Abstract
Symbolic mathematical software has become inexpensive enough that it is almost practical to require each first-year calculus student to purchase it. But having a program do math for you might not be learning, and does not teach you how to apply it. This paper examines research issues related to learning calculus including cognitive processes, the role of proficiency and a proposed causal model.

Current trends in calculus curriculum reform are reviewed including the need for better measurements of learning effects. John R. Anderson's ACT-R theory is proposed as a comprehensive cognitive learning theory and basis for the causal model.

The primary theoretical contribution is advocating proficiency as a construct for measuring learning in both individual and collaborative environments. Proficiencies (such as subject mastery, missing elements, and faulty elements) are both causes and effects, and are therefore modeled as control feedback mechanisms (more similar to an instrument panel than a direct outcome). Based on the causal model and precise measurements of proficiency, an experiment is proposed to study the impact of symbolic math software and collaborative processes in the teaching of calculus to university pre-business students.

1. Introduction
Mathematics education is being transformed by use of graphics calculators and symbolic math software. Students solve and graph complex calculus problems by entering formulas and selecting solution symbols. To be successful, students need limited knowledge of algebraic manipulations. This technological transformation raises a fundamental question.

Is this "real" calculus, just a lazy method, or worse yet a technological fad? Is it appropriate for training students to apply math? Can it make any contribution to doing "real" or theoretical math? In part, this paper develops a causal model with which this question can be evaluated.

Another emerging educational trend is student collaboration in problem solving. Group projects in a class have been traditionally encouraged and are a major thread in modern calculus pedagogy (including the Harvard Consortium Calculus [10]). Opportunities for out-of-class cooperation have greatly increased during recent years. Enabled with Internet access and starting in elementary school, students participate in educational discussion groups and form web-based partnerships for solving homework assignments. Twenty years ago, students prepared reports by copying or rephrasing an encyclopedia article. From a learning perspective they found the appropriate article, transcribed the words, and added new or transitional sentences to demonstrate content knowledge. Today, one merely needs to cut and paste the article from an electronic version of the encyclopedia or worse yet from a web site of a fellow student or researcher. Students can be impelled to learn much more by using group techniques to extend, compare, and contrast views. This paper also considers the impact of collaborative processes on learning.

Calculus requires complex learning processes. As the culmination of elementary and secondary mathematics programs, calculus proficiency at the university level is a prerequisite for most hard sciences and business economics curriculums. Teaching calculus is difficult because students have diverse math backgrounds and different algebraic manipulation skills. Simple algebraic errors result in incorrect answers and confusion; moderate to advanced calculus problems require algebraic manipulation skills beyond many students' capacity. This complex learning environment is the context chosen to study impacts of symbolic math tools and collaborative learning.

Background
The opportunity to study and evaluate a calculus learning environment resulted from ongoing curriculum revision efforts in an introduction to business calculus class at a large Southwestern university. The curriculum changes are driven by national, state and local interests. National interest is derived from the Calculus Reform efforts [1]. State interest comes from the legislature's mandates to raise educational standards and increase teaching loads of tenured professors. Local interests come from the Mathematics Department as service provider, the College of Business as recipient of students who successfully complete the
prerequisites, and students themselves who want to qualify for Business School. The Mathematics Department supports two different types of introductory courses. The first is a five semester hour class for students planning to enroll in mathematics or hard science degree programs. This class is rigorous and demands proficiency in algebraic manipulations. The second, a three semester hour class, is not as rigorous, and focuses on students applying to business or social science degree programs. The latter class has been used as a means of screening students and consequently grades tend to be bimodal. Students either receive satisfactory grades, given reduced learning expectations, or relatively low grades, which make them ineligible for advanced standing in their degree program.

The students in these two classes tend to differ as well in motivation and interests. The majority of "hard science" students perceive value in math itself and see themselves actively using it in their future studies. Most of the pre-business students take math only because it is required, need to be shown the relevance to business, and pursue only a good grade rather than long-term learning.

