Computing Planar Intertwines

Arvind Gupta*
School of Computing Sciences
Simon Fraser University
Burnaby, BC, Canada V5A 1S6

Russell Impagliazzo
Department of Computer Science
University of Toronto
Toronto, Ontario, Canada M5S 1A4

Abstract

The proof of Wagner's Conjecture by Robertson and Seymour gives a finite description of any family of graphs which is closed under the minor ordering, called the obstructions of the family. Since the intersection and the union of two minor closed graph families is again a minor closed graph family, an interesting question is that of computing the obstructions of the new family given the obstructions for the original two families. It is easy to compute the obstructions of the intersection but, until very recently, it was an open problem to compute the obstructions of the union. We show that if the original families are planar then the obstructions of the union are no larger than $n^{O(n^2)}$ where n is the size of the largest obstruction of the original family.

1 Introduction

Robertson and Seymour's proof of Wagner's conjecture [RSb] raises some interesting computational questions. An immediate corollary of the theorem is that for every family of graphs closed under minors (called lower ideals), the set of minimal graphs outside the family is finite. This set of graphs is called the obstruction set of the family. Robertson and Seymour also show that for every fixed graph H, there is a polynomial time algorithm which checks if a graph G contains H as a minor [RSa]. Therefore, every lower ideal has a polynomial time membership test.

It would thus seem that the Robertson and Seymour proof should give powerful techniques for devising graph-theoretic algorithms. However, the non-constructive nature of some parts of this work make it, in general, difficult to apply to some problems. For example, consider the problem of deciding whether a graph is embeddable in 3-space without a knot. It is easy to see that this class of graphs is closed under minors and therefore can be recognized in polynomial time. However, there is no specific polynomial time algorithm, or in fact even a recursive algorithm, known for this problem.

Some research has centered on devising constructive proofs of certain aspects of the Robertson and Seymour work. A general format for such work is to start with a specific lower ideal or a class of lower ideals and, for these ideals, either compute their obstruction sets or find an alternate polynomial time algorithm. For example, Fellows and Langston have identified a large number of such ideals and investigated different lines of attack in finding algorithms for these problems [FL88, FL89]. In [MRS88], Motwani, Raghunathan and Saran found the obstruction set for the problem of linkless embeddings of graphs in 3-dimensional space. Like knotless embeddings, no algorithm for this problem was previously known. More recently, Bodendiek and Wagner [BW89] gave upper bounds on the size of obstruction sets for fixed genus graphs. In [DR91], Djidjev and Reif give improved bounds for the same problem.

In this paper we are interested in operations under which lower ideals are closed. In particular, lower ideals are closed under both finite unions and intersections. In light of this, a natural problem is that of determining the obstructions of the new lower ideal in terms of the obstructions of the original ideals. For intersections the situation is quite easily resolved; the new obstructions are a subset of all the original obstructions.

For unions, the problem can be reduced to the following: *Given two graphs G_1 and G_2, what are the minimal graphs under the minor ordering containing both G_1 and G_2 as minors?* These minimal graphs are called the *intertwines* of G_1 and G_2. Now the obstructions of the union are obtained by computing

*Research performed while the author was a NSERC Post Doctoral Fellow in the Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Canada.

all the intertwines of all pairs from the original two obstruction sets.

A related problem is that of computing the topological embedding intertwines of two graph. That is, given two graph G_1 and G_2 what are the minimal graphs under the topological embedding relation which contain both G_1 and G_2 as topological embeddings. An upper bound on the size of the largest topological embedding intertwine gives an upper bound on the minor intertwines. The topological embedding intertwines have a number of properties which make their study simpler.

In 1976, Lovàsz conjectured that there were only a finite number of topological embedding intertwines. In 1978, Migram and Ungar [Ung78] independently also made the same conjecture and went on to compute the intertwines for a few specific examples. Robertson and Seymour's work directly implies that the minor intertwine must be finite. They also prove that the topological embedding intertwine is finite [RSb].

Until recently, no recursive bounds on the sizes of either minor or topological embedding intertwines were known, even for very restricted cases. Here, we obtain a bound of $n^{O(n^2)}$ for the size of planar intertwines of two planar graphs. This directly gives a doubly exponential time algorithm for finding the planar obstructions of the union of two lower ideals of graphs. In related work, we also obtain a triply exponential bound on the size of the planar topological embedding intertwine of planar graphs [GI91].

Very recently, Seymour and Thomas [ST91] have given a general bound on the sizes of topological embedding intertwines using techniques developed in the Robertson-Seymour papers. In general, their bound is an iterated tower of 2's whose height is an iterated tower of 2's of height n. However, they give a bound of $2^{2^{poly(t,n)}}$ for topological embedding intertwines having tree-width $\leq t$ which, together with other results of Robertson, Seymour and Thomas yields a bound of $2^{2^{2^{2^{poly(n)}}}}$ for the case considered here, namely planar minors.

The outline of the paper is as follows. In the next section, we introduce basic definitions and results about graph minors. In section 3, we give a number of technical results which will be useful in the main theorem. Section 4 contains the main result. Finally, in section 5, we present open problems.

2 Preliminaries

Our graphs are undirected and simple, that is, we do not allow multiple edges or self-loops. Our results can easily be extended to non-simple graphs. For G a graph, $V(G)$ and $E(G)$ will denote its vertex and edge set respectively. For $a, b \in \mathbb{N}$, we will denote by $[a, b]$ the set $\{a, a+1, \ldots, b\}$.

We will mainly be concerned with the minor relation on graphs. A graph H is a *minor* of a graph G if by performing a sequence of vertex deletions, edge deletions and edge contractions on G we obtain a graph isomorphic to H. We will use the following characterization of the minor relation.

Lemma 2.1 *Let H and G be graphs. Then, H is a minor of G if and only if there is an injective function $\mu : V(H) \rightarrow \{$subgraphs of $G\}$, such that*

1. *For every $v \in V(H)$, $\mu(v)$ is a connected non-null subgraph of G.*
2. *For $v, w \in V(H)$, if $v \neq w$ then $\mu(v) \bigcap \mu(w) = \emptyset$.*
3. *For each $e \in E(H)$, $e = \{v, w\}$, there is an $e' = \{x, y\} \in E(G)$, such that $x \in \mu(v)$ and $y \in \mu(w)$.*

We will call μ the *minor embedding* of H into G.

