frac{\partial L}{\partial T'} - \frac{\partial L}{\partial T} = 0;
\end{aligned} \tag{LE}$$

Pontryagin optimality principle (POP): at each point of the optimal trajectory the Lagrangian must reach its minimum as a function of control variables:

$$\begin{aligned}
&L(q_j(p), q'_j(p), T(p), T'(p), u_j(p), p) =\\
&= \min_{u_j \geq 0} L(q_j(p), q'_j(p), T(p), T'(p), u_j, p)
\end{aligned} \tag{POP}$$

Transversality conditions

$$\left.\frac{\partial L}{\partial q'_j}\right|_{p=p_0} = \frac{\partial l}{\partial q_j(p_0)}; \quad \left.\frac{\partial L}{\partial q'_j}\right|_{p=p_1} = -\frac{\partial l}{\partial q_j(p_1)}.$$

Complimentarity conditions

$$\begin{aligned}
&\mu q_j(p_0) = 0;\\
&\mu \geq 0;\\
&\eta\left[q_j(p_1) - W_j\right] = 0;\\
&\eta \geq 0.
\end{aligned}$$

Equations (LE) are:

$$\begin{aligned}
&\frac{d\psi_j}{dp} = \lambda + (1-\beta)\left[\bar{T} - T\right] + T'\left[C'_j - p\right];\\
&q'_j\left[p - C'_j\right] + \beta q_j = \lambda D'_t.
\end{aligned} \tag{34}$$

The POP could be reduced to the consideration of the only term in the Lagrangian that depends on the control function u_j:

$$\min_{u_j \geq 0}\left[-u_j\psi_j\right] = -\max_{u_j \geq 0}\left[u_j\psi_j\right] \tag{35}$$

Equation (35) indicates that if the adjoint function is positive, $\psi_j > 0$, then there exists no optimal value of the

[1] In general, the set of conditions should also include $T'(p) \geq 0$. However, this follows automatically from the assumption that $D_T > 0;\ D_p \leq 0;\ S' \geq 0$ and a requirement that $q'_j = u_j \geq 0$.

control function u_j, hence the optimality requires $\psi_j \le 0$. The control function could be positive if only the adjoint function is equal to zero. Finally, if the adjoint function is negative, the optimal control function must be at zero. In other words, if the supply function q_j is growing on some interval of prices, the adjoint function must remain equal to zero on that interval. If the adjoint function is negative on some interval of prices, the supply function must remain constant on that interval.

Consider the case of the growing supply function. Since the adjoint function is zero on the interval of growth, its derivative must also be zero on that interval. Combining that with equations (34), we get

$$\begin{aligned} D_t'\lambda &= T'D_t'\left[p - C_j'\right] - (1-\beta)D_t'\left[\bar{T} - T\right] \\ &= q_j'\left[p - C_j'\right] + \beta q_j. \end{aligned} \quad (36)$$

On the other hand, by differentiating the "supply equals demand" equation, we derive that

$$T'D_t' = q_j' + S_j' - D_p'. \quad (37)$$

Substitution of (37) into (36) yields the following equation:

$$S_j' = \frac{\beta q_j + (1-\beta)(\bar{T} - T)D_t'}{p - C_j'} + D_p' \quad (38)$$

which proves statement 2) of the theorem.

If we substitute (37) into (LE) and get rid of λ using the second equation in the (LE) group, we can obtain the following differential equation for the adjoint function which as one can see holds regardless of whether the supply function is growing or flat:

$$\begin{aligned} \frac{d\psi_j}{dp} &= (1-\beta)(\bar{T} - T) - \\ &+ \frac{\beta q_j - [p - C_j'(q_j)][S_j' - D_t'(T,p)]}{D_t'(T,p)} \end{aligned} \quad (39)$$

In the first case, the right hand side is equal to zero. In the second case, when supply is constant, the adjoint equation takes the following form:

$$\frac{d\psi_j}{dp} = (1-\beta)\left[\bar{T} - T\right] + \left[p - C_j'(q_j^*) + \frac{\beta q_j^*}{S_j'(p) - D_p'}\right]T' \quad (40)$$

where q_j^* is the level at which the supply function remains flat.

This proves statement 3) of the theorem.

Statement 4) is obvious, it could be interpreted as the formula to compute the IPDF given all supply functions.

Finally, the transversality conditions (TC) result in the following:

$$\mu = -\psi_j(p_0); \qquad \eta = -\psi_j(p_1). \quad (41)$$

By combining (41) with the complimentarity conditions (CC), we get

$$\begin{aligned} &\psi_j(p_0)q_j(p_0) = 0;\ \psi_j(p_0) \le 0; q_j(p_0) \ge 0; \\ &\psi_j(p_1)\left[q_j(p_1) - W_j\right] = 0;\ \psi_j(p_1) \le 0; q_j(p_1) \le W_j. \end{aligned} \quad (42)$$

That proves Statement 5) of the Theorem.

Let us assume that the supply function has a vertical jump from q_- to q_+ at price *p*. This jump could be optimal only if $\psi_j(p) = 0$. Given that the adjoint function cannot be positive, the latter is possible only if $\psi_j'(p-) \ge 0$ and $\psi_j'(p+) \le 0$. Using (39), we get that

$$\begin{aligned} \beta q_- &\ge [p - C_j'(q_-)][S_j'(p-) - D_p'] - (1-\beta)(\bar{T} - T)D_t' \\ \beta q_+ &\le [p - C_j'(q_+)][S_j'(p+) - D_p'] - (1-\beta)(\bar{T} - T)D_t' \end{aligned}$$

Since $q_+ > q_-$ and $\beta > 0$, the above two inequalities imply that the $S_j'(p+) > S_j'(p-)$, and moreover

$$\frac{S_j'(p+) - D_p'}{S_j'(p-) - D_p'} > \frac{p - C_j'(q_-)}{p - C_j'(q_+)}. \quad (43)$$

If $\beta = 0$, the same inequality will hold if $C_j'(q_-) < C_j'(q_+)$. Otherwise, the inequality (43) would have to be restated as a non-strict (with a $\ge$ sign).

Q.E.D.