Discussion Note on Logic for Group Decisions

Rahul Chattergy

University of Hawaii, 2540 Dole St. Honolulu, Hawaii 96822

ABSTRACT

A group decision support system uses a knowledge base created by the active participation of the group members. Uncertainties can be introduced in such a group knowledge base from at least two sources. Hard uncertainties can result from a lack of knowledge on the part of every group member about some state of affairs in the world. Soft uncertainties can result from a lack of consensus among the group members. A logical infrastructure is necessary to support query processing and drawing inferences from a group knowledge base. It is shown that a natural formulation of logic in this case leads to a many-valued logic which has significant differences from traditional logic. Some basic operations in this logic are derived here. Much more work remains to be done in the future.

INTRODUCTION

Let us consider a group of agents involved in distributed decision making. Each agent has its own logic system supporting its own personal knowledge base. A group knowledge base in its simplest form is assumed to be a collection of propositions variously related to the individual knowledge bases of the agents. To ascertain the truth status of a proposition in the group knowledge base, we query each agent in the group and try to synthesize all their responses in a rational manner. A group knowledge base, being a collective knowledge base, provides a more complete description of the state of affairs in the world than provided by a component knowledge base of an agent.

An agent's response may be either true or false or don't know. Hence, the truth status of a proposition in a group knowledge base may be marked either true, or false, or both, or none depending on how we synthesize the individual responses. A four-valued logic such as this was originally developed by N. Belnap, Jr. [1,2]. The logical operators were defined by Belnap using the theory of continuous functions on lattices. This is a particularly elegant approach and the lattice structure sheds light on what is possible in such a logic system. Rescher [5] also uses a similar logic system in his discussion of cognitive dynamics. We have provided an alternate derivation of Belnap's results which is more intuitive and follows smoothly from our knowledge of traditional logic. A similar approach is often used by Rescher [4] in his discussion of many-valued logic.

Since we have several responses from different agents to synthesize, it is possible to weigh these responses. This allows various degrees of "bothness" or differences of opinion about the truth status of a proposition. Belnap mentioned this case as a possible extension of his work. We have attempted to account for this case by creating a new logical lattice to support operations on the group knowledge base. Since the acceptance or rejection of the truth status of a proposition by a majority is intimately connected with this lattice, we call it a plurality lattice. The common conjunction and disjunction operations are defined by means of the meet and the join operations of the lattice. In the general case the lattice is neither modular nor distributive. However, an abridged version is modular and ultimately leads to Belnap's lattice which is both modular and distributive.

At this point a note of clarification may serve a useful purpose. A proposition in a group knowledge base does not have a truth value but merely a truth status marked by some symbol. It is not possible for a group knowledge base to know if a proposition is true; it merely reports what it has been told by the agents in the group.

FOUR-VALUED LOGIC

Let us consider a logic system where an atomic proposition can be marked either true(T), or false(F), or both(B), or none(N). Following Belnap ([1,2]), we can describe this situation by means of an approximation lattice as shown in figure 1.

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Belnap’s work shows that if a many-valued logic can be visualized as a lattice, then the meet and the join operation on that lattice can serve as the & and the v operators defined in the manner discussed above. In this manner we see that the & & spirit and the v v spirit marking with an unknown logic marking x produces x (false), and (ii) the v of a false (true) marking with an unknown logic marking x produces x (true).

Next letting x stand for B and N respectively in (ii) and (iii) we have T & B = B, T & N = N, F v B = B, and F v N = N. Considering what B and N stand for in the context of a group knowledge base we conclude that B & B = B, B v B = B, N & N = N, and N & N = N. Finally, since a mark B means that a proposition has been marked both T and F by distinct agents, we see that B & N can be interpreted either as T & N = N or F & N = F. Now given a choice between N and F we choose F since it has more information content than N in the approximation lattice of figure 1. Hence, we conclude B & N = F. In a similar manner we derive B v N = T.

The definitions of the & and the v operators obtained in the manner discussed above are shown in figures 2a and 2b.

The same results were derived by Belnap[1,2] from considerations of continuous functions on lattices. It is not difficult to see that the & and the v operations defined in this manner are respectively the meet (v) and the join (v) operations of the logic lattice shown in figure 3.