Definition: A family of graphs $\mathcal{L}$ is a *lower ideal* if whenever a graph $G \in \mathcal{L}$ and $H \leq_m G$ then $H \in \mathcal{L}$. The *obstruction set*, $\mathcal{O}$, of a lower ideal $\mathcal{L}$ is the minimal set of graphs (with respect to the minor ordering) not in $\mathcal{L}$. Then, a graph $G \notin \mathcal{L}$ if and only if for some $H \in \mathcal{O}$, $H \leq_m G$.

We can characterize Robertson and Seymour's result on graph minors in terms of obstruction sets of lower ideals.

Theorem 2.2 (Robertson-Seymour) *The obstruction set of every lower ideal is finite.*

Related to the minor relation is the topological embedding relation on graphs. Let us call an edge with one endpoint having degree at most 2 a 2-edge. A graph H is *topologically embedded* in a graph G, $H \leq_e G$, if by performing a sequence of vertex deletions, edge deletions and 2-edge contractions on G, we obtain a graph isomorphic to H. We use the following characterization of this relation.

Lemma 2.3 *Let H and G be graphs. Then, H is topologically embedded in G if and only if there is a pair of injective functions (τ, τ') such that*

1. $\tau : V(H) \to V(G)$. *We call* $\tau(V(H))$ *the* roots *of* H.

2. $\tau' : E(H) \to \{\text{simple paths in } G\}$.

3. *If* $e = \{x, y\} \in E(H)$ *then* $\tau'(e)$ *has endpoints* $\tau(x)$ *and* $\tau(y)$.

4. *For* $e, e' \in E(H)$, $e \neq e'$, $\tau'(e)$ *and* $\tau'(e')$ *are internally vertex disjoint.*

We will call the pair (τ, τ') the *topological embedding* of H into G.

Clearly if $H \leq_e G$ then $H \leq_m G$. The converse holds in general only if H is trivalent. We are interested in the following relationship between these two relations.

Lemma 2.4 *For every graph* H *there is a finite family of graphs* $H_1, H_2, \ldots, H_k$, $H \leq_m H_i$ *for every* i, *such that for any graph* G, $H \leq_m G$ *if and only if for some* i, $H_i \leq_e G$. *Furthermore, every* H_i *has the same genus as* H *and* $|V(H_i)| \leq O(|E(H)|)$.

Proof (sketch): Suppose H and G are graphs such that $H \leq_m G$ and let μ be the minor embedding of H into G which for every $v \in V(H)$ minimizes the size of $\mu(v)$. Then, for every $v \in V(H)$, $\mu(v)$ is a tree with leaves at most the degree of v in H. Furthermore, the number of internal vertices of $\mu(v)$ with degree at least 3 is bounded by the number of leaves of $\mu(v)$. Let $\mu'(v)$ be the tree obtained from $\mu(v)$ by contracting all 2-edges. Then, the H_i we are looking for is H such that for every vertex v we substitute $\mu'(v)$ where the leaves of $\mu'(v)$ are used to construct the adjacencies of v. Now each of the H_i's is obtained by substituting for each vertex v of G a tree with at most degree of v leaves and no internal degree 2 vertices. Since the total number of possible trees for each degree is bounded the result follows. The genus and size conditions are easy to verify. **QED**

We will refer to $H_1, \ldots H_k$ in Lemma 2.4 as the *expansions* of H.

2.1 Unions of Lower Ideals

Clearly lower ideals are closed under finite unions and intersections. Given the obstruction set of two lower ideals, one can ask for the obstruction set of their intersection or union. For intersection, the situation is quite straight-forward.

Lemma 2.5 *Let* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *be lower ideals with obstructions sets* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *respectively. Then the obstruction set of* $\mathcal{L}_1 \bigcap \mathcal{L}_2$ *is the set of minimal graphs in* $\mathcal{O}_1 \bigcup \mathcal{O}_2$.

The case of the union is more complicated.

Definition: For graphs G_1 and G_2 the *intertwine set* of G_1 and G_2, $\mathcal{I}(G_1, G_2)$, is

$$\{G : \ G_1, G_2 \leq_m G \text{ and for every } H \leq_m G, H \neq G, \text{ either } G_1 \not\leq_m H \text{ or } G_2 \not\leq_m H\}$$

Note that $\mathcal{I}(G_1, G_2)$ can, in general, have several elements; it is not difficult to construct examples where $|\mathcal{I}(G_1, G_2)| \geq 2^{O(|E(G_1)|+|E(G_2)|)}$.

Lemma 2.6 *Let* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *be lower ideals with obstructions sets* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *respectively. Then the obstruction set of* $\mathcal{L}_1 \bigcup \mathcal{L}_2$ *is a subset of* $\bigcup\{\mathcal{I}(G_1, G_2) : G_1 \in \mathcal{O}_1, G_2 \in \mathcal{O}_2\}$.

Proof: Let H be a graph, $H \notin \mathcal{L}_1 \bigcup \mathcal{L}_2$. Then, $H \notin \mathcal{L}_1$ and $H \notin \mathcal{L}_2$ and therefore there is a $G_1 \in \mathcal{O}_1$ and $G_2 \in \mathcal{O}_2$ such that $G_1, G_2 \leq_m H$. But, then by the definition of $\mathcal{I}(G_1, G_2)$, there is an $H' \in \mathcal{I}(G_1, G_2)$ such that $H' \leq_m H$. **QED**

Notice that if $\{G_1\}$ and $\{G_2\}$ are the obstruction sets of $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, then $\mathcal{I}(G_1, G_2)$ is exactly the obstruction set of $\mathcal{L}_1 \bigcup \mathcal{L}_2$.

We can similarly define the *topological embedding intertwine* of two graphs, $\mathcal{I}_e(G_1, G_2)$, as the set of minimal graphs (under topological embedding) which contain G_1 and G_2 as topological embeddings. Using Lemma 2.4 we obtain:

Lemma 2.7 *Let* G_1 *and* G_2 *be graphs and* $H \in \mathcal{I}(G_1, G_2)$. *Then there are graphs* G_1', G_2', *expansions of* G_1 *and* G_2 *respectively, such that* $H \in \mathcal{I}_e(G_1', G_2')$.

Let G_1, G_2, G_1', G_2', H be as in the above lemma. If (τ_1, τ_1') is the embedding of G_1' in H and (τ_2, τ_2') is the embedding of G_2', then the *terminals* of H are $\tau_1(V(G_1')) \bigcup \tau_2(V(G_2'))$, that is the roots of G_1' plus the roots of G_2'. The *non-terminals* of H are all vertices which are not terminals.

