Cutwidth Approximation in Linear Time*

(Extended Abstract)

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Abstract

Graph width metrics have been widely studied for their relevance to VLSI design. Examples include cutwidth, pathwidth, bandwidth and several others that arise in circuit layout. When the width is bounded, graphs that satisfy these metrics can often be recognized by finite lists of obstruction tests. One of the most foundational tests is to determine whether K_4 is immersed in a graph. In this paper, we present for the first time a fast, practical algorithm to perform this test, and discuss its relevance to cutwidth and other metrics.

1 Introduction

Layout permutation problems abound in VLSI design theory. These can often be described in terms of graph width metrics. Familiar metrics include cutwidth, pathwidth, bandwidth and many others. Although deciding whether a graph satisfies one of these metrics is $\mathcal{NP}$-complete in general, several of them can be decided in polynomial time as long as the relevant metric is bounded.

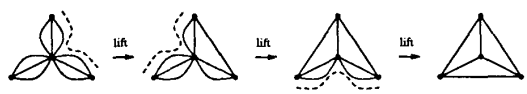
Cutwidth, for example, can in principle be decided in $O(n^2)$ time for any fixed width using a finite but unknown list of immersion tests. Pathwidth, also amenable to this approach, can be decided analogously for any fixed width with a finite number of minor tests. Bandwidth, on the other hand, is not currently known to permit any such technique. We refer the reader to [4] for detailed information on this subject.

In this paper, we focus on cutwidth and the immersion order. The *cutwidth* of $G = (V, E)$ is defined to be the maximum number of edges from E that must be cut between consecutive vertices in any linear layout of V. A graph H is *immersed* in a graph G if a graph isomorphic to H can be obtained from G by a series of these two operations: taking a subgraph and lifting[1] a pair of adjacent edges.

The aforementioned tests are for "obstruction" containment. An obstruction can be viewed in this setting as a forbidden graph. Thus, when one knows all obstructions to cutwidth k, one knows a characterization for the family of graphs that have cutwidth k or less.

Unfortunately, very little is known about practical immersion tests or the nature of cutwidth obstructions. One problem is that multiple edges are important, making immersion tests much more "slippery" than, say, many well-known minor tests. As with other width metrics and other orders, however, complete graphs are obstructions. Testing for K_1 and K_2 are of course trivial. Testing for K_3 is easy: K_3 is immersed in any graph of order three or more unless the graph is a tree with no pair of multiple edges incident on a common vertex.

The first really difficult test, and the one we devise here, is that for K_4. Ours is the first practical linear-time algorithm known for this task. Observe that K_4 is an obstruction for cutwidth three, because any arrangement of its vertices on a line will require a cut of four edges. Moreover, those graphs with no immersed K_4 are necessarily series-parallel[2], a class of graphs frequently encountered in circuit layout. But K_4 can be immersed in a series-parallel graph. As a simple example, consider the star graph with three rays, each ray with three edges:



In the next section we state relevant definitions and a few useful technical lemmas. In Section 3 we present our linear-time K_4 test. In the final section we discuss its application in recognizing graphs of bounded cutwidth and other problems relevant to VLSI design.

2 Preliminaries

Let $G = (V, E)$ denote a connected series-parallel multi-graph with n vertices and m edges. Without

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[1] A pair of edges uv and vw, with $u \neq v \neq w$, is *lifted* by deleting the edges uv and vw and adding the edge uw.

[2] A graph with no immersed K_4 can contain no topological K_4, which is one of several equivalent characterizations of series-parallel graphs.

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loss of generality, we assume m is $O(n)$. (G can contain at most $2n-3$ distinct edges to be series-parallel; at most six copies of any edge are required for K_4 containment.)

A *cut vertex* (*cut edge*) of G is a vertex (edge) whose removal disconnects G. A pair of vertices is *biconnected* if they can be separated by no cut vertex. A *biconnected component* of G is the subgraph induced by a maximal subset of pairwise biconnected vertices. A pair of edges, neither of which is a cut edge, is a *cut edge pair* if the removal of both disconnects G. A pair of vertices is *three-edge-connected* if they can be separated by no cut edge or cut edge pair. A *three-edge-connected component*[3] of G is a graph $G' = (V', E')$ for which:

1) $V' \subseteq V$,
2) $u, v \in V' \Rightarrow u$ and v are three-edge-connected in G,
3) $u \in V'$ and $v \in V - V' \Rightarrow u$ and v are not three-edge-connected in G, and
4) E' contains all edges induced by V' plus a set of *virtual edges* which join each pair of vertices in V' that appear in the same cut edge pair.

Several technical lemmas are required. We list three of them here. The first two permit us to consider only a graph's three-edge-connected components. The third lemma is well-known.

Lemma 1. *K_4 is immersed in G if and only if K_4 is immersed in some three-edge-connected component of G.*
Proof: The proof is by induction, and can be generalized to any three-edge-connected graph immersed in G. We omit the details. □

Lemma 2. *Each three-edge-connected component of a series-parallel graph is series-parallel.*
Proof: Omitted. □

Lemma 3. *Any series-parallel graph contains at least two vertices with at most two neighbors.*

3 The Algorithm

Our K_4 test, algorithm immerse, proceeds in two stages. We first invoke algorithm components, which breaks G into three-edge-connected components. Then we invoke algorithm test on each component until either a K_4 is encountered or all components have been eliminated (Lemma 1).