Figure 4 shows such a lattice of markings for M = 4 agents in a group.

The lattice shown in Figure 4 has three parallel threads. These threads stretch between two nodes: accepted by all as true (T), and accepted by all as false (F). The node marked N is the marking for all propositions which cannot be either accepted or rejected as true and the don’t know. The other two threads show the two possible paths for moving between acceptance by all as true and acceptance by all as false. On the rightmost thread of the lattice n1 > n2 and it is called the accepted-by-a-majority as true thread. On the center thread n1 < n2 and it is called the accepted-by-a-majority as false thread. Since the lattice models acceptance by a majority, we call it a plurality lattice.

The response of an agent can be either true, or false, or don’t know. These responses are then synthesized in the following manner.

Strategy for synthesizing responses:
1: If all responses are don’t knows, then p is marked N.
2: If some responses are don’t knows and the others are evenly divided between the true and the true, then p is marked N. This case may occur whenever M = k is divisible by two, where k is the number of don’t knows.
3: If all responses are true (false), then p is marked T(F).
4: Otherwise p is marked (n1, n2, n3) where n1 is the number of true responses, n2 is the number of false responses, and n3 is the number of don’t knows.

The markings so obtained from the responses are now arranged in a lattice. The simplest way to describe the construction of such a lattice is to give an example. Figure 4 shows such a lattice of markings for M = 4 agents in a group.

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**PLURALITY LATTICE**

In a group knowledge base, let M denote the number of agents reporting to the base. Each agent has its own knowledge base, and for simplicity we assume each such knowledge base to consist of a set of atomic propositions. The operations on such a knowledge base are governed by an agent’s own logic system. A group knowledge base K is therefore a purported knowledge base where the propositions are viewed from an epistemic distance (Rescher [5, p.84]).

The fact that a plurality lattice is nonmodular and nondistributive is obvious from the construction shown in Figure 4. By collapsing the center and the right threads each into a single node we can create a modular lattice but then we lose the finer distinctions of the truth markings.

It follows that a many-valued logic based on a plurality lattice is bound to be nontraditional in many ways. Here we list some of these differences and similarities. Derivations are given in the appendix.
Note that a plurality lattice consist of the truth markings of propositions and not of the propositions themselves. However, we shall often write of a proposition \( p \) being on a particular thread of a plurality lattice when we mean its truth marking to be on that thread. Use of different symbols is possible but cumbersome and often renders simple results quite opaque.

**Condition of negation:**
For every proposition \( p \) marked with \((n_1, n_2, M_{n_1-n_2})\) its denial \( \neg p \) is marked \( N \), then so is \( \neg p \). If \( p \) is not marked \( N \), then \( p \) and \( \neg p \) are on different threads of a plurality lattice. Hence, \( p \land \neg p \) is \( F \).

**Condition of conjunction:**
\( p \land q \) is \( T \) if \( p \) is \( T \) and \( q \) is \( T \). If \( p \) and \( q \) are on different threads of a plurality lattice, then \( p \land q \) is \( F \). However, if \( p \land q \) is \( F \), then it does not follow that either \( p \) or \( q \) is \( F \).

**De Morgan's laws:**
The following laws of De Morgan are valid in plurality logic.

(a): \( \neg(p \lor q) = \neg p \land \neg q \).
(b): \( \neg(p \land q) = \neg p \lor \neg q \).

**Distributive laws:**
The two distributive laws are as follows.

(a): \( p \land (q \lor r) = (p \land q) \lor (p \land r) \).
(b): \( p \lor (q \land r) = (p \lor q) \land (p \lor r) \).

In plurality logic, the distributive laws are not valid in general. They fail if \( p, q, \) and \( r \) have distinct truth markings on three distinct threads of the lattice. Law (a) also fails if \( q \) and \( r \) are on different threads, \( p \) is on the same thread as \( q(r) \), and \( p\text{-}q(r) \). A similar condition for the failure of law (b) also exists.

**IF...THEN**
To complete our logic we must consider the definition of logical consequence. Suppose following the definition of material implication in traditional logic we define logical consequence \( (p \rightarrow q) \) in plurality logic as \( p \rightarrow q \). If \( p \) and \( q \) are on the same thread of a plurality lattice, then by construction \( p \rightarrow q = T \) and furthermore, \( \neg q \rightarrow q = T \). Hence, \( p \) and \( q \) are equivalent. Clearly this approach trivializes the notion of logical consequence.