2.2 Grids and Planar Graphs

We denote a $k \times \ell$ grid by $\mathcal{G}_{k,\ell}$ We will need the following result of Schnyder [Sch90].

Theorem 2.8 *For every planar graph* G *such that* $|V(G)| = n$, $G \leq_m \mathcal{G}_{n-2,n-2}$.

Although this specific bound is non-trivial to prove, by considering a rectilinear drawing of G, it is not difficult to see that there is some polynomial f such that, for every planar graph G of size n, $G \leq_m \mathcal{G}_{f(n),f(n)}$.

Let G be a planar graph with some fixed embedding on the plane. Let C be a simple circuit of G. Then, the plane with C removed divides into two regions, an infinite one and a finite one. We call the finite region the disc induced by C and denote it $\Delta(C)$. $\Delta(C)$ will not contain C itself.

3 Technical Results

Let G_1 and G_2 be graphs where $|E(G_1)|+|E(G_2)| = m$. Consider a minor intertwine of G_1 and G_2, say H. Let G'_1 and G'_2 be the topological expansions of G_1 and G_2 given in Lemma 2.7. Throughout this section assume that G_1, G_2, G'_1, G'_2 and H are all fixed and that for $i = 1, 2$, (τ_i, τ'_i) is the topological embedding of G'_i in H. We begin by defining a labelling on $E(H)$ and $V(H)$.

Lemma 3.1 *Let $e \in E(H)$ such that neither endpoint is a terminal. Then, there is exactly one $e' \in E(G'_1) \bigcup E(G'_2)$ such that e is in the image of e'.*

Proof: If there is no edge e' then we can delete e from H contradicting the minimality of H. Furthermore, clearly there cannot be two edges of $E(G'_1)$ (respectively $E(G'_2)$) whose image contains e since τ'_1 (respectively τ'_2) maps edges of G'_1 (G'_2) to internally vertex disjoint paths. Suppose $e'_1 \in E(G'_1)$ and $e'_2 \in E(G'_2)$ both contain e in their image. But then H with e contracted contains both G_1 and G_2 as minors and is thus not minimal. **QED**

For e, e' as in Lemma 3.1, we denote e' by $l(e)$.

Lemma 3.2 *Every non-terminal of H is in the image of exactly one edge of G'_1 and one edge of G'_2.*

Proof: Let v be a non-terminal of H. First, v can be in the image of at most one edge of G'_1 and G'_2 since the image of edges are internally disjoint paths. Now, there must be at least one edge $e_1 \in E(G'_1)$ such that v is in $\tau'_1(e_1)$ otherwise we can either delete v or v has degree 2 and we can contract one of the edges through v and similarly for $E(G'_2)$. **QED**

If v is a non-terminal, let $e_1 \in E(G'_1)$ and $e_2 \in E(G'_2)$ as in Lemma 3.2. Then, we write $l_1(v) = e_1$ and $l_2(v) = e_2$. We note the following corollary of Lemma 3.1 and Lemma 3.2.

Corollary 3.3 *Let v be a non-terminal of H. Then, v has degree 4.*

Now, suppose we also assume that G_1, G_2, G'_1, G'_2 and H are all planar and that we have fixed some planar drawing of H. This induces a fixed planar drawing of G'_1 and G'_2. We begin by studying circuits of H.

Lemma 3.4 *(Refer to figure 1) Let $e_1, e_2 \in E(G'_1)$ and $f_1, f_2 \in E(G'_2)$. Suppose there is an edge $\bar{e}_1$ of $\tau'_1(e_1)$, an edge $\bar{e}_2$ of $\tau'_1(e_2)$, a subpath P_1 of $\tau'_2(f_1)$ and a subpath P_2 of $\tau'_2(f_2)$ such that $P_1, \bar{e}_1, P_2, \bar{e}_2$ forms a simple cycle C of H. If any vertex of H is in $\Delta(C)$ then there is a terminal of H in $\Delta(C)$.*

Proof: First, $f_1 \neq f_2$ otherwise H is not minimal since we can reroute along $\bar{e}_1$. Let v be a non-terminal in $\Delta(C)$ such that $l_2(v) = g_2$. Since H is planar, and the endpoints of $\tau'_2(g_2)$ are not in $\Delta(C)$, the path $\tau'_2(g_2)$ must cross C at least 2 times. If $g_2 \neq f_1, f_2$ then by Lemma 3.2, $\tau'_2(g_2)$ cannot cross C. If $g_2 = f_1$ or $g_2 = f_2$ then the path $\tau'_2(g_2)$ is not simple. **QED**

3.1 Jumps

Here we consider edges of G'_i ($i = 1, 2$) whose images are long paths in H. For the remainder of this section let $e \in E(G_1)$ and $f_1, \ldots, f_k \in E(H)$ such that

1. each f_i has endpoints x_i and y_i where x_i is on $\tau'_1(e)$;
2. $x_1, x_2, \ldots, x_k$ occur in that order on $\tau'_1(e)$ (ie. ordered by distance from one endpoint);
3. the y_i are all distinct; and
4. if z, z' are the endpoints of $\tau'_1(e)$ then by viewing $\tau'_1(e)$ as a directed path in the plane from z to z' any edge intersecting $\tau'_1(e)$ is in one of two orientations which we will call "up" and "down". All edges f_i have the same orientation with respect to $\tau'_1(e)$.

If $e \in E(G_1)$ and x is an endpoint of $\tau'_1(e)$ and u, v are two other vertices on $\tau'_1(e)$ then we say that relative to x, u is *left* (resp. *right*) of v on $\tau'_1(e)$ if x, u, v (resp. x, v, u) occur in that order on $\tau'_1(e)$.

Definition: A *jump* on $f_1, \ldots, f_k$ is an interval $[j_1, j_2]$, $1 \leq j_1 < j_2 \leq k$, such that $l_1(y_{j_1}) = l_1(y_{j_2}) = a$ for some $a \in E(G'_1)$; the subpath of $\tau'_1(a)$ with endpoints y_{j_1} and y_{j_2} contains no other

y_i; and $j_2 > j_1 + 1$. The jump is *maximal* if there is no other jump $[j'_1, j'_2]$ such that $[j_1, j_2] \subseteq [j'_1, j'_2]$ (see Figure 2).

With every jump $[j_1, j_2]$ we associate a disc $\Delta(j_1, j_2)$ bounded by the circuit which starts at y_{j_1}, follows $\tau'_1(a)$ to y_{j_2}, goes to x_{j_2}, follows $\tau'_1(e)$ to x_{j_1} and returns to y_{j_1}.