In the College of Business, a faculty committee reviews mathematics prerequisites. Requirements are changing from traditional calculus to an emphasis on statistics, business problem solving, and computer competency skills. Last year, the College of Business requested the introduction of math software and collaborative problem solving skills into the business calculus curriculum. The assumptions underlying this approach are that symbolic math and collaborative problem solving will produce students who (1) score better on common exams, (2) understand business applications of calculus better, (3) exhibit more computer competence, (4) work better in teams, (5) have greater confidence in their ability to apply calculus to business problems, and (6) have greater confidence in their ability to use computers to solve problems. In addition, teachers will alter their behavior. They will among other things (1) focus on calculus, with less review of algebra and other previously studied material, (2) introduce more business applications, (3) develop more computer competence themselves, (4) encourage student teamwork, and (5) spend less time preparing and checking problems for tests, quizzes and other assignments after a brief learning period [2].

Pilot Studies

Two pilot studies have been conducted, partly to work out the myriad problems using symbolic math software (MathCAD) throughout the term for all work in a classroom. Over 100 students were also surveyed concerning motivation, comfort with mathematics, technological skills, prior skills, course content, and personal class expectations. The initial pilot was well received by students. Table 1 and 2 show there were fewer drop-outs, no bi-modal grade distribution and slightly better performance. Pilot study students perceived greater learning in business problem solving and computer competency skills. During Spring 1997, two pilot classes were taught. The second classroom involved one new teaching team, consisting of an instructor and a technical advisor. Although the resulting analysis is pending, second semester findings are not expected to be significantly different than the first semester.

| Table 1: Quantitative Measures |
|---------|--------|--------|
| Grades  | n      | 23     | 557    |
| Course  | 2.30   | 1.88   |
| GPA     | 2.84   | 2.68   |
| \(P(\chi^2)=.66\) | P      | \(P=.01-\) | \(P=.01-\) |
| “Success” | Successes | 18  | 355    |
| \(P(\chi^2)=.09\) | Failures | 5   | 226    |

| Table 2: Measures of Perception |
|-----------------------------|--------|--------|
| Perception Measures          | Pre    | Post   | P     |
| Subject                     |        |        |       |
| How useful is calculus in   | 5.64   | 7.86   | 0+    |
| business? (9 items)         |        |        |       |
| How much calculus do you    | 36.3   | 48.8   | 0+    |
| know? (12 scored items)     |        |        |       |
| Activity and Results         |        |        |       |
| Expected Final Grade?        | 4.12   | 4.12   | 0+    |
| Time outside class?          | 3.18   | 2.41   | .20   |
| Participation in class       | 3.82   | 3.41   | .04   |
| Confidence (5 point scale)   |        |        |       |
| Solving business problems    | 2.0    | 3.2    | 0+    |
| Using computers              | 2.3    | 4.05   | 0+    |
| Teamwork                     | 3.68   | 3.84   | .29   |

Initially, we hoped to demonstrate by experiment that symbolic mathematics software belonged in the classroom. However, discussions between the College of Business and the Mathematics Department resulted in a consensus that use of symbolic math tools (whether graphics calculators or software) is unavoidable and not the primary question. Instead, the more pressing question is how to measure

\[1\] MathCAD is a registered trademark of MathSoft Corp.
learning effects. This question is also gaining national recognition [3]. Symbolic math tools and collaborative learning are considered to be factors in a complex solution space. Other major factors include applied problem solving and long-term impact of teaching methods. The issue has greater importance because of the expense and effort required to re-tool curricula and instructors for these changes.

2. Cognitive Learning

Cognitive processes underlie mathematics and have been well studied. One theory with a rich history in mathematics is John R. Anderson's ACT-R. It posits interactions between declarative and production memories though cognitive elements (goals, declarative facts, general productions, specific productions and operational facts). ACT-R is a comprehensive cognitive theory and led to development of computer simulation and intelligent tutoring software. The appendix describes how we apply ACT-R to individual and group learning. [4-8]

Few would question that solving math problems requires cognitive processes. But when we try to teach students, we want to measure their assimilation of the relevant cognitive elements. The ACT-R theory has merit for us because it gives us a precise way to identify and categorize the elements that are essential to the successful use of calculus in solving applied problems.