Note that \( p \rightarrow q \) is just another proposition in plurality logic and as such, it is not necessarily marked either \( T \) or \( F \). We should define \( p \rightarrow q \) in such a way that allows for truth markings other than \( T \) or \( F \). Let us denote the truth marking of a proposition \( p \) by \( /p/ \). One possible approach is to use Gödel implication:

\[
\begin{align*}
&\text{if } /p/ \leq /q/ \\
/p \rightarrow q/ &= /q/ \\
&\text{if } /p/ > /q/ \\
&\text{otherwise}
\end{align*}
\]

If the only truth markings are \( T \) and \( F \), then Gödel implication becomes material implication. Also by definition \( p \rightarrow q \) applies only if \( /p/ \) and \( /q/ \) are comparable on the plurality lattice. It is obvious that \( /p/ \rightarrow /q/ = T \). Several other properties of Gödel implication are given in the appendix.

Next we consider the problem of assigning a marking to \( q \) when \( /p/ \) and \( /p \rightarrow q/ \) are given. It follows from the definition of \( /p \rightarrow q/ \) that (i) if \( /p/ = /q/ \), then \( /q/ = /p/ \), or \( /p/ > /q/ \) and (ii) if \( /p/ > /p \rightarrow q/ \), then \( /q/ = /p/ \). No markings can be assigned to \( q \) if \( T > /p/ \rightarrow q/ > /p/ \).

On plurality lattice, Gödel implication gives us the following result. If \( /p/ \rightarrow /q/ = T \), then \( \neg q \rightarrow /p/ = T \), \( /q/ \rightarrow /p/ = /p/ \), and \( /p/ \rightarrow \neg q/ = \neg /q/ \).

**SEMANTIC CONSEQUENCE**
By definition \( q \) is a valid semantic consequence of \( \{p_i, i=1,...,n\} \) if and only if for every assignment of truth markings to \( \{p_i, i=1,...,n\} \), the truth marking of \( q \) is no less than the smallest truth marking assigned to \( \{p_i, i=1,...,n\} \). The notation used for a valid semantic consequence is \( \{p_i, i=1,...,n\} \Rightarrow q \). Also \( p_i \) is used to denote \( p \land \ldots \land p \).

**Theorem I:** \( \{p_i, i=1,...,n\} \Rightarrow q \) if \( /p_i/ \rightarrow /q/ = T \) for all truth marking assignments to \( \{p_i, i=1,...,n\} \) and \( q \).

Given \( \{p_i, i=1,...,n\} \Rightarrow q \), it follows from the definition of a valid semantic consequence that \( /q/ \geq \min_{\{p_i, i=1,...,n\}} /p_i/ \) for all truth marking assignments to \( \{p_i, i=1,...,n\} \) and \( q \). Hence, \( /q/ = /p/ \) and by the definition of Gödel implication \( /p/ \rightarrow /q/ = T \) for all truth marking assignments to \( \{p_i, i=1,...,n\} \) and \( q \).

Given that for all truth marking assignments to \( \{p_i, i=1,...,n\} \) and \( q \), \( /p_i/ \rightarrow /q/ = T \), it follows from the definition of Gödel implication that \( /q/ = /q/ \). Hence, \( /q/ = \min_{\{p_i, i=1,...,n\}} /p_i/ \) for all truth marking assignments to \( \{p_i, i=1,...,n\} \) and \( q \), and by definition, \( \{p_i, i=1,...,n\} \Rightarrow q \).

**DISCUSSION**
Plurality logic is devised to deal with operations on knowledge bases where (a) lack of knowledge and/or (b) lack of consensus are critical factors. So why not use other logics already in existence?