Lemma 3.5 *Suppose that for* $1 \leq i \leq k$, $l_1(y_i) = a$ *for some* $a \in E(G_1)$. *Then there is a sequence* $1 \leq i_1 < i_2 < \cdots < i_\ell \leq k$ *such that* $y_{i_1}, y_{i_2}, \ldots, y_{i_\ell}$ *occur in that order on* $\tau'_1(a)$ *and* $\ell \geq \frac{k}{5}$.

Proof (Sketch): Suppose $[j_1, j_2]$ is a jump on $f_1, \ldots, f_k$. Then at least one endpoint of $\tau'_1(a)$ must occur in $\Delta(j_1, j_2)$ since all y_i between y_{j_1} and y_{j_2} are on $\tau'_1(a)$. Thus there are at most 2 maximal jumps.

Let $[j_1, j_2]$ be a jump, $1 \leq j_1 < j_2 \leq k$, so that exactly one endpoint w of $\tau'_1(a)$ lies in $\Delta(j_1, j_2)$. Without loss of generality assume that w, j_2, j_1 occur in that order on $\tau'_1(a)$. Then, we claim that there is an ℓ, $j_1 \leq \ell \leq j_2$, so that starting at w, the vertices $y_\ell, y_{\ell-1}, \ldots, y_{j_1}$ occur in that order and similarly for vertices $y_\ell, y_{\ell+1}, \ldots, y_{j_2}$. The proof is by induction on $j_2 - j_1$.

If $[j_1, j_2]$ does not properly contain any jump then the path from w passes through $y_{j_1+1}, \ldots, y_{j_2}$ in that order and we let $\ell = y_{j_1+1}$.

Otherwise let $[j'_1, j'_2]$ be the unique maximal jump in the interval $[j_1 + 1, j_2]$. Uniqueness is guaranteed since w must lie in any such jump. Therefore, the vertices $y_{j'_2}, y_{j'_2+1}, \ldots, y_{j_2}$ occur in that order starting at w since there are no jumps in the interval $[y_{j'_2}, y_{j_2}]$. Furthermore, $j'_1 = j_1+1$ since otherwise an endpoint of $\tau'_1(a)$ would lie in $[j_1, j_2]$ but not in $[j'_1, j'_2]$. By induction there is an ℓ in $[j'_1, j'_2]$ such that $y_\ell, y_{\ell+1}, \ldots, y_{j'_2}$ occur in that order starting at w and similarly for $y_\ell, y_{\ell-1}, \ldots, y_{j'_1}$. Thus, $y_\ell, \ldots, y_{j'_2}, \ldots, y_{j_2}$ occur in that order. Furthermore, starting at y_{j_1+1} and going to y_{j_1}, if $\tau'_1(a)$ passes through a y_i then $i > \ell$. Thus, $\tau'_1(a)$ starting at w passes through $y_\ell, y_{\ell-1}, \ldots, y_{j_1}$ in that order.

If the jump $[j_1, j_2]$ contains two endpoints, then no maximal jump properly inside the interval $[j_1, j_2]$ contains both endpoints. Thus, we can apply the claim to maximal jumps inside $[j_1, j_2]$ to yield an ℓ_1, ℓ_2, ℓ_3, $j_1 \leq \ell_1 \leq \ell_2 \leq \ell_3 \leq j_2$, such that $y_{\ell_1}, y_{\ell_1-1}, \ldots, y_{j_1}$, $y_{\ell_1}, y_{\ell_1+1}, \ldots, y_{\ell_2}$, $y_{\ell_3}, y_{\ell_3-1}, \ldots, y_{\ell_2}$, and $y_{\ell_3}, y_{\ell_3+1}, \ldots, y_{j_2}$ each occur in that order.

Finally if we consider all y_i such that i is not inside a jump. Then $\tau'_1(a)$ must pass consecutively through these y_i. Thus, the interval $[1, k]$ can be partitioned into at most 5 pieces such that the y_i's occur consecutively on $\tau'_1(a)$ in each piece and the result follows. **QED**

We can generalize Lemma 3.5 to allow for arbitrary labels on the y_i.

Lemma 3.6 *There is an* $a \in E(G_1)$ *and* $1 \leq i < j \leq k$ *such that* $\tau'_1(a)$ *goes consecutively through* $y_i, y_{i+1}, \ldots, y_j$ *and* $j - i$ *is at least* $\frac{k}{O(m^2)}$ *and there are no terminals in* $\Delta(y_\ell, y_{\ell+1})$ *for* $i \leq \ell < j$.

Proof (Sketch): Let $a \in E(G_1)$ such that $\tau'_1(a)$ passes through $y_{i_1}, y_{i_2}, \ldots, y_{i_\ell}$ where $1 \leq i_1, i_2, \ldots, i_\ell \leq k$ and $\ell \geq \frac{k}{m}$. By Lemma 3.5, there are $j_1, \ldots, j_{\ell'}$, $\{j_1, \ldots, j_{\ell'}\} \subseteq \{i_1, \ldots, i_\ell\}$, such that $1 \leq j_1 < j_2 < \cdots < j_{\ell'}$ and $\ell' \geq \frac{k}{5m}$. Consider the discs $\Delta(j_r, j_{r+1})$ for $1 \leq r < \ell'$. By Lemma 3.4, at most $2m$ of these discs contains a vertex of H since each disc which contains a vertex of H contains a terminal. Therefore,.there must be $\frac{k}{5m(2m+1)}$ consecutive empty discs giving the required interval. **QED**

We next look at the types of intersections two paths can make on the plane. Our presentation here is informal but it can be formalized. For G a planar graph suppose P_1 is a path in G, $P_2 = v_1, v_2, v_3$ is a path of G of length 3 and P_1 and P_2 have only the vertex v_2 in common where v_2 is not as endpoint of P_1. Let w be an endpoint of P_1 and consider a fixed planar drawing of G. Then, relative to w there are exactly three different ways in which P_1 and P_2 can intersect in the drawing (see Figure 3). We call these three types of intersections a $\bigvee$-intersection, a $+$-intersection and a $\bigwedge$-intersection relative to w. Furthermore, we will call v_1 and v_3 the *ends* of the intersections relative to w and more specifically for a $\bigvee$-intersection, we will call v_1 the *left-end* and v_3 the *right-end*.