Note that for most instructors it is not sufficient to measure outputs such as the number of correct answers on an exam or assignment. This may be a rough indication of learning, but it suffers from the following failings. (1) It gives no indication of the cognitive processes employed, which could in theory range from a guess to a detailed analysis. Which parts does the student know? (2) It eliminates feedback as a mechanism for correcting the student. (3) No indication of the mode of failure is given. Evaluators, therefore, rely on reading and analysis of the solution method employed ("show your work" for "partial credit") when assessing work for evidence of correct cognitive processes, and write notes to identify "bugs" in the student's reasoning or "missing" elements.

ACT-R is precise enough for mathematics to allow creation of an automated intelligent tutoring program in CommonLisp. It helps students learn to solve problems by correcting them when they introduce bugs or leave out key productions. We use ACT-R to derive the specific cognitive elements applied or required so that a standard, less subjective assessment of cognitive knowledge can be obtained.

We focus on three elements of the ACT-R theory: (1) goals, (2) productions, and (3) applications. Goals are used to search for productions and are processed in a last-in-first-out sequence. If a goal can not be processed directly it is decomposed and the sub-goals are processed first. Learning to formulate goals is very important in the process of learning to apply math. Productions are behavioral in nature; they take the form (IF a set of conditions exist THEN a course of action is taken). Applications refer to the declared and operational facts that drive and result from productions. The cognitive process is iterative and cyclical and is illustrated in Figure 1.

In the calculus problem and solution of Figures 2, and 3, we illustrate ACT-R analysis of a cognitive math process and identify in Figure 4 the three basic elements (goals, productions, applications).

Find the price per unit \( p \) that produces the maximum profit \( P \):

Cost Function: \( C = 2400x + 5200 \)
Demand Function: \( p = 600 - 0.4x^2 \)

Source: [9] from exercise 3.5 #10 on page 217

Figure 2 - A calculus problem example.
Find the price per unit \((p)\) that produces the maximum profit \((p)\)

CORRECT SOLUTION

1. Let \(x = \# \) of units \(C = \) total costs in \$  
   \(p(x) = \) demand function in $ per unit as a function of \(x\)

2. Given problem statement:
   Cost function:  
   \(C(x) := 2400x + 5200\)
   Demand function:
   \(p(x) := 6000 - 0.4x^2\)

3. Create primary equation for profit \(P\)
   \(P(x) := p(x)x - C(x)\)
   \(P(x) \rightarrow (6000 - 0.4x^2)x - 2400x - 5200\)

4. Determine feasible domain
   \(rv := 0, 1, \ldots, 100\)

   No negative quantities of \(x\) allowed.

5. Find critical number
   Because relative max must happen at critical number
   \(P'(x) := \frac{d}{dx}P(x)\)
   \((P'(x)) \rightarrow -1.2x^2 + 3600\)
   \(-1.2x^2 + 3600 \geq 0\)
   \((-54.7722557505166) \leq 54.772\)
   \(54.7722557505166 \leq 54.772\)

   The only solution of interest is the positive 54.772.

   Note there are no critical numbers with an undefined \(P'\)
   because the function is a polynomial.

6. Determine whether critical point is a max:
   Because value changes from positive to negative this critical point
   is a relative max.
   \(P(54) = 100.8\)
   \(P'(55) = -30\)
   \(P'(x) := \frac{d}{dx}P(x)\)
   \(P'(54.77225575051661346) = -131.453\)

   Because second derivative is negative this must be a relative max.