Algorithm components first finds the biconnected components of G using the method of [14]. The biconnected components are processed to eliminate all cut edge pairs. Each biconnected component of a series-parallel graph is a two-terminal series-parallel graph. These graphs can be defined recursively and have a simple structure [15]. We use properties of two-terminal series-parallel graphs to remove all cut edge pairs and add the appropriate virtual edges. Thus this algorithm exploits heavily the structure of series-parallel graphs, and hence is considerably simpler than a technique suggested [12] based on finding triconnected components with [6].

Pseudocode for components is lengthy and subtle. We omit it from this presentation.

Algorithm test is the heart of our method, and is described in the pidgin Algol that follows. Its input is series-parallel (Lemma 2). We proceed by examining vertices with at most two neighbors (Lemma 3). At each iteration, some such vertex is selected. The vertex is deleted (after deleting or lifting its incident edges) if we can determine that it is not contained in every copy of K_4 should K_4 be present. Otherwise, the vertex is marked. We assume all vertices are initially unmarked. As the algorithm progresses, a vertex may be marked then later unmarked again as its neighborhood changes.

Theorem 1. *Algorithm* immerse *correctly decides whether K_4 is immersed in an input graph.*
Proof: The analysis, especially for test, is based on a number of cases. We will not present it here. □

Theorem 2. *Algorithm* immerse *runs in $O(n)$ time.*
Proof: Straightforward (recall that graphs of interest have a linear number of edges). □

Immersion containment may alternatively be viewed in the following manner. A graph H is immersed in a graph G if H can be injected into G so that (1) the images of the vertices of H are distinct vertices of G and (2) the images of the edges of H are pairwise edge disjoint paths in G connecting the appropriate image vertices. Such an image of H is termed its *model* with respect to G.

A natural extension to detecting an immersed graph is finding one of its models. Assuming K_4 is indeed immersed, algorithm test can easily be modified to return a model of K_4. The first step is to locate four image vertices. This is trivial and can be done in constant time. Six edge-disjoint paths are then located by lifting edges at vertices with only one or two neighbors and storing the sequence of lifts used. Details are forthcoming in the full version of this paper.

[3] Such a component is not in general a subgraph.

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algorithm test
delete all but three copies of edges incident on vertices with one neighbor
if any articulation vertex has degree seven or more
    then report "yes" and halt
while there are unmarked vertices with fewer than three neighbors do
    if there are fewer than four vertices
        then report "no" and halt
    while there is a vertex v with exactly one neighbor w do
        delete v and every copy of the edge vw
        unmark w if it is marked
        end while
    if there is an unmarked vertex v with exactly two neighbors u and w
        then
            assume the multiplicity of uv is no less than that of vw
            if u or v is an articulation vertex
                then
                    lift all possible pairs of edges uv and vw,
                    delete any remaining copies of uv, and delete v
                else
                    if there is only one copy of edge vw
                        then
                            lift uv and vw
                            delete any remaining copies of uv
                            delete v
                            if the degree of u exceeds three
                                then report "yes" and halt
                                else unmark u if it is marked
                                end if
                        else mark v
                        end if
                end if
        end if
    end while
if there is a vertex with degree five or more
    then report "yes" and halt
end test
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4 Discussion

The cutwidth metric has appeared in various layout settings (see, for example, [2, 3, 7, 8]). Integer weights are often placed on edges in these applications. Such weights can directly be modeled by multiple edges for the purpose of immersion testing. This is a significant feature of the immersion order; multiple edges can be ignored in the topological and minor orders. Deciding whether a graph has small cutwidth is a natural problem in the layout process. Not only are graphs that represent real circuits often series-parallel, but more generally they tend to be sparse (with at most a linear number of edges) and of bounded degree (due to limitations on porting or fan-in/fan-out). These are necessary, but not sufficient, conditions for bounded cutwidth.

A fast K_4 immersion test is a fundamental method. It can be employed to indicate whether a graph is likely to have cutwidth three (a K_4 minor test can be used analogously to predict pathwidth two [9]). Although the presence of an immersed K_4 guarantees that a graph cannot have cutwidth three, its absence merely approximates the cutwidth at three. Such an absence says nothing about how to find a layout of width three even if many exist. To solve this, our algorithm can be used in conjunction with previously-studied *self-reduction* techniques [1, 5] to try to find a layout in $O(n^2)$ time.

For completeness, we observe that there is a much less practical way to test for K_4 in linear time. It is possible in principle to use the method sketched in [16] to obtain a tree-decomposition of width two, and then to use the dynamic programming formulation of [13]

on the decomposition. This two step procedure runs in $O(n)$ time, although the constant of proportionality is extremely high. Cutwidth can even be decided directly, although the time complexity is exponential in the width [11].

In addition to cutwidth, other combinatorial problems relevant to circuit design can benefit from a fast K_4 test. For example, a variety of other *load factor* [4] problems can be decided by a finite list of immersion tests, including K_4. Another candidate, though one only indirectly approachable with this method, is graph bisection. Note that bounded cutwidth is a sufficient, but not necessary, condition for bounded graph bisection. For problems such as these, there is interest in devising fast tests for other important graphs [10]. It will be interesting to see how quickly this general approach makes the transition from theory to practice.

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