Lemma 3.7 *Let* $e \in E(G_1)$ *and* w *be an endpoint of* $\tau'_1(e)$. *Suppose a path* $W = v_1, v_2, v_3$ *is a* $\bigvee$*-intersection with* $\tau'_1(e)$ *relative to* w *and* $l_1(v_1) = l_1(v_3) = a$. *Consider the subpath* P *of* $\tau'_1(a)$ *between* v_1 *and* v_3. *Then, if the disc bounded by the circuit* P, v_3, v_2, v_1 *contains no terminals, there is a path* $W' = w_1, u, w_2$ *such that* u *is on* P *and* W' *is a* $\bigvee$*-intersection with* P *relative to* v_1. *We call* W' *a* child of W *(see Figure 4).*

Proof: Since H is minimal, $|P| > 2$ and there is a non-terminal u on P. Suppose $l_1(u) = a$ where $a \in E(G_1)$. Let $\tau'_1(a) = P_1, w_1, u, w_2, P_2$ where P_1

and P_2 are subpaths of $\tau_1'(a)$. Since by Lemma 3.4 no part of $\tau_1'(a)$ can be inside the disc bounded by the circuit P, v_3, v_2, v_1, either w_1, u, w_2 or w_2, u, w_1 is the required $\bigvee$-intersection. **QED**

Using Lemma 3.7, we define a $\bigvee$-intersection P' to be a *descendant* of a $\bigvee$-intersection P if there is a sequence of $\bigvee$-intersections $P_1 = P, P_2, \ldots, P_k = P'$ such that for $1 < i \leq k$, P_i is a child of P_{i-1}.

4 Planar Intertwines

In this section we prove the main result, namely

Theorem 4.1 *For planar graph G_1 and G_2 and H a planar intertwine of G_1 and G_2, if $|E(G_1)|+|E(G_2)| = m \geq 2$ then $|E(H)| = m^{O(m^2)}$.*

This theorem follows directly from Theorem 2.8 and the following lemma:

Lemma 4.2 *Let G_1 and G_2 be planar graphs and H be a planar topological embedding intertwine of G_1 and G_2. Let $m = |E(G_1)| + |E(G_2)|$. If $|E(H)| > m^{O(m^2)}$ then $\mathcal{G}_{m,m} \leq_m H$.*

The remainder of this section constitutes a proof of Lemma 4.2. Let G_1 and G_2 be planar graphs and H be as in Lemma 4.2. Let (τ_1, τ_1') and (τ_2, τ_2') be the topological embeddings of G_1 and G_2 into H respectively.

We begin with an outline of the proof. Suppose H is large. Then the image of some edge of G_1 is a long path in H. Let v be an endpoint of this path. Then, every non-terminal of this path must have either a $\bigvee$, $+$, or $\bigwedge$-intersection relative to v with a subpath of an edge of G_2. Suppose there are a large number of $\bigvee$-intersections relative to v. Then, by applying Lemma 3.6 there is some edge e of G_1 such that $\tau_1'(e)$ goes consecutively through the endpoints of a large fraction of the $\bigvee$-intersections. Now, by Lemma 3.7, each $\bigvee$-intersection gives rise to a new $\bigvee$-intersection on $\tau_1'(e)$. Continuing we construct a grid.

The only problem occurs if we encounter the same edge of G_1 more than once. In that case if we are, for the most part, using a different part of the image of that edge then we can continue constructing the grid. Otherwise, we show that the paths corresponding to the edges of G_1 which we have encountered so far can be rerouted, thus contradicting the minimality of H. The argument for the case where there are many $\bigwedge$-intersections is symmetric. If there is no edge whose image has many $\bigvee$-intersections or $\bigwedge$-intersections, then the image of some edge has many $+$-intersections. Now, the basic idea is the same but a few technical details are different.

Assume that $|E(H)| > m^{O(m)}$. Since every non-terminal of H is labelled by some edge of G_1, then for some edge e of G_1, $\tau_1'(e)$ is a path of length $\geq m^{O(m)}$ in H. We begin by considering the case where there are at least $m^{O(m)}$ $\bigvee$-intersections on some $\tau_1'(e)$. The case of $\bigwedge$-intersections is similar and we will not discuss it.

Definition: (Refer to figure 5). Let $k, \ell \in \mathbf{N}$. A planar graph G is a $k \times \ell$ $\bigvee$-grid, $\bigvee_{k,\ell}$ if the following holds: There are k vertex disjoint simple paths in G, $P_1, \ldots, P_k$ where P_i has endpoints e_i and f_i so that

1. there are vertices $v_{i,j}$ for $(i,j) \in [1,k] \times [1,\ell]$, $u_{i,j}, w_{i,j}$ for $(i,j) \in [2, k+1] \times [1,\ell]$ which are all distinct;
2. $P_1 = e_1, Q_{1,1}, v_{1,1}, Q_{1,2}, v_{1,2}, \ldots, v_{1,\ell}, Q_{1,\ell}, f_1$ where $Q_{1,j}$ are arbitrary paths of G;
3. for $1 < i \leq \ell$, $P_i = e_i, Q_{1,1}, u_{i,1}, U_{i,1}, v_{i,1}, W_{i,1}, w_{i,1}, Q_{1,2}, \ldots, Q_{i,\ell}, f_i$ where $Q_{i,j}, U_{i,j}, W_{i,j}$ are arbitrary paths of G of length ≥ 1;
4. for $(i,j) \in [1,k] \times [1,\ell]$ there is an edge from $v_{i,j}$ to $u_{i+1,j}$ and $w_{i+1,j}$; and
5. there is no vertex v' on P_1, v' between $v_{1,1}$ and $v_{1,\ell}$ and distinct from the $v_{1,j}$ and e_1, f_1 such that there are two edges from v' to P_2.

Definition: Suppose H is the planar intertwine of two graphs G_1 and G_2. A $\bigvee$-grid in H is *monochromatic* if for every P_i there is an edge $a_i \in E(G_1)$ such that P_i is a subpath of $\tau_1'(a_i)$.

A $\bigvee$-grid is illustrated in Figure 5. The idea is to inductively build up $\bigvee_{n,n}$ since $\mathcal{G}_{n,n} \leq_m \bigvee_{n,n}$. If we are ever stuck we show that H was not minimal. Therefore it is sufficient to prove the following:

Lemma 4.3 *Suppose that H contains a monochromatic $k \times \ell$ $\bigvee$-grid as a subgraph, $k \leq m$. Then, H contains a monochromatic $(k+1) \times \frac{\ell}{m^{O(1)}}$ $\bigvee$-grid as a subgraph.*

It suffices to prove this lemma since, by assumption, H contains a monochromatic $1 \times m^{O(m)}$ $\bigvee$-grid.