7. Solve for price per unit:
   \(p(54.772) = 4.8 \times 10^3\)

Figure 3 - MathCAD Solution to Calculus Problem

Goals

Goals \{ find price per unit that produces maximum profit, define variables, define problem statement, create primary equation, determine feasible domain, find critical number, verify maximum critical point, solve \}

Productions:

General Productions \{ profit = revenues - cost, quadratic equation graphing, IF 1st derivative=0 THEN critical number, IF 2nd derivative<0 then relative maximum, IF 2nd derivative>0 then relative minimum, IF 1st derivative just before critical number is positive AND 1st derivative just after critical number is negative THEN critical number is a relative maximum, IF 1st derivative just before critical number is negative AND 1st derivative just after critical number is positive THEN critical number is a relative minimum, algebraic solution \}

Specific Production \{ evaluate \(P(x)\), \(P(x) \rightarrow\), if \(x < 0\) then \(P(x)\) has no meaning, \(P'(x) :=\), \(P''(54.772) :=\), \(p(54.772) :=\) \}

Applications:

Declarative Facts \{ cost function: \(C(x)=2400x+5200\), demand function: \(p(x)=6000-0.4x\) \}

Operational Facts \{ \(x\) is \# of units, cost function is total costs in $, demand function is price in $ at a given quantity, \(P(x)\) is profit function in $, \(P(x)\rightarrow p(x)x-C(x)\), \(x \geq 0\), \(P'(x)\) graph, \(P'(x)=0\), \(x=54.772\), \(P''(54.772)=-131.453\), \(p(x)=4,800\), \(P(x)\rightarrow p(x)x-C(x)\), \(x=-3000\), \(x < 0\), \(p(x)\) does not exist \}

Figure 4 - ACT-R Elements of Problem

3. Proficiency and Measurement

The dictionary definition of proficiency is (1) advancement in knowledge or skill and (2) the quality or state of being proficient. It is clear that a more proficient student has (or can recall) a larger set of goal-setting strategies and general productions which can apply to varied problems or opportunities. Observe also that "learning" math involves increasing this portfolio and that having a larger portfolio gives the student a better chance of solving a greater variety of problems, either new applications or differently phrased problems.

We equate the size of this portfolio with proficiency of the student. A time-sequence of measurements of the size should increase. So our interest as educators is in developing this proficiency.

Proficiency is, in our view, a surrogate for the ability to perform the desired cognitive processes. Measuring it demands a better tool than simply the solutions to problems: with the cognitive model we know what data to capture as evidence. From our perspective, measurement of proficiency requires that the grader identify cognitive elements (goals, productions and facts) and determine whether logical sequencing and reasoning is recorded. To do this, just like the grader above, we need to observe use of these elements in correct relations, not just the solution output.
4. Collaborative Learning

The nature of a collaborative learning environment has not been precisely defined in the literature and is not presented here. Instead, a collaborative experiment is proposed.

Traditional Teaching Interactive Teaching Group Enabled Teaching

T = Teacher S = Student G = Group

Figure 5 - Collaborative Learning Environments

Figure 5 shows a range of collaborative environments. Collaboration in a traditional (lecture) teaching environment exists if the teacher is considered a collaborator, though the collaboration is "one way" - admittedly an extreme definition which verges on no collaboration. More collaboration is required in an interactive teaching situation. Information exchanges are increased over traditional teaching, but verbal exchanges are limited to one person at a time and it is possible that students who do not engage or that are otherwise excluded will not learn. Group-enabled environments greatly increase collaborative learning and information exchanges; for instance the students may work on a project as a group and occasionally seek help from the teacher. Symbolic math software is complementary in both interactive teaching and group-enabled teaching, changing the tools, not the collaborative nature. Our pilot studies use interactive environments to some extent. Classrooms are equipped with individual computers and access to the symbolic math software. While listening to a lecture, students are encouraged to simultaneously solve problems being presented by the instructor. Students often work in pairs and explain steps to each other. One of the instructors is very comfortable with this format and provides extra examples to the students before the start of class. On average students perceived that time spent in class greatly reduces their learning efforts outside of class. The instructors want to move to an even greater collaborative environment in which immediate feedback can be provided, and much more joint including collaborative use of MathCAD.