**Why not use fuzzy logic?** Local and fuzzy logics are discussed by Bellman and Zadeh in [6]. It appears that in fuzzy logic (see p.115 of [6]) \( \text{Truth} \) is a linguistic variable with fuzzy values \( \{\text{true, false, not true, very true, not very true, very false, not very true, not very false, ...}\} \). Whatever these fuzzy values may signify (eg. \( \text{false} \) being different from \( \text{not true} \)), it is not the model of a group knowledge base we have in mind. For us, each and every proposition in a group knowledge base is either true or false. Their truth values may be momentarily uncertain and we have to deal with truth status markings, but we have every hope that eventually these values will be known. Hence, plurality logic is designed to blend into traditional logic when uncertainties are removed.
From the discussion in [6] it also appears that fuzzy logic may lack self-consistency. For example, $\neg p \land p$ may turn out to be not very true rather than false. By construction this cannot happen in plurality logic unless $p$ is marked $N$.

**Why not use probabilistic logic?** In a group decision situation, the truth values of propositions are uncertain because of either a lack of knowledge or a lack of consensus among the agents or both. What is needed is an acceptance rule to tell us which proposition to accept as true. The most obvious application of probability calculus to create such a rule does not work. Consider using a threshold value such that if the probability of a proposition being true exceeds this threshold, it is accepted as true. Then the Lottery paradox ([3] p.222, [7] p.176, and [8] p.57) shows that, the computation of truth values of propositions can be isolated from the rest of the knowledge base. The law of contradiction as well as the law of excluded middle (it is not dependent upon two-valuedness) hold for future-contingent propositions. Also the set of propositions marked $N$ is closed under negation, conjunction, and disjunction. Hence, if so desired these propositions can be isolated from the rest of the knowledge base. The law of contradiction as well as the law of excluded middle (it is not dependent upon two-valuedness) hold for future-contingent propositions. From the construction of a plurality knowledge acquisition system where the truth value of a proposition may not be known in the future. So it is reasonable to set $p/N$, $q/N$, and $\neg p/N$, $\neg q/N$, $p \lor q/N$, $p \land q/N$, $p \land \neg q/N$, $p \lor \neg q/N$, $\neg p \land \neg q/N$, $\neg p \lor q/N$, $\neg q \lor p/N$, $\neg p \land \neg q/N$, $p \lor q/N$, $p \land q/N$, $\neg p \land \neg q/N$, $p \lor \neg q/N$, $\neg q \lor p/N$. Hence, $p = /p/q = /q/N$. However, $/p \land q/N$ is known to be $F$ and also $/p \lor q/N$ is known to be $T$. To account for this situation we can split $N$ into $N_1$ and $N_2$ with $/p = N_1$ and $/q = N_2$.

Plurality logic can provide the logical infrastructure of any knowledge acquisition system where the truth value of a proposition can not be guaranteed. Rescher [5] provides extensive discussions on such knowledge bases. Although plurality logic deals only with truth markings, truth values may be incorporated as $(M,0,0)$ or $(0, M,0)$ markings.

**REFERENCES**


**APPENDIX**

We derive some of the simple results of plurality logic mentioned in the text. From the construction of a plurality lattice we see that if $p$ is a proposition accepted by a majority as true, then its denial $\neg p$ is accepted by a majority as false. Also if $p$ is marked $N$, then so is $\neg p$.

**Self-contradiction:**

(a) If $p$ is not marked $N$, then by the property of a lattice $\land$ operation, and the structure of the plurality lattice $p \land \neg p = F$.

(b) Otherwise $p \land \neg p = N$.

**De Morgan's laws:**

Let us consider the validity of $\neg (p \land q) = \neg p \lor \neg q$ first.

(a) Assume $p$ and $q$ are on the same thread and without loss of generality $p \geq q$. Then $\neg p$ and $\neg q$ are on the other thread and $\neg p \leq q$. It follows that $p \land q = q$ and $\neg (p \land q) = \neg q$. On the other hand $\neg p \lnot q$ is $\neg q$. Hence, $\neg (p \land q) = \neg p \lor \neg q$.

(b) If $p$ and $q$ are both marked $N$ then so are $\neg p$ and $\neg q$. Hence, their $\land$ and $\lor$ are also marked $N$.