For $a \in E(G_1)$, suppose u, v are vertices on $\tau_1'(a)$. Then, we denote the subpath of $\tau_1'(a)$ between u and v by $T_a(u,v)$. Now, suppose we have found a monochromatic $k \times \ell$ $\bigvee$-grid as a subgraph of H. Let $l_1(e_i) = a_i$. Then by Lemma 3.6, there is some $a \in E(G_1)$ such that $\tau_1'(a)$ goes consecutively through $\ell' = \frac{\ell}{m^{O(1)}}$ of the vertices in $u_{k+1,1}, w_{k+1,1}, u_{k+1,2}, w_{k+1,2}, \ldots, u_{k+1,\ell}, w_{k+1,\ell}$, say $u_{k+1,r}, \ldots, w_{k+1,s}$ where $s - r \geq \ell'$. Thus, by Lemma 3.4, the discs bounded by $u_{k+1,j}, T_a(u_{k+1,j}, w_{k+1,j}), w_{k+1,j}, v_{k,j}$, $(r \leq j \leq s)$, do not contain any vertex of H. Let e_{k+1} be the vertex of $\tau_1'(a)$ adjacent to $u_{k+1,r}$ such that $e_{k+1}, u_{k+1,r}, w_{k+1,r}$ occur in that order in $\tau_1'(a)$ and similarly let f_{k+1} be the vertex of $\tau_1'(a)$ adjacent to $w_{k+1,s}$ such that $u_{k+1,s}, w_{k+1,s}, f_{k+1}$ occur in that order in $\tau_1'(a)$.

By Lemma 3.7, for each pair $u_{k+1,t}, w_{k+1,t}$, $r \leq t \leq s$, there is a $\bigvee$-intersection with $\tau_1'(a)$ relative to e_{k+1}, say $u_{k+2,t}, v_{k+1,t}, w_{k+2,t}$ where $u_{k+2,t}$ is the left-end and $w_{k+2,t}$ the right-end of the intersection.

If $T_{a_{k+1}}(e_{k+1}, f_{k+1}) \bigcap T_{a_i}(e_i, f_i) = \varnothing$ for $1 \leq i \leq k$ then the above construction yields the required $\bigvee$-grid. If $T_{a_{k+1}}(e_{k+1}, f_{k+1}) \bigcap T_{a_i}(e_i, f_i) \neq \varnothing$ then $i = 1$ since H is planar. Also, the endpoint of $\tau_1'(a_1)$ closer to e_1 than to f_1 must be the same as the endpoint of $\tau_1'(a_1)$ closer to e_{k+1} than to f_{k+1} because otherwise by the planarity of H, $T_{a_1}(e_{k+1}, f_{k+1}) \bigcap T_{a_1}(e_1, f_1) = \varnothing$.

Since the remainder of the proof only involves vertices $u_{i,j}, v_{i,j}, w_{i,j}$ such that $r \leq j \leq s$, we rename these vertices $u_{i,j-r+1}, v_{i,j-r+1}, w_{i,j-r+1}$. Notice that for $1 \leq j \leq \ell'$, the $\bigvee$-intersection $u_{k+1,j}, v_{k,j}, w_{k+1,j}$ is a descendant of the intersection $u_{2,j}, v_{1,j}, w_{2,j}$. Furthermore, $u_{k+1,j}, w_{k+1,j}$ are on $\tau_1'(a_1)$.

Without loss of generality , suppose that $u_{k+1,1} \in T_{a_1}(v_{1,1}, v_{1,k})$. The case where $u_{k+1,\ell'} \in T_{a_1}(v_{1,1}, v_{1,k})$ is symmetric.

If, relative to e_1, $u_{k+1,k}$ occurs to the right of $v_{1,\ell'}$ then it is sufficient to ignore the first k $\bigvee$-intersections on each path $T_{a_i}(e_i, f_i)$ that were in our grid. Thus, we obtain our $\bigvee$-grid by taking our original grid and taking the subgraph induced by $u_{i+1,j}, v_{i,j}, w_{i+1,j}, T_{a_i}(u_{i+1,j}, u_{i+1,j+1})$ where $1 \leq i \leq k$ and $k+1 \leq j \leq \ell'$.

Now suppose $u_{k+1,k}$ occurs to the left of $v_{1,\ell'}$. We will show that in this case H is not minimal. For each $\bigvee$-intersection $u_{k+1,i}, v_{k,i}, w_{k+1,i}$, $(1 \leq i \leq k)$, there is at least one $\bigvee$-intersection $u_{2,j}, v_{1,j}, w_{2,j}$, $(1 \leq j \leq \ell')$, such that $v_{1,j}$ occurs between $u_{k+1,i}$ and $w_{k+1,i}$. This follows from Lemma 3.7 and the fact that there are no $\bigvee$-intersections between $u_{2,j}, v_{1,j}, w_{2,j}$ and $u_{2,j+1}, v_{1,j+1}, w_{2,j+1}$ where $1 \leq j < \ell'$. (see the definition of a $\bigvee$-grid). For each $\bigvee$-intersection $u_{k+1,j}, v_{k,j}, w_{k+1,j}$, $(1 \leq j \leq k)$, choose a $\bigvee$-intersection $u_{2,p_j}, v_{1,p_j}, w_{2,p_j}$ such that v_{1,p_j} occurs between $u_{k+1,j}$ and $w_{k+1,j}$ on $\tau_1'(a_1)$ (see Figure 6).

We next define a set of internally vertex disjoint paths $R_1, R_2, \ldots, R_k$ such that

1. R_i has endpoints $v_{i,1}$ and $v_{i,\ell'}$;
2. for every vertex v on these paths, v is on $T_{a_i}(e_i, f_i)$ for some i; and
3. there is some edge g of H such that g is on $T_{a_i}(e_i, f_i)$ for some i, but g is not on any of the R_i.

Such a set of paths contradicts the minimality of H since $H \backslash \{g\}$ contains both G_1 and G_2 as embeddings.