5. Causal Model

We propose a causal model of math learning (shown in Figure 6) which consists of five major components. The people (players) are individual students, their teacher and their peers. Causes include motivation (vested interest, expectations, personal commitment), problem opportunities ("real-life" problems, homework, exam questions, projects), and tools (calculators, symbolic math software, intelligent tutoring, pencil & paper, group decision software). The cognitive processes have already been described and include goals, productions and applications. Effects include solutions to the problems and are sometimes called productivity.

Proficiency includes subject mastery, goal setting strategies, knowledge of productions and applications and knowledge about the use of tools. Proficiency functions as both a cause and an effect, and therefore is shown as a feedback loop. Existing components engender new ones through the ACT-R cognitive process. Measuring proficiency is our goal.

6. Experimental Design

We now present an experimental design based on our model and experience. During the beginning of the
semester, emphasis is placed on individual learning skills and cognitive processes. In this period, the instructor identifies the cognitive elements (goal-setting strategies, general productions) and provides feedback on bugs and missing productions. After time for students to become proficient with symbolic math software, the experiment begins with a regularly scheduled exam on material that has been covered.

In grading, proficiency is assessed by identifying bugs and missing productions for each test question. This can be easily done when MathCAD is the test medium because the evidence appears right in the MathCAD file. A software tool could actually read the MathCAD file and with a knowledge base of what goals and productions are required for the problem determine most of what is missing and buggy. For paper solutions, the proficiency assessment would be made by a grader, using the knowledge base of required proficiencies, and the results input by keyboard or menu into a database.

Then students are assigned to pairs based on their exam scores and on non-overlapping bugs and missing elements. Students with higher grades are paired with students with lower grades so that the averages are as close as possible, subject to the constraint that the proficiencies are mismatched as desired. Project problems are then assigned which are similar to some of the test questions. Students are also told that they will be re-tested during the next exam on problems similar to their group assigned problems.

To provide maximum incentive for collaborative efforts, students are told that for both tests they will receive a single grade which is the higher of the two test scores, and further they will be given a grade bonus based on the difference in average test scores for both students. Thus a high scoring student is rewarded for helping the lower scoring student improve if they maintain or improve their own score. The lower scoring student benefits by receiving a higher score and the additional bonus from the improved group average. This is adequate motivation for the group activity.

The collaborative experiment lets us measure proficiency before and after collaboration. We know they will learn; each problem element is covered at least three times (first test, group project and second test). But we want evidence that such peer-to-peer collaboration enhances proficiency. By collecting the data marked with who contributed it, we will be able to observe actual transfers of cognitive elements such as goals and productions. Of course it would be desirable to automate the collection of this cognitive trace data. Building a tool using MathCAD for collaborative work opens this as a possibility for the first time. Even without MathCAD, a recording program a human grader can use will enable execution of the experiment. We can document and quantify the transfer of the cognitive components of proficiency from group member to group member. This would be significant evidence of the success or failure of the group process in building proficiency.

### 7. Conclusions

This research stream studies the impact of symbolic math software and collaborative learning environments on the learning of business calculus. A causal model of math learning using John R. Anderson's ACT-R theory represents the cognitive learning processes. It provides a method for capturing and measuring proficiency, a set of precise cognitive elements which at the same time represent the current state of a cognitive process and which are inputs that catalyze subsequent learning. This method can be applied in mathematical processes such as calculus as Anderson and others have shown.

An intelligent tutoring system like John R. Anderson's ACT-R simulation software provides more detailed individual feedback and could be the basis for an individual learning environment. We make simplifying assumptions using selected cognitive elements to identify mastered and missing cognitive elements and bugs.

The study of symbolic math software and collaborative processes in the learning of business calculus is a study of technology diffusion. But we see these as an implement for engineering the application of cognitive learning theory in group learning settings. We can formulate tests of the absorption of cognitive concepts and trace the effectiveness of a group in enhancing proficiency. To show how, we have proposed a precise experiment to directly measure the gains in a collaborative calculus environment.

An obvious application of this model is to conduct the collaborative learning project and experiment like the one proposed above, which we intend to do in the 1997-1998 school year. Significant work remains to carefully construct two exams and collaborative projects that include overlapping cognitive processes and determine which elements are used and in what sequences. The pilot will validate the experimental process and identify common bugs and missing elements as well. The process of constructing these tools and using them to assess work will provide helpful feedback to the instructor and the Math Department concerning student proficiencies and deficiencies, which might be addressed by various techniques.