If $p$ is marked $N$ and $q$ is not marked $N$ we consider two cases $q = T$ and $q = F$. In the first case $p \land q = N$ and $\neg p \lor q = N$. In the second case $p \land q = F$ and $\neg p \lor q = F$. Hence, De Morgan's law is valid in these cases.
(c) Assume p and q are on different threads. Then so are $\neg p$ and $\neg q$. It follows that $p \land q$ is F and $\neg (p \land q)$ is T. Also $\neg p \lor \neg q$ is T. Hence, $\neg (p \land q) = \neg p \lor \neg q$.

This completes the proof of $\neg (p \land q) = \neg p \lor \neg q$.

The validity of $\neg (p \lor q) = \neg p \land \neg q$ can be demonstrated in a similar manner.

**Distributive laws:**

On a plurality lattice, the distributive laws do not hold in general. Let us consider $p \land (q \lor r) = (p \land q) \lor (p \land r)$ first. It is simpler to state the conditions under which this law fails. It fails if p, q, and r are on three distinct threads of the lattice. It also fails if q and r are on different threads, p is on the same thread as q(r), and p > q(r). Next we derive the conditions under which the law is still valid.

(a) Assume q and r are on the same thread and p is on a different thread. Then $p \land (q \lor r)$ is F, $p \land q$ is F, and $p \land r$ is F. Since FvF is F, the law is valid in this case.

(b) Assume q, r, and p are on the same thread. If $(q \lor r) \leq p$, then $p \land (q \lor r) = (q \lor r)$, $p \land q = q$, and $p \land r = r$. The law follows immediately.

Next assume that $p \leq (q \lor r)$. Then we have two cases to consider.

(i) Let $q \leq p \leq r$. Then $p \land (q \lor r) = p$, $p \land q = q$, and $p \land r = p$. Hence, the law is valid.

(ii) Let $p \leq q \leq r$. Then the same derivations as in the previous paragraph are still valid except $p \land q = p$. The law follows immediately again.

The conditions for the validity of the other distributive law $p \lor (q \lor r) = (p \lor q) \lor (p \lor r)$ can be derived in a similar manner.

**Gödel implication:**

Designated and antidesignated truth value sets are $D^+ = \{T\}$, and $D^- = \{F\}$ respectively.

**Transitivity:** Assume $p \rightarrow q \in D^+$ and $q \rightarrow r \in D^-$. Then by definition of Gödel implication $\neg q \in D^+$, and $\neg r \in D^-$. Hence $\neg r \in D^+$ and by definition $p \rightarrow r \in D^+$.

Assume $T > p \rightarrow q > q > r$. Then by definition of Gödel implication $p \rightarrow q \in D^+$, and $q \rightarrow r = r = [q \rightarrow r]$.

**Nonsymmetry:** Assume $q \rightarrow p \in D^+$ and $p < q \in D^-$. Since $q > p$, $p \rightarrow q = T \in D^+$. However, $q \rightarrow p = /p \in D^-$. 

**Modus ponens:** To establish that q follows from p and p $\rightarrow q$ in plurality logic we need to show that $(p \land (p \rightarrow q)) \rightarrow q$ is a tautology. Assume $(p \land (p \rightarrow q)) \rightarrow q \in D^+$. Then from the definition of Gödel implication $(p \land (p \rightarrow q)) \rightarrow q$ implies $(p \land q) = q$ and hence a contradiction. Hence, $(p \land (p \rightarrow q)) \rightarrow q \in D^-$. 

The following properties are easy to verify from the definition of Gödel implication.

**Modus tollens:** 1. Whenever $p \rightarrow q \in D^+$ and $q \in D^+$, then $p \in D^-$. 2. Whenever $p \rightarrow q \in D^+$ and $q \in D^-$, then $p \in D^-$.

**Designation enhancement:** 1. If $p \in D^+$ and $q \in D^-$, then $p \rightarrow q \in D^+$. 2. If $p \in D$ and $q \in D^-$, then $p \rightarrow q \in D$.

**Designation degradation:** 1. If $p \in D^+$ and $q \in D^-$, then $p \rightarrow q \in D^-$. 2. If $p \in D^-$ and $q \in D^-$, then $p \rightarrow q \in D^-$. 

Whenever $p > q$, then $r \rightarrow p > r \rightarrow q$. Whenever $q > r$, then $p \rightarrow r > q \rightarrow r$. Whenever $q > p$, then $p \rightarrow r > q \rightarrow r$.