For $2 \leq i \leq k+1$ and $1 \leq j \leq \ell'$, define the *successor* of a vertex $w_{i,j}$, $S(w_{i,j})$, as follows:

$$S(w_{i,j}) = \begin{cases} w_{i+1,j+1} & \text{if } i < k+1 \text{ and } j < \ell' \\ w_{2,p_j+1} & \text{if } i = k+1 \text{ and } p_j < \ell' \\ v_{i,\ell'} & \text{otherwise} \end{cases}$$

Furthermore, define the *link* of $w_{i,j}$, $L(w_{i,j})$, as follows:

$$L(w_{i,j}) = \begin{cases} T_{a_i}(w_{i,j}, v_{i,j+1}), w_{i+1,j+1} & i < k+1, j < \ell' \\ T_{a_1}(w_{i,j}, v_{1,j+1}), w_{2,p_{j+1}} & i = k+1, p_j < \ell' \\ T_{a_i}(w_{i,j}, v_{i,\ell'}) & \text{otherwise} \end{cases}$$

From the definition of successor and link, we can immediately conclude the following lemma.

Lemma 4.4 *Let $1 \leq i, i' \leq k+1$ and $1 \leq j, j' \leq \ell'$. If $w_{i,j} \neq w_{i',j'}$ then $S(w_{i,j}) \neq S(w_{i',j'})$ and $L(w_{i,j})$ is internally vertex disjoint from $L(w_{i',j'})$.*

For $1 \leq i \leq k$ and $1 \leq t \leq \ell'$ define a path $R(i,t)$ as follows:

$$\begin{aligned} R(i,0) &= (v_{i,1}, w_{i+1,1}) \\ R(i,t+1) &= R(i,t), L(S^{(t)}(w_{i,1})) \\ R(i,k+1) &= R(i,k), T_{a_i} S^{(k+1)}(w_{i,1}) \end{aligned}$$

Now, let $R_i = R(i, k+1)$. We show that $R_1, \ldots, R_k$ satisfy the required properties.

Lemma 4.5 *For $1 \leq i \leq k$, let the i^{th} vertex sequence be given by:*

$$w_{i,1}, S(w_{i-1,1}), S^{(2)}(w_{i-2,1}), \ldots, S^{(j)}(w_{i-j,1}), \ldots, S^{(i-2)}(w_{2,1}), S^{(i-1)}(w_{k+1,1}), \ldots, S^{(k-1)}(w_{i-1,1})$$

Then, the elements of the i^{th} vertex sequence appear in order on P_i starting at e_i.

Proof (Sketch): First notice that for any i and $1 \le j < j' \le \ell'$, the vertices $S(w_{i,j})$ is to the left of $S(w_{i,j'})$ with respect to $e_{(i mod k)+1}$. We prove the lemma simultaneously for all i for prefixes of length t, $1 \le t \le k+1$, of the vertex sequences.

For $t = 1$, the claim is obvious. Now, suppose the lemma holds for $t = r$. Then, by the induction hypothesis for $i' = i-1$ if $i \ne 1$ and $i' = k$ when $i = 1$, the elements of the prefix of the i'^{th} vertex sequence appear in order on $P_{i'}$ starting at $e_{i'}$. Therefore, their successors appear in order on P_i starting at e_i. But their successors are exactly the 2^{nd} through $(r+1)^{st}$ elements of the i^{th} vertex sequence. Furthermore, w_{p_1} is the left-most w on P_i with respect to e_i. **QED**

From Lemma 4.4 it immediately follows that the R_i are vertex disjoint. Furthermore, by definition every vertex on these paths is a vertex of $T_{a_i}(e_i, f_i)$ for some i. Finally, if g is the edge of $\tau_1'(a_1)$ with one endpoint $v_{1,1}$ and the other endpoint to the right of $v_{1,1}$ relative to e_1, then g is not used in any of the R_i. Thus H is not minimal.

Now suppose that there is no edge of either G_1 or G_2 whose image has at least $m^{O(m)}$ $\bigvee$-intersections. Therefore there must be some edge of G_1 or G_2 whose image has at least $m^{O(m)}$ $+$-intersections. We follow the same proof outline as in the $\bigvee$-intersection case. The first complication in this part of the proof is that when we construct a new level of the grid we must make sure that there are no $\bigvee$-intersections between the $+$-intersections. This involves dividing by $m^{O(m)}$ at each level of the grid thus giving the larger bound. Also, the rerouting strategy when new levels of the grid intersect previous levels is quite different.

5 Conclusions and Open Problems

There is a large amount of work which remains to be done in this area. There is a vast gap between the upper bounds and known lower bounds. The best lower bound for even topological embedding of general graphs is $O(n^3)$ as compared to the upper bound of a tower of tower of 2's as given by Seymour and Thomas [ST91]. Although it is probably not difficult to obtain slight improvements in the lower bounds, no natural candidate for the asymptotic complexity of this function presents itself. Further investigation of intertwine bounds may give insight into the enormous constants found in parts of the Robertson and Seymour proof.

There are also a number of other natural operations on lower ideals. For example, consider the following problem suggested by Fellows and Langston. Let $\mathcal{F}$ be a lower ideal and define $\mathcal{F}+1$ to be:

$$\{G : \exists v. G \backslash v \in \mathcal{F}\}.$$

Then, $\mathcal{F}+1$ is a lower ideal but it is unknown how the obstructions of $\mathcal{F}+1$ relate to those of $\mathcal{F}$. In particular, for $\mathcal{F}$ the family of planar graphs, $\mathcal{F}+1$ is the set of *apex* graphs, and its obstructions are currently unknown.

Acknowledgements

We thank Stephen Cook, Charles Rackoff and Neil Robertson for a number of interesting and useful discussions.

References

[BW89] R. Bodendiek and K. Wagner. Solution to König's graph embedding problem. *Math. Nachr.*, 140:251–272, 1989.

[DR91] Hristo Djidjev and John Reif. An efficient algorithm for the genus problem with explicit construction of forbidden subgraphs. In 31^{st} *Symposium on Foundations of Computer Science*, pages 337–347, 1991.

[FL88] M. Fellows and M. Langston. Nonconstructive tools for proving polynomial-time decidability. *Journal of the Association for Computing Machinery*, 35(3):727–739, July 1988.

[FL89] M. Fellows and M. Langston. On search, decision and the efficiency of polynomial-time algorithms. In 21^{st} *ACM Symposium on Theory of Computing*, pages 501–512, 1989.

[GI91] A. Gupta and R. Impagliazzo. Intertwining planar graphs with topological embeddings, 1991. in preperation.

[MRS88] R. Motwani, A. Raghunathan, and H. Saran. Constructive results from graph minors: linkless embeddings. In 29^{th} *Symposium of Foundations of Comp. Sci.*, pages 398–409, 1988.