Use of intelligent tutoring and assessment software designed along these lines would provide a means to help measure individual proficiency as well as give feedback.
Instead of giving tests, individuals would use the tutoring system; then the tutoring data would be used to form groups matching complementary skills and performance levels in a "real-time" manner throughout the semester. This could enhance the effect of grouping students for cooperative activity.

One hypothesis hotly debated is whether use of business related problem sets makes a difference for business students. Empirical evidence from Duke University [3] suggests that the greater relevance of the problems creates greater motivation or vested interest in learning. It would be interesting to evaluate over a long term whether this relevance has an effect on a student's problem solving skills in future semesters. Follow up studies with precise measurements of cognitive process learning such as we suggest would be very informative.

The study of collaborative processes in education is wide open for research. We hope to have shown how to provide quantitative data on student proficiency and how it was achieved, by tracking the group work process, much as conventional electronic meeting systems can track individual contributions and position modifications through analysis of text. Math learning at this level lends itself well to this approach because the cognitive process though complex is rather standard, according to ACT-R.

Many math teaching experts believe that group processes have great importance in enhancing learning. Our model allows actual measurement of the effects through the group interaction. Such research offers the potential to increase understanding of the exact mechanism by which the group activities support learning, no small step in our opinion.

References


Appendix - ACT-R Theory in brief

Figure A-1 is a representation of John R. Anderson's ACT-R theory for an individual student.

According to ACT-R theory, cognitive memories are divided into declarative and production memory. Schemas are represented by different shaped cognitive elements and their grouping in either production or declarative memory. When a problem is introduced (E0), the declarative memory is provided with facts, goals, and possibly general productions. This is not to say that the entire set of appropriate declarative elements are present or possible. Cognitive processes are required to "fill-in" the missing pieces.

Given the initial declarative elements, the brain instantiates operational facts (E1) in the production memory and pushes the primary goal (E2) to a search for specific productions. Specific productions are composed (E3) from general productions from either short-term or long-term declarative memory. Action (E4) is taken on the specific production and the related facts to derive new operational facts. If the new operational facts provide conclusive evidence, then the goal is popped (E5) off of the stack. When the primary goal is popped, the problem is solved (E6). The cognitive process are iterative and sub-goals are usually required.
A much more complex cognitive process occurs when one or more specific productions are combined via analogy processes (E7) to form general productions. This analogy process, summarizing several observations into a theorem (production), is not what we are training students to do in elementary calculus. Within the ACT-R computer simulation program and in learning environments, it is difficult to replicate or measure analogy processes.

Figure A-2 represents ACT-R theory for group-enabled problem solving. The problem might be perceived differently by each group member and invoke different declarative memories. Assume that the collaborative process is similar in terms of cognitive elements. In group processes key cognitive elements must be communicated (or recorded) and the individual cognitive processes contribute to the group solution. In the worst case (from a group participation standpoint), a single individual maps their solution and the other members validate or accept the solution. In a well-designed learning process, all members contribute to the process as it occurs and draw on the subject mastery of each member. In an ideal situation, the collective memories of the group are bug-free and contain the cognitive elements necessary to solve the problem. Assuming the collaborative process is not allowed to stop until a correct solution is derived, both group and individual cognitive processes are required to identify missing elements and correct bugs. In most cases corrective processes occur within the group; however, if necessary teacher or other instructional intervention can occur (another form of collaboration!).

The group-enabled graphical representation reduces an even more complex set of processes into an intuitively understandable representation. Ideally, as shown by the dotted lines it is possible to trace individual contributions to the group-solution. Symbolic math software and group-enabled computer tutoring software might enable such a tracing mechanism. Even if individual contributions are not tagged, the group solution will be graded and provide feedback to both the group and its members.

Figure A-2 Metagraph of Group-Enabled ACT-R