[RSa] N. Robertson and P. Seymour. Graph Minors XIII. The disjoint paths problem. in preparation.

[RSb] N. Robertson and P. Seymour. Graph Minors XV. Wagner's conjecture. in preparation.

[Sch90] Schnyder. Embedding planar graphs on the grid. In 1st *ACM-SIAM Symposium on Discrete Algorithms*, 1990.

[ST91] Paul Seymour and Robin Thomas. Personal communication, 1991.

[Ung78] P. Ungar. Dissections and intertwinings of graphs. *American Mathematical Monthly*, 85(8):664–666, 1978.

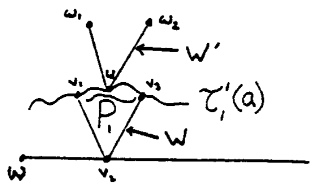
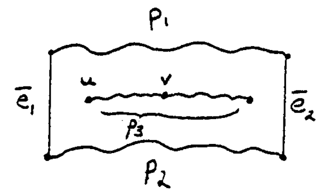


Figure 4



(u is a terminal node)

Figure 1

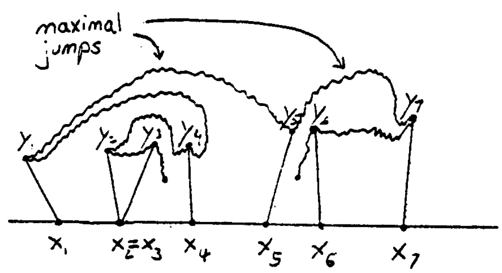


Figure 2

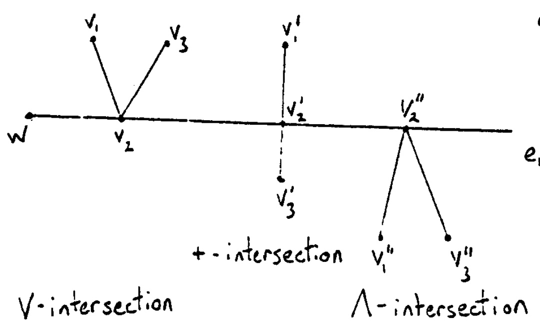


Figure 3

A V-grid

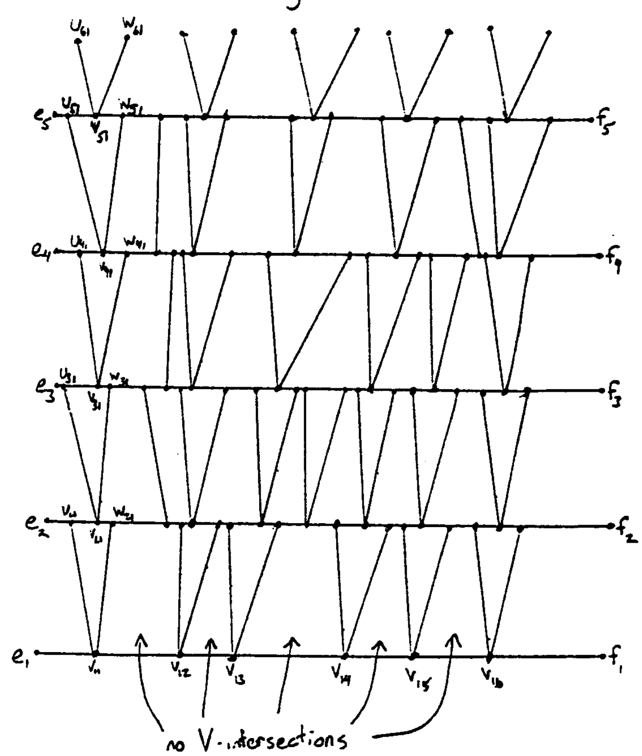


Figure 5

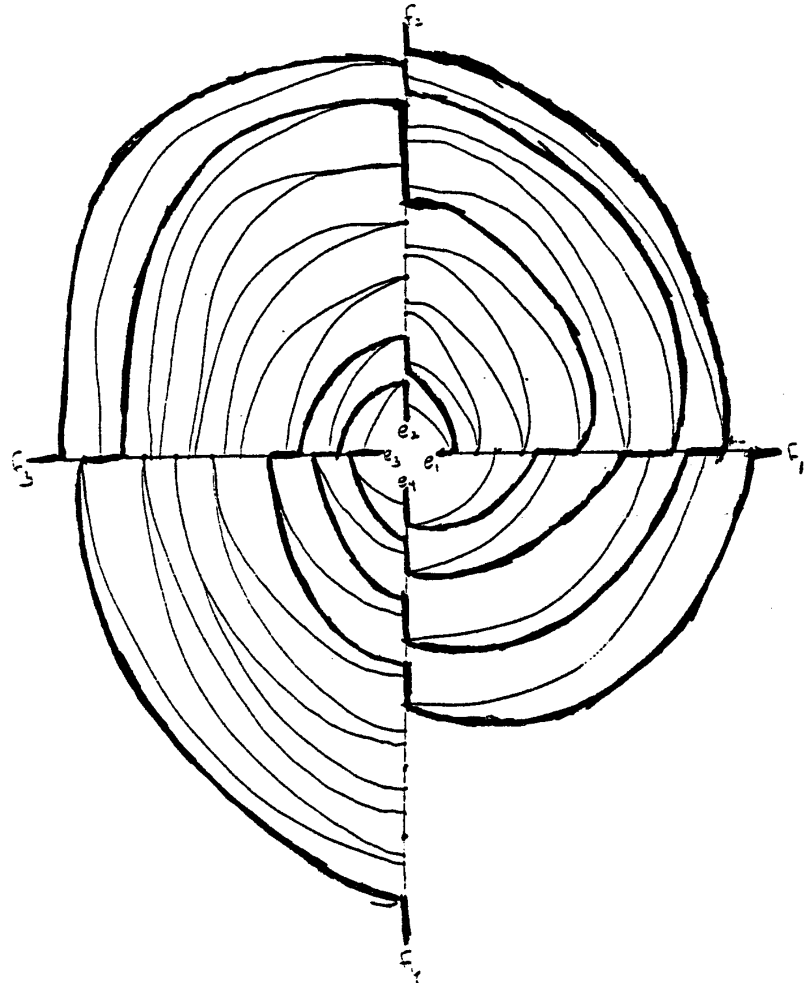


Figure 6: Using a spiral to